ON q-GENERATING FUNCTIONS AND CERTAIN FORMULAS OF DAVID ZEITLIN[1]

BY

H. M. SRIVASTAVA

1. Introduction

Recently, Zeitlin [12] proved the generalized formula

$$R_n(x) = \left[\frac{Q(x)}{Q(y)}\right]^{n/p} \sum_{k=0}^{n} \binom{n}{k} [\Delta_{1/p}(MQ)]^{n-k} \cdot R_k(y), \tag{1.1}$$

where as well as in what follows, for the sake of brevity,

$$\Delta_\nu(UV) \equiv U(x)[V(y)/V(x)]^\nu - U(y),$$

and if

$$E[z] = \textstyle\sum_{n=0}^{\infty} z^n/n\,!, \qquad G[z] = \sum_{n=0}^{\infty} g_n\, z^n, \qquad g_n \neq 0, \tag{1.2}$$

then $R_n(x)$ is defined by the generating function

$$E[M(x)t]G[Q(x)t^p] = \textstyle\sum_{n=0}^{\infty} (t^n/n\,!)R_n(x) \tag{1.3}$$

and explicitly,

$$R_n(x) = \textstyle\sum_{k=0}^{[n/p]} (n\,!/(n-kp)\,!)[Q(x)]^k[M(x)]^{n-kp} g_k, \tag{1.4}$$

provided $M(x) \neq 0$ and $Q(x) \neq 0$ are real functions, and $p = 1, 2, 3, \cdots$.

When $M(x) = x$, $p = 2$ and $Q(x) = \frac{1}{4}(x^2 - 1)$, (1.1) reduces to the result (2.9) derived earlier by Chaudhuri [4, p. 553]. On the other hand, a choice of x^α for $M(x)$, where α is a free parameter, sets (1.1) as the special case $\lambda = 0$ of our recent formula (2.9) in [11, p. 455].

It may be of interest to extend (1.1) and certain other results in [11] and [12] to the following form:

THEOREM 1. *Let $E[z]$ and $G[z]$ be defined as in* (1.2), *and let*

$$E[M(x)t^m]G[Q(x)t^p] = \textstyle\sum_{n=0}^{\infty} (t^n/(\lambda+1)_n)\Psi_n^{(\lambda)}(x), \tag{1.5}$$

where $M(x) \neq 0$, $Q(x) \neq 0$ are real functions, and m, p are positive integers. Then

$$\Psi_{mn}^{(\lambda)}(x) = \left[\frac{Q(x)}{Q(y)}\right]^{mn/p} \sum_{k=0}^{n} \binom{\lambda+mn}{mn-mk} \frac{(mn-mk)!}{(n-k)!} \cdot [\Delta_{m/p}(MQ)]^{n-k}\Psi_{mk}^{(\lambda)}(y) \tag{1.6}$$

Received September 19, 1968.

[1] Presented at the 654th meeting of the American Mathematical Society at Los Angeles, California on March 23, 1968; cf. Notices Amer. Math. Soc., vol. 15 (1968), p. 341, Abstract 654-17.

and

$$\Psi_{mn}^{(\lambda)}(x) = \left[\frac{Q(x)}{Q(y)}\right]^{mn/p} \sum_{k=0}^{n} \binom{\lambda + mn}{mk} \frac{(mk)!}{k!} \cdot [\Delta_{m/p}(MQ)]^k \Psi_{mn-mk}^{(\lambda)}(y). \tag{1.7}$$

For $\lambda = 0$, (1.6) reduces to Zeitlin's result (1.1) when $m = 1$, and it gives us his more general formula (6.2) in [12] in the special case $\lambda = 0$. Also when $m = 1$, a choice of x^α for $M(x)$ will set (1.6) and (1.7) equivalent, respectively, to the formulas (2.9) and (2.10) in [11, p. 455].

2. Proof of Theorem 1

Setting $t = [Q(y)]^{1/p}z$ in (1.5), we have

$$E(M(x)[Q(y)]^{m/p}z^m)G[Q(x)Q(y)z^p] = \sum_{n=0}^{\infty} \frac{[Q(y)]^{n/p}}{(\lambda+1)_n} \Psi_n^{(\lambda)}(x)z^n, \tag{2.1}$$

which, on interchanging x and y, gives us

$$E(M(y)[Q(x)]^{m/p}z^m)G[Q(x)Q(y)z^p] = \sum_{n=0}^{\infty} \frac{[Q(x)]^{n/p}}{(\lambda+1)_n} \Psi_n^{(\lambda)}(y)z^n. \tag{2.2}$$

From (2.1) and (2.2) it follows that

$$\sum_{n=0}^{\infty} \frac{[Q(y)]^{n/p}}{(\lambda+1)_n} \Psi_n^{(\lambda)}(x)z^n = E([M(x)[Q(y)]^{m/p} - M(y)[Q(x)]^{m/p}]z^m) \cdot \sum_{n=0}^{\infty} \frac{[Q(x)]^{n/p}}{(\lambda+1)_n} \Psi_n^{(\lambda)}(y)z^n,$$

and on equating coefficients of z^n from both sides we obtain the desired result (1.6). A reversal of the order of summation in (1.6) will then yield the formula (1.7).

3. Generating functions for $\Psi_n^{(\lambda)}(x)$

In our generating function (1.5) if we replace t by $tz^{1/m}$, multiply both sides by $z^{\nu-1}E[-z]$, and take their Laplace transforms, using the elementary integral

$$\int_0^\infty z^{\xi-1}E[-\eta z]\,dz = \Gamma(\xi)\eta^{-\xi}, \qquad \operatorname{Re}(\xi) > 0, \quad \operatorname{Re}(\eta) > 0,$$

we shall obtain

THEOREM 2. *If* $\Psi_n^{(\lambda)}(x)$ *are generated by* (1.5), *then for arbitrary* ν,

$$[1 - M(x)t^m]^{-\nu} H\left(Q(x)\left[\frac{t^m}{1 - M(x)t^m}\right]^{p/m}\right) = \sum_{n=0}^{\infty} \frac{\Gamma(\nu + n/m)}{(\lambda + 1)_n} t^n \Psi_n^{(\lambda)}(x) \tag{3.1}$$

provided

$$H(z) = \sum_{n=0}^{\infty} \Gamma(\nu + np/m) g_n z^n \qquad (g_n \neq 0)$$

A special form of Theorem 2 appears in Example 21, p. 186 in [10], with $M(x) = x$, $Q(x) = x^2 - 1$, $m = 1$, $p = 2$, which, in turn, extends Theorem 46, p. 134 of [10].

For $m = 1$, (3.1) assumes the form[2]

$$[1 - M(x)t]^{-\nu} J\left(Q(x)\left[\frac{t}{1 - M(x)t}\right]^{p}\right) = \sum_{n=0}^{\infty} \frac{(\nu)_n}{(\lambda + 1)_n} t^n R_n^{(\lambda)}(x), \tag{3.2}$$

where

$$J(z) = \sum_{n=0}^{\infty} (\nu)_{np} g_n z^n \qquad (g_n \neq 0)$$

and

$$R_n^{(\lambda)}(x) = \sum_{k=0}^{[n/p]} \frac{(\lambda + 1)_n}{(n - kp)!} g_k [Q(x)]^k [M(x)]^{n-kp}. \tag{3.3}$$

It is not difficult to give a direct proof of (3.2). Indeed from (3.3) we notice that, for arbitrary ν,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(\nu)_n}{(\lambda + 1)_n} t^n R_n^{(\lambda)}(x) &= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} (\nu)_n t^n g_k [Q(x)]^k \frac{[M(x)]^{n-kp}}{(n - kp)!} \\ &= \sum_{k=0}^{\infty} (\nu)_{kp} t^{kp} g_k [Q(x)]^k \sum_{n=0}^{\infty} \frac{(\nu + kp)_n}{n!} [M(x)t]^n \\ &= [1 - M(x)t]^{-\nu} \sum_{k=0}^{\infty} (\nu)_{kp} g_k [Q(x)]^k \left[\frac{t}{1 - M(x)t}\right]^{kp},\end{aligned}$$

and (3.2) follows immediately.

The role of Theorem 2 lies in the fact that, if $G[z]$ is a specified hypergeo-

[2] It may be of interest to note that in a subsequent paper (Yokohama Math. J., vol. 17 (1969), pp. 65–71) the author gave a generalization of the generating function (3.2). For other generalizations, including those that would involve polynomial systems in several complex variables, one may refer to his forthcoming papers (Proc. Nat. Acad. Sci. U.S.A., vol. 67 (1970), pp. 1079–1080; J. Math. Anal. Appl., vol. 34 (1971); etc.).

metric function, it gives for $\Psi_n^{(\lambda)}(x)$ a class (ν being arbitrary) of generating functions involving a hypergeometric function of superior order. For instance, on comparing the ultraspherical generating function

$$(3.4)\quad E[xt]\,{}_0F_1[-;\ \alpha+1;\ \tfrac{1}{4}(x^2-1)t^2] = \sum_{n=0}^{\infty}(t^n/(2\alpha+1)_n)C_n^{\alpha+1/2}(x)$$

with (1.5), we get $M(x) = x$, $Q(x) = \frac{1}{4}(x^2-1)$, $m = 1$, $p = 2$, $\lambda = 2\alpha$, $g_n = 1/n!(\alpha+1)_n$, and $\Psi_n^{(\lambda)}(x) = C_n^{\alpha+1/2}(x)$, and our theorem yields

$$(3.5)\quad (1-xt)^{-\nu}{}_2F_1\left[\begin{matrix}\frac{1}{2}\nu,\ \frac{1}{2}\nu+\frac{1}{2}; & \dfrac{(x^2-1)t^2}{(1-xt)^2}\\ \alpha+1; & \end{matrix}\right] = \sum_{n=0}^{\infty}\frac{(\nu)_n}{(2\alpha+1)_n}C_n^{\alpha+1/2}(x)t^n,$$

a formula due to Brafman [1, p. 945].

Similarly as an immediate consequence of Theorem 2, we can derive the divergent generating function for generalized Hermite polynomials of Gould and Hopper [6, p. 58] in the form

$$(3.6)\quad (1-xt)^{-\nu}{}_rF_0\left[\begin{matrix}\frac{\nu}{r},\ \frac{\nu+1}{r},\ \cdots,\ \frac{\nu+r-1}{r}; & \left(\dfrac{rt}{1-xt}\right)^r\cdot h\\ \text{———}; & \end{matrix}\right] \approx \sum_{n=0}^{\infty}\frac{(\nu)_n}{n!}g_n^r(x,h)t^n,$$

and since

$$(3.7)\quad g_n^2(2x,-1) = H_n(x),$$

(3.6) reduces to Brafman's formula [1, p. 948]

$$(3.8)\quad (1-2xt)^{-\nu}{}_2F_0\left[\begin{matrix}\frac{1}{2}\nu,\ \frac{1}{2}\nu+\frac{1}{2}; & -\dfrac{4t^2}{(1-2xt)^2}\\ \text{———}; & \end{matrix}\right] \approx \sum_{n=0}^{\infty}\frac{(\nu)_n}{n!}H_n(x)t^n$$

for Hermite polynomials. Another interesting specialization of (3.6) will yield a class of generating functions for certain polynomials, analogous to the Hermite polynomials, introduced by Carlitz [3].

In general, if we give $G[z]$ the hypergeometric form

$$G[z] = {}_pF_\sigma\left[\begin{matrix}\alpha_1,\ \cdots,\ \alpha_p; \\ \beta_1,\ \cdots,\ \beta_\sigma;\end{matrix}\ z\right],$$

and for the sake of simplicity, let $m = 1$, so that (1.5) becomes

$$(3.9)\quad E[M(x)t]\,{}_\rho F_\sigma\left[\begin{matrix}\alpha_1, \cdots, \alpha_\rho;\\ \beta_1, \cdots, \beta_\sigma;\end{matrix}\, Q(x)t^p\right] = \sum_{n=0}^{\infty} \frac{[M(x)t]^n}{n!}$$

$$\cdot\ {}_{\rho+p}F_\sigma\left[\begin{matrix}-\frac{n}{p}, -\frac{(n-1)}{p}, \cdots, -\frac{(n-p+1)}{p}, \alpha_1, \cdots, \alpha_\rho;\\ \beta_1, \cdots, \beta_\sigma;\end{matrix}\, \left(\frac{-p}{M(x)}\right)^p Q(x)\right];$$

then Theorem 2 gives us

$$(3.10)\quad [1 - M(x)t]^{-\nu}$$

$$\cdot\ {}_{\rho+p}F_\sigma\left[\begin{matrix}\frac{\nu}{p}, \frac{\nu+1}{p}, \cdots, \frac{\nu+p-1}{p}, \alpha_1, \cdots, \alpha_\rho;\\ \beta_1, \cdots, \beta_\sigma;\end{matrix}\, \left(\frac{pt}{1 - M(x)t}\right)^p Q(x)\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} [M(x)t]^n$$

$$\cdot\ {}_{\rho+p}F_\sigma\left[\begin{matrix}-\frac{n}{p}, -\frac{(n-1)}{p}, \cdots, -\frac{(n-p+1)}{p}, \alpha_1, \cdots, \alpha_\rho;\\ \beta_1, \cdots, \beta_\sigma;\end{matrix}\, \left(\frac{-p}{M(x)}\right)^p Q(x)\right]$$

for arbitrary parameter ν, and $p = 1, 2, 3, \cdots$.

Known special cases of (3.10) include Chaundy's result (25), p. 62 in [5], where $M(x) = 1$, $Q(x) = -x$, $p = 1$, and the relatively recent formula (55), p. 187 in [2], where $M(x) = 1$, $Q(x) = (-p)^{-p}x$, which, in turn, reduces to the equation (24), p. 947 of [1] when $p = 2$.

Next we consider the double series

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{n=0}^{\infty} \frac{t^n}{(\lambda+1)_n} \Psi_{n+k}^{(\lambda)}(x) = \sum_{n=0}^{\infty} \frac{t^n}{(\lambda+1)_n} \Psi_n^{(\lambda)}(x) \sum_{k=0}^{n} \frac{(-\lambda-n)_k}{k!} \left(-\frac{z}{t}\right)^k.$$

The inner series on the right is a binomial when $\lambda = 0$, and we thus have

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{n=0}^{\infty} \frac{t^n}{n!} \Psi_{n+k}(x) = \sum_{n=0}^{\infty} \frac{(t+z)^n}{n!} \Psi_n(x)$$

$$= E[M(x)(t+z)^m] G[Q(x)(t+z)^p]$$

$$= \sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{[M(x)]^\nu}{\nu!} [Q(x)]^n g_n \sum_{k=0}^{\nu m+np} \binom{\nu m+np}{k} t^{\nu m+np-k} z^k$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{n=[k/p]}^{\infty} t^{np-k} [Q(x)]^n g_n \sum_{\nu=0}^{\infty} \frac{(\nu m+np)!}{(\nu m+np-k)!} \frac{[M(x)t^m]^\nu}{\nu!},$$

and, therefore, on equating coefficients of $z^k/k!$, we obtain

THEOREM 3. *If $\Psi_n(x)$ are generated by (1.5) with $\lambda = 0$, then for $k = 0, 1, 2, \cdots$,*

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \Psi_{n+k}(x) = \sum_{n=[k/p]}^{\infty} \frac{t^{np-k}}{(np-k)!} (np)!\,[Q(x)]^n g_n \tag{3.11}$$
$$\cdot\, {}_mF_m \left[\begin{matrix} \dfrac{np+1}{m}, \dfrac{np+2}{m}, \dots, \dfrac{np+m}{m} & ; \\ & M(x)t^m \\ \dfrac{np-k+1}{m}, \dfrac{np-k+2}{m}, \dots, \dfrac{np-k+m}{m}; & \end{matrix} \right].$$

4. Recurrence relations for $\Psi_n^{(\lambda)}(x)$

If we set

$$F(x, t) = E[M(x)t^m]G[Q(x)t^p]$$

and take first partial derivatives, we obtain

$$\partial F/\partial x = t^m M'(x)F(x, t) + t^p Q'(x)E[M(x)t^m]G'[Q(x)t^p] \tag{4.1}$$

and

$$\partial F/\partial t = mt^{m-1}M(x)F(x, t) + pt^{p-1}Q(x)E[M(x)t^m]G'[Q(x)t^p], \tag{4.2}$$

whence it follows that

$$pQ(x)\partial F/\partial x - tQ'(x)\partial F/\partial t = t^m[pQ(x)M'(x) - mQ'(x)M(x)]F(x, t). \tag{4.3}$$

Since

$$G'[z] = \textstyle\sum_{k=0}^{\infty} kg_k z^{k-1},$$

therefore on equating coefficients of t^n in (4.1), (4.2), and (4.3), we have the following result:

THEOREM 4. *If $\Psi_n^{(\lambda)}(x)$ is defined by* (1.5), *then for $n \geqq m$,*

$$\begin{aligned} &\frac{\partial}{\partial x} \Psi_n^{(\lambda)}(x) - (-)^m(-\lambda - n)_m M'(x)\Psi_{n-m}^{(\lambda)}(x) \\ &\qquad = \sum_{k=0}^{[n/p]} kg_k[Q(x)]^{k-1}Q'(x)(\lambda + 1)_n \frac{[M(x)]^{(n-kp)/m}}{[(n - kp)/m]!}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} &\Psi_n^{(\lambda)}(x) - \frac{n}{m}(-)^m(-\lambda - n)_m M(x)\Psi_{n-m}^{(\lambda)}(x) \\ &\qquad = \sum_{k=0}^{[n/p]} pkg_k[Q(x)]^k \frac{(\lambda + 1)_n}{n} \frac{[M(x)]^{(n-kp)/m}}{[(n - kp)/m]!}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} &pQ(x)\partial(\Psi_n^{(\lambda)}(x))/\partial x - nQ'(x)\Psi_n^{(\lambda)}(x) \\ &\qquad = (-)^m(-\lambda - n)_m[pQ(x)M'(x) - mQ'(x)M(x)]\Psi_{n-m}^{(\lambda)}(x). \end{aligned} \tag{4.6}$$

The last result readily yields

$$\begin{aligned} &pQ(x) \sum_{n=m}^{\infty} \frac{t^{n-m}}{(\lambda + 1)_n} \frac{\partial}{\partial x} \Psi_n^{(\lambda)}(x) \\ &\qquad = Q'(x) \sum_{n=0}^{\infty} \frac{t^n}{(\lambda + 1)_{n+m}} (n + m)\Psi_{n+m}^{(\lambda)}(x) \\ &\qquad\quad + [pQ(x)M'(x) - mQ'(x)M(x)] \sum_{n=0}^{\infty} \frac{t^n}{(\lambda + 1)_n} \Psi_n^{(\lambda)}(x) \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} pQ(x) \sum_{n=m}^{N} & \frac{(-)^m}{(-\lambda - n)_m} t^n \frac{\partial}{\partial x} \Psi_n^{(\lambda)}(x) \\ & = Q'(x) \sum_{n=m}^{N} \frac{(-)^m}{(-\lambda - n)_m} n t^n \Psi_n^{(\lambda)}(x) \\ & \quad + t^m [pQ(x)M'(x) - mQ'(x)M(x)] \sum_{n=0}^{N-m} t^n \Psi_n^{(\lambda)}(x), \end{aligned} \tag{4.8}$$

where $N \geqq m$.

The special cases $m = 1$, $\lambda = 0$ of these results appear in [12, p. 1060]. For $m = p = 1$, $\lambda = 0$, (4.5) and (4.6) reduce to relations satisfied by Appell polynomials, with $M(x) = x$, $Q(x) = 1$, while (4.4), (4.5), and (4.6) correspond to known formulas for the polynomials generated by $E[t]G[xt]$ (cf. [10, p. 132]) if we set $M(x) = 1$, $Q(x) = x$.

5. Further extensions

Since

$$\begin{aligned} & E[t^m]E[M(x)t^m]G[Q(x)t^p] \\ & = \sum_{n=0}^{\infty} \frac{t^n}{(\lambda + 1)_n} \left[\sum_{k=0}^{[n/m]} \binom{\lambda + n}{mk} \frac{(mk)!}{k!} \Psi_{n-mk}^{(\lambda)}(x) \right], \end{aligned} \tag{5.1}$$

from (1.5) and (1.6) it follows that

$$\begin{aligned} & \sum_{k=0}^{n} \binom{\lambda + mn}{mn - mk} \frac{(mn - mk)!}{(n - k)!} \Psi_{mk}^{(\lambda)}(x) \\ & = \left[\frac{Q(x)}{Q(y)}\right]^{mn/p} \sum_{k=0}^{n} \binom{\lambda + mn}{mn - mk} \frac{(mn - mk)!}{(n - k)!} \\ & \cdot \left[\frac{[1 + M(x)][Q(y)]^{m/p} - [1 + M(y)][Q(x)]^{m/p}}{[Q(x)]^{m/p}} \right]^{n-k} \\ & \cdot \sum_{j=0}^{k} \binom{\lambda + mk}{mk - mj} \frac{(mk - mj)!}{(k - j)!} \Psi_{mj}^{(\lambda)}(y); \end{aligned} \tag{5.2}$$

and since

$$\begin{aligned} & E^2[M(x)t^m]G^2[Q(x)t^p] \\ & = \sum_{n=0}^{\infty} \frac{t^n}{(\lambda + 1)_n} \left[\sum_{k=0}^{n} \binom{\lambda + n}{n - k} \frac{(n - k)!}{(\lambda + 1)_{n-k}} \Psi_{n-k}^{(\lambda)}(x) \Psi_k^{(\lambda)}(x) \right], \end{aligned} \tag{5.3}$$

our formula (1.6) gives us

$$\begin{aligned} & \sum_{k=0}^{mn} \binom{\lambda + mn}{mn - k} \frac{(mn - k)!}{(\lambda + 1)_{mn-k}} \Psi_{mn-k}^{(\lambda)}(x) \Psi_k^{(\lambda)}(x) \\ & = \left[\frac{Q(x)}{Q(y)}\right]^{mn/p} \sum_{k=0}^{n} \binom{\lambda + mn}{mn - mk} \frac{(mn - mk)!}{(n - k)!} [2\Delta_{m/p}(MQ)]^{n-k} \\ & \cdot \sum_{j=0}^{mk} \binom{\lambda + mk}{mk - j} \frac{(mk - j)!}{(\lambda + 1)_{mk-j}} \Psi_{mk-j}^{(\lambda)}(y) \Psi_j^{(\lambda)}(y). \end{aligned} \tag{5.4}$$

In view of these relations it seems worthwhile to prove the following general result on lines parallel to those of Theorem 1.

THEOREM 5. *Let $E[z]$ and $G[z]$ be defined by* (1.2), *and let*

$$(5.5) \quad E[M(x)t^m + N(x)t^l]G[Q(x)t^p] = \sum_{n=0}^{\infty} (t^n/(\lambda+1)_n)S_n^{(\lambda)}(x),$$

where $M(x)$, $N(x)$, and $Q(x) \neq 0$ are real functions, and l, m, p are positive integers.

Then

$$(5.6) \quad \begin{aligned} S_n^{(\lambda)}(x) = \left[\frac{Q(x)}{Q(y)}\right]^{n/p} \sum_{k,j\geqq 0}^{mk+lj\leqq n} \binom{\lambda+n}{mk+lj} (mk+lj)\,! \\ \cdot \frac{[\Delta_{m/p}(MQ)]^k}{k\,!} \frac{[\Delta_{l/p}(NQ)]^j}{j\,!} S_{n-mk-lj}^{(\lambda)}(y). \end{aligned}$$

6. The basic analogue

Following Jackson [9] and Hahn [8] let the basic binomial $[a+b]_n$ abbreviate the n-rank product

$$(a+b)(a+qb)(a+q^2b)\cdots\cdots(a+q^{n-1}b), \qquad |q|<1,$$

so that since

$$[a+b]_n = a^n\left[1+\frac{b}{a}\right]_n = [a+b]_{n-1}(a+q^{n-1}b);$$

it follows that

$$[a+b]_0 = 1$$

and

$$[a+b]_{-n} = 1/(a+bq^{-1})\cdots(a+bq^{-n}) = q^{(1/2)n(n+1)}b^{-n}/[1+aq/b]_n.$$

Also let [7, p. 6]

$$\begin{bmatrix}\alpha\\k\end{bmatrix} = \frac{(1-q^\alpha)(1-q^{\alpha-1})\cdots(1-q^{\alpha-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)} = \frac{[1-q^{\alpha-k+1}]_k}{[1-q]_k},$$

and denoting the q-exponential function by

$$(6.1) \qquad e_q[z] = \lim_{k\to\infty} 1/[1-z]_k = \sum_{n=0}^{\infty} z^n/[1-q]_n,$$

we have

$$e_q[z]/e_q[\zeta] = \sum_{k=0}^{\infty} [z-\zeta]_k/[1-q]_k.$$

The method of proof of Theorem 1 can now be invoked to establish the following q-analogue of Theorem 5 of the preceding section.

THEOREM 6. *With $e_q[z]$ defined by* (6.1) *and $G[z]$ by* (1.2), *let*

$$(6.2) \quad e_q[M_q(x)t^m]e_q[N_q(x)t^l]G[Q(x)t^p] = \sum_{n=0}^{\infty} (t^n/[1-q^{\lambda+1}]_n)S_{q,n}^{(\lambda)}(x),$$

where $M_q(x)$ and $N_q(x)$ depend, in general, on both the base q and the argument x, $|q|<1$, $Q(x)\neq 0$ is a real function, and l, m, p are posisive integers.

Then

$$S_{q,n}^{(\lambda)}(x) = \left[\frac{Q(x)}{Q(y)}\right]^{n/p} \sum_{k,j\geqq 0}^{mk+lj\leqq n} \begin{bmatrix}\lambda + n \\ mk + lj\end{bmatrix} \cdot [1 - q]_{mk+lj} \tag{6.3}$$
$$\cdot \frac{[\Delta_{m/p}(M_q Q)]_k}{[1 - q]_k} \frac{[\Delta_{l/p}(N_q Q)]_j}{[1 - q]_j} S_{q,n-mk-lj}^{(\lambda)}(y).$$

For $N_q(x) \to 0$ the last theorem will provide us with a basic analogue of Theorem 1. On the other hand, if $M_q(x)$, $N_q(x)$, and $S_{q,n}^{(\lambda)}(x)$ are replaced by $(1 - q)M(x)$, $(1 - q)N(x)$ and $(1 - q)^n S_n^{(\lambda)}(x)$ respectively, its limiting case when $q \to 1$ gives us Theorem 5.

It may be of interest to remark that the analysis of §§3 and 4 when applied to the generating function (5.5) and its q-analogue (6.2) in a rather straightforward manner will lead us to the generalizations of Theorems 2, 3, and 4.

Acknowledgment. The author wishes to thank his colleague, Professor Henry W. Gould for helpful discussions during the course of the present investigation.

REFERENCES

1. F. BRAFMAN, *Generating functions of Jacobi and related polynomials*, Proc. Amer. Math. Soc., vol. 2 (1951), pp. 942–949.
2. ———, *Some generating functions for Laguerre and Hermite polynomials*, Canadian J. Math., vol. 9 (1957), pp. 180–187.
3. L. CARLITZ, *The product of certain polynomials analogous to the Hermite polynomials*, Amer. Math. Monthly, vol. 64 (1957), pp. 723–725.
4. J. CHAUDHURI, *On the generalization of a formula of Rainville*, Proc. Amer. Math. Soc., vol. 17 (1966), pp. 552–556.
5. T. W. CHAUNDY, *An extension of hypergeometric functions*, Quart. J. Math. (Oxford), vol. 14 (1943), pp. 55–78.
6. H. W. GOULD AND A. T. HOPPER, *Operational formulas connected with two generalizations of Hermite polynomials*, Duke Math. J., vol. 29 (1962), pp. 51–63.
7. W. HAHN, *Über Orthogonalpolynome, die q-Differenzengleichungen genügen*, Math. Nachr., vol. 2 (1949), pp. 4–34.
8. ———, *Beiträge zur Theorie der Heineschen Reihen. Die 24 Integrale der hypergeometrischen q-Differenzengleichung. Das q-Analogon der Laplace-Transformation*, Math. Nachr., vol. 2 (1949), pp. 340–379.
9. F. H. JACKSON, *The Application of basic numbers to Bessel's and Legendre's functions*, Proc. London Math. Soc. (2), vol. 2 (1904), pp. 192–220.
10. E. D. RAINVILLE, *Special functions*, Macmillan, New York, 1960.
11. H. M. SRIVASTAVA, Finite summation formulas associated with a class of generalized hypergeometric polynomials, J. Math. Anal. Appl., vol. 23 (1968), pp. 453–458. (Cf. Abstract 68T-132, Notices Amer. Math. Soc., vol. 15 (1968), p. 231.)
12. DAVID ZEITLIN, *On generating functions and a formula of Chaudhuri*, Amer. Math. Monthly, vol. 74 (1967), pp. 1056–1062. See also reviews in Math. Rev., vol. 36 (1968), pp. 1052–1053 # 5413 and Zentral. Math., vol. 153 (1968), pp. 394–395 by H. M. Srivastava.

WEST VIRGINIA UNIVERSITY
MORGANTOWN, WEST VIRGINIA
UNIVERSITY OF VICTORIA
VICTORIA, BRITISH COLUMBIA