COHOMOLOGY OPERATIONS IN LOCAL COEFFICIENT THEORY

BY
JERROLD SIEGEL[1]

In [4] we developed an obstruction theory for not necessarily simply connected spaces. Fundamental to the application of this theory is the computation of the group $H^n(L_\pi(\varphi_1, m), \varphi_2))$ which corresponds to the group of cohomology operations of type $(m, n, \varphi_1, \varphi_2)$. We showed that this group could in theory be computed by a modified Serre spectral sequence. But, except for special cases no attempt was made to compute this spectral sequence.

In the present work we explore this computation question. We present an alternate formulation of our spectral sequence as the spectral sequence of a bigraded module. We then explore the two filtrations that such a module presents. The usual filtration contains little new information. However, the second filtration makes possible extensive computations, moreover, it exhibits the relation between the equivariant cohomology theories and our investigations.

We would like to thank the referee for his many helpful comments especially with regard to Section 2 below.

1. Rigid bundles

In this section we introduce a category whose objects are basically fibre bundles with discrete group, but whose maps are subject to a special condition. We begin by recalling some information from [6].

Let $F \subseteq B \to X$ be a fibre bundle with group G. We assume throughout that X is path connected. In general, given $x_0, x_1 \in X$ and a path $C : I \to X$ with $C(0) = x_0$ and $C(1) = x_1$ we may lift the path (not necessarily uniquely) to $C^{\#} : I \times F_{x_0} \to B$ such that $C^{\#} : t \times F_{x_0} \subseteq F_{x_t}$ is a homeomorphism.

1.1 LEMMA [5, 13.2]. *If the topology of the group G is totally disconnected then translation of fibres along curves is a unique operation.*

1.2 LEMMA [5, 13.3]. *If the topology of G is totally disconnected let C_1 and C_2 be paths from x_0 to x_1 and x_1 to x_2 respectively. Let $C_1 \circ C_2$ be the standard composition from x_0 to x_2; then*

$$C_2^{\#}(1 \times C_1^{\#}(1 \times F_{x_0})) = (C_1 \circ C_2)^{\#}(1 \times F_{x_0}) \to F_{x_2}.$$

1.3 LEMMA [6, 13.4]. *If the topology of G is totally disconnected. Let C and C' be paths from x_0 to x_1 that are homotopic by an end point preserving homotopy then* $C^{\#}(1 \times F_{x_0}) = C'^{\#}(1 \times F_{x_0})$.

Received September 19, 1968.

[1] During this work the author was supported by a National Science Foundation contract.

Motivated by the above we define a category that allows application of the theorem on acyclic models [5]. For x_0, $x_1 \in X$ let $P(x_0, x_1)$ be the space of paths from x_0 to x_1. Let $P_h(x_0, x_1)$ be the set of equivalence classes of $P(x_0, x_1)$ under end point preserving homotopy.

1.4 DEFINITION. We define $\mathcal{RB}$, the category of Rigid *Bundles* as follows: The objects of $\mathcal{RB}$ are pairs (B, τ) where B is a bundle

$$F \subseteq B \xrightarrow{p} X$$

over a path connected base pointed space X. (Note: "bundle" simply means locally a product). τ is a function which satisfies the properties 1.1–3. More specifically, for each x_0, $x_1 \in X$ we are given

$$\tau : P_h(x_0, x_1) \times F_{x_0} \to F_{x_1}$$

a homeomorphism for each $c \in P_h(x_0, x_1)$ satisfying the following.

(a) For $c \in P_h(x_0, x_1)$ and $c' \in P_h(x_1, x_2)$ we have

$$\tau_{c'}\,\tau_c = \tau_{c \circ c'} : F_{x_0} \to F_{x_2}$$

where $\tau_c(x) = \tau(c, x)$.

(b) τ is continuous. That is, let $h_0 : U_0 \times F \to B$ and $h_1 : U_1 \times F \to B$ be local homeomorphisms. Let $P(U_0, U_1)$ be the space of paths starting in U_0 and ending in U_1. Consider the map

$$\rho : P(U_0, U_1) \to C(F, F)$$

defined by the formula $\rho(p)(x) = \pi_2 h_1^{-1} \tau_{[p]} h_0(p(0), x)$. We assume ρ is continuous where $C(F, F)$ is the space of continuous functions F to F.

The morphisms of our category, written $\bar{f} : (B, \tau) \to (B', \tau')$ are fibre maps that commute with τ and τ'. That is, pairs of maps (f, f'), f' base pointed, such that

(1) $p'f = f'p$
(2) $\tau_{f'(c)} f = f\tau_c : F_{x_0} \to F_{f'(x_1)}$ where $c \in P_h(x_0, x_1)$

1.5 *Remarks.* (a) The map f is completely determined by f' and $f_{x_0} : F_{x_0} \to F_{f'(x_0)}$

(b) Given a fibre bundle B with totally disconnected group there is a natural choice for τ making (B, τ) into a rigid bundle.

(c) The only rigid bundles over simply connected spaces are product bundles.

We now define two functors from $\mathcal{RB}$ to $\mathcal{C}$, the category of chain complexes, and show that they are naturally chain homotopic.

1.6 DEFINITION. Let $T : \mathcal{RB} \to \mathcal{C}$ be defined by $T(B, \tau) = C_*(B, M)$ the singular complex of the total space of the bundle with coefficients in a module over a commutative ring with unit R.

We now define $T' : \mathcal{RB} \to \mathcal{C}$. Let $F = F_{\bar{x}}$ and let $\pi = \pi_1(X, \bar{x})$ then $C_*(F, M)$ is a left $R[\pi]$-complex by the formula $g \cdot [c_q'] = [\tau(g) c_q']$ where $c_q' : \Delta^q \to F$ is the map corresponding to the generator $[c_q']$. Note that the boundary homomorphism d_F commutes with the operations of π.

Now let $\bar{C}_*(X, M) \subseteq C_*(X, M)$ be the subcomplex generated by singular simplices sending vertices to $\bar{x}$. Let $[\bar{c}_p] \in \bar{C}_*(X, M)$ and let $g[\bar{c}_p] \in \pi$ be the class represented by the loop corresponding to the leading edge of $\bar{c}_p$. We define $\bar{d}$ on $\bar{C}_*(X, M) \otimes_R C_*(F, M)$ by the formula

$$(1.7) \quad \bar{d}_n([\bar{c}_p] \otimes [c_q']) = [\sigma_0 \bar{c}_p] \otimes g[\bar{c}_p] \cdot [c_q'] + \textstyle\sum_{i=1}^{n} (-1)^i [\sigma_i \bar{c}_p] \otimes [c_q']$$

where the σ_i are face operators.

We note that $\bar{d}(1 \otimes d_F) = (1 \otimes d_F)\bar{d}$.

1.8 Definition. $T' : \mathcal{RB} \to \mathcal{C}$ is now defined as follows:

$$T'(B, \tau) = \bar{C}_*(X, M) \otimes_R C_*(F, M)$$

with the usual grading and usual total differential associated with $\bar{d}$ and $(1 \otimes d_F)$. $T'(\bar{f})$ is the obvious tensor product.

1.9 Theorem. *The functors T and T' are naturally chain homotopy equivalent.*

Proof. The proof is based on Spanier's version of the theorem of acyclic models [5]. We sketch the necessary details. Let

$$X \times Y \xrightarrow{\pi_1} X$$

be the product bundle over X. Let $\tau_{X,Y}$ be the obvious product translation then the models for T are $(\Delta^q \times \Delta^q, \tau_{\Delta^q,\Delta^q})$ and the models for T' are $(\Delta^p \times \Delta^q, \tau_{\Delta^q,\Delta^q})$. Clearly the models for $T(T')$ are acyclic in $T'(T)$.

T_n is free on $[d_n]$ where $d_n : \Delta^n \to \Delta^n \times \Delta^n$ is the diagonal map.

T_n' is free on $[1_p] \otimes [1_q]_{p+q=n}$ where $1_p : \Delta^p \to \Delta^p$ is the identity map as is 1_q.

Freeness in each case follows from an examination of 1.5a.

It should be noted that if we use cubical singular homology above, there is a natural filtration preserving choice for the chain equivalence $T' \to T$. We map $\bar{c}_p \otimes c_q'$ to the $p + q$ cube c_{p+q} in B defined by

$$c_{p+q}(t_1, \cdots t_p, t_{p+1}, \cdots t_{p+q}) = \tau_{c^\sim} c'(t_{p+1}, \cdots, t_{p+q}),$$

where $c^\sim$ is the class of the path $p : I \to X$ defined by $p(t) = \bar{c}_p(tt_1, \cdots, tt_p)$.

We now cast the above into the setting of [4]. In particular, we assume that we are giving a fibration

$$F \subseteq B \xrightarrow{p} (X, x_0)$$

over a path connected space X. Assume also we are given a local system of R-module $\bar{\mathfrak{M}}$ over X. We set $\mathfrak{M} = p^* \bar{\mathfrak{M}}$ (Note $\mathfrak{M}$ is trivial in each fibre).

1.10 DEFINITION. (a) Assume we are given a rigid bundle (B, τ) and a local system $\mathfrak{M}$ on X. We define $\mathcal{C}^*(F, \mathfrak{M})$ to be the local system of complexes over X with $\mathcal{C}^*(F, \mathfrak{M})_x = \mathcal{C}^*(F_x, M_x)$. Let $b(t) : I \to X$ be a path from x_0 to x_1. We define

$$b^* : C^*(F_{x_2}, M_{x_2}) \to C^*(F_{x_1}, M_{x_1})$$

to be the composition of

$$\tau^*_{[b]} : C^*(F_{x_2}, M_{x_2}) \to C^*(F_{x_1}, M_{x_2}) \quad \text{and}$$

$$\mathfrak{M}^*(b^{-1}) : C^*(F_{x_1}, M_{x_2}) \to C^*(F_{x_1}, M_{x_1}).$$

As in 1.6, δ_F the coboundary, commutes with the actions of the b^*.

(b) Letting $F = p^{-1}(x_0)$ and $M = M_{x_0}$, we form the double complex $\bar{C}^p(X, C^q(F, M))$. The differentials are δ^q_F, and $\bar{\delta}^p$, the coboundary in local coefficient theory.

1.11 THEOREM. *Let (B, τ) and $\mathfrak{M}$ be as in 1.10. There is a filtration-preserving natural chain map between $C^*(B, \mathfrak{M})$ and the total complex of $\bar{C}^*(X, C^*(F, M))$ filtered on the first index. This chain map induces an isomorphism of the corresponding spectral sequences on the E_2-level.*

2. The natural filtration

In this section we briefly examine the natural filtration. Little new will be gleaned from this filtration. In fact, this section serves more to illustrate the difficulties presented by the usual filtration than ways to circumvent them.

2.1 DEFINITION. A first quadrant spectral sequence $(E_r^{p,1}, H^n(S))$ si called d-split if for each k there is a spectral sequence $({}_kE_r^{p,q}, H^n({}_kS))$ and spectral sequence maps

$$\alpha_k : {}_kE_r^{p,q} \to E_r^{p,q}, \qquad \beta_k : E_r^{p,q} \to {}_kE_r^{p,q},$$

$$\alpha_k : H^n({}_kS) \to H^n(S), \qquad \beta_k : H^n(S) \to H^n(S_k)$$

satisfying

(1) $\beta_k \alpha_k = 1$,

(2) $\alpha_k : {}_kE_r^{p,q} \cong E_r^{p,q}$, $q + d < k$ and $r \geq 0$,

(3) ${}_kE_r^{p,q} = 0$, $q > k$ and $r \geq 0$.

2.2 LEMMA. *A sufficient condition for a spectral sequence to be d-split is the existence for each k of a spectral sequence $({}_kE_r^{p,q}, H^n({}_kS))$ and maps α_k and β_k satisfying* (1), (2) *and* (3) *of* 2.1 *for* $r = 0$.

Proof. Clearly we must only check (2). Assume we have verified (2) for some $r \geq 0$; we show (2) holds for $r + 1$. Consider the diagram

$$\begin{array}{ccccc} E_r^{p-r,q+r-1} & \to & E_r^{p,q} & \to & E_r^{p+r,q-r+1} \\ \beta_k \downarrow\uparrow \alpha_k & & \beta_k \| \wr \alpha_k & & \beta_k \downarrow\uparrow \alpha_k \\ {}_kE_r^{p-r,q+r-1} & \to & {}_kE_r^{p,q} & \to & {}_kE_r^{p+r,q-r+1}. \end{array}$$

Since $E_{r+1}^{p,q}$ and ${}_kE_{r+1}^{p,q}$ are the cohomology of the middle terms of the horizontal sequences a simple diagram chase completes the proof.

2.4 THEOREM. *A d-split spectral sequence satisfies the condition $E_{3+d}^{p,q} = E_\infty^{p,q}$ for all p, q. If in addition $d = 0$ and ${}_kE_2^{p,k} = 0$ for $k \leq n + 1$ then $E_2^{p,q} = EE_\infty^{p,q}$ for $q \leq n$ and $H^t(s) \cong \sum_{p+q=t} \oplus E_2^{p,q} t \leq n$.*

Proof. The proof of the first part of the theorem follows by examining the following diagram

$$(*)\qquad \begin{array}{ccccc} E_r^{p-r,q+r-1} & & E_r^{p,q} & \xrightarrow{d_r^{p,q}} & E_r^{p+r,q-r+1} \\ \downarrow & & \Vert\wr & & \Vert\wr \\ 0 = {}_kE_r^{p-r,q+r-1} & & {}_kE_r^{p,q} & \rightarrow & {}_kE_r^{p+r,q-r+1} \end{array}$$

where $r \geq 3 + d$ and $k = q + d + 1$.

For the second part we notice that the condition ${}_kE_2^{p,k} = 0$ for $k \leq n + 1$ in conjunction with $(*)$ for $r = 2$, implies $E_2^{p,q} = E_3^{p,q}$ $q \leq n$. Now the first part of this theorem with $d = 0$ implies $E_3^{p,q} = E_\infty^{p,q}$. The splitting follows from a straight forward examination of the composition series for $H^t(S)$ in relation to the various composition series for $H^t(S_k)$, $k \leq t \leq n$. One uses the maps α_k and β_k to establish the splitting.

The above theorems may now be applied to the situation of Section 1. As before we have (B, τ) and $\mathfrak{M}$; we also let $\pi = \pi_1(X, x_0)$.

We give several applications of the above theorem. Unfortunately, as mentioned above, none of the applications are useful in a general way.

In each case a 0-splitting will be obtained by considering an appropriate splitting of $C_*(F, M)$.

2.4 THEOREM. *Suppose R and $R[\pi]$ are principal ideal domains (P.I.D.) and $C_*(F, R)$ is free as an $R[\pi]$-module; then the spectral sequence of 1.11 is an $E_3 = E_\infty$ spectral sequence.*

Moreover, if $C^*(F, M]$ is free as an $R[\pi]$-module then the spectral sequence is an $E_2 = E_\infty$ spectral sequence and

$$H^t(B, \mathfrak{M}) \cong \sum_{p+q=t} \oplus H^p(X, \mathcal{H}^q(F, \mathfrak{M}))$$

Proof. Consider the following commutative diagram of $R[\pi]$-*modules.*

$$(2.5)\qquad \begin{array}{ccccccc} 0 \leftarrow C_0(F, R) \leftarrow \cdots & \xleftarrow{d_F^{k-1}} & C_{k-1}(F, R) & \xleftarrow{i} & B_{k-1}(F, R) & \leftarrow 0 \cdots & 0 \\ \bar\beta \Vert\wr \bar\alpha & & \beta \Vert\wr \bar\alpha & & \bar\beta_k \upharpoonleft\downharpoonright \bar\alpha_k & & \upharpoonleft\downharpoonright \\ 0 \leftarrow C_0(F, R) \leftarrow & & \leftarrow C_{k-1}(F, R) & \xleftarrow{d_F} & C_k(F, R) & \leftarrow & C_{k-1}(F, R) \end{array}$$

where $B_{k-1}(F, R) =$ image d_F^k, $i\bar\alpha_k = d_F$ and $\bar\beta_k$ is any lifting of $\bar\alpha_k$ with $\bar\alpha_k \bar\beta_k = 1$.

$\bar\beta_k$ exists since $B_{k-1}(F, R)$ is $R[\pi]$-free being a submodule of a free $R[\pi]$-module. The construction of the 0-splitting is now easy. The ${}_kE$ spectral sequence is that of 1.11 with the top line of 2.5 in place of $C_*(F, R)$.

If $C^*(F, M)$ is itself $R[\pi]$-free, the splitting map may be defined directly on $C^*(F, M)$ giving the necessary additional conditions needed for an application of the second part of 2.3.

2.6 *Remarks.* The conclusions of 2.4 hold with the following slightly altered hypothesis; $C_n(F, R)$ (resp. $C^n(F, R)$) is free only for $n > 0$. $E_2^{p,0} \cong E_\infty^{p,0}$ for all p. In 3.5–9 below we see that the conditions on $C_n(F, R)$ and $E^{p,0}$ always are present in the study of local cohomology operations, and the condition on $C^n(F, R)$ is true in a large number of situations. However, the assumption that $R[\pi]$ is a P.I.D. is quite strong. In fact, usually we may infer 2.4 more directly (see [4, 2.13]) when this condition is present.

As can be seen from the above a 0-splitting can be produced by constructing an $R[\pi]$-splitting of the chain complex $C_*(F, R)$ then passing to cohomology. Another situation where this is possible is the following generalization of the Kunneth formula.

2.7 THEOREM. *Let $B = X \times F$ and let R be a P.I.D.; then the spectral sequence for B is an $E_3 = E_\infty$ spectral sequence.*

Proof. Consider the diagram (2.5) as a splitting of $C_*(F, R)$ considered as a complex of R-modules. This is possible since R is a P.I.D. Since the action of π is assumed to be trivial it is automatically a splitting of $C_*(F, R)$ as a complex of $R[\pi]$-modules. Now all goes as in 2.4.

2.8 *Remarks.* As in 2.4, we may improve 2.6 by insisting that $R[\pi]$ be a P.I.D. and M be $R[\pi]$-free but again this has limited application.

We leave unstated theorems employing full use of the concept of d-splitting since they involve obvious statements about the global dimension of R and $R[\pi]$ and again are not particularly useful.

3. The second filtration

The major problem in applving 2.4 of the previous section is the strong algebraic condition needed for $R[\pi]$. In this section we show that in our situation we can make do with a condition that is more often present.

3.1 DEFINITION (2]. Let Λ be a ring; we say Λ is *self injective* if Λ is injective when considered as a left Λ-module. (Note Λ need not be commutative.)

3.2 *Examples of self injective commutative rings.*

(a) Let R be any field

(b) Let $R = Z_n$, the integers mod n for any n.

3.3 THEOREM [2]. *Let R be a commutative ring. Let π be a finite group. R is self injective iff $R[\pi]$ is.*

We will use 3.3 in conjunction with

3.4 LEMMA. *Let $M = \Lambda \oplus \cdots \oplus \Lambda$ be the free Λ-module on n generators; then Λ self injective implies M injective.*

3.5 *Remarks.* We now apply the results of Section 1 and the algebra above to the computation of the group $H^n(L_\pi(\varphi_1, m), \varphi_2(\hat{L}))$.

To quickly review the situation presented in [4], we have a bundle with cross-section $K(G_1, m) \subseteq L_\pi(\varphi_1, m) \to K(\pi, 1)$, where φ_1 and φ_2 are actions of π on R-modules G_1 and G_2 respectively and φ_1 induces a base point preserving action of π on $K(G_1, m)$. $\hat{L}$ is the universal π bundle over $K(\pi, 1)$. We note that because of the actions of π on $K(G_1, m)$ and G_2 we can speak of $H^n_{ev}(K(G_1, m), G_2)$ the n^{th} equivariant cohomology group with respect to this action. (See [7].)

We let

$$0 \leftarrow R \xleftarrow{d_0} \Pi_0 \leftarrow \Pi_1 \xleftarrow{d_1} \cdots$$

be a free $R[\pi]$ resolution of R. We let Π_* be the associated unaugmented complex.

3.6 Theorem. *The cohomology group $H^n(L_\pi(\varphi_1, m), \varphi_2(\hat{L}))$ may be computed from the total complex associated with the double complex*

$$\mathrm{Hom}_{R(\pi)}(\Pi_*, C^*(K(G_1, m), G_2)).$$

Proof. 1.11 and elementary considerations about the spaces $K(\pi, 1)$.

We now state the main computational theorem.

3.7 Theorem. *Suppose the complex $C^*(K(G_1, m), G_2)$ is injective except in dimension* 0; *then*

$$H^n(L_\pi(\varphi_1, m), \varphi_2(\hat{L})) \cong H^n_{ev}(K(G_1, m), G_2) \oplus H^n(K(\pi, 1), \varphi_2(\hat{L}))$$

Proof. Instead of filtering the double complex of 3.6 on p we filter on the second index q and denote this spectral sequence by $'E_2^{p,q}$. We note the following consequences of the injectivity of $C^q(K(G_1, m), G_2)$ $q > 0$.

(1) $'E_r^{p,q} = 0$ for r, p, q all > 0.

On the other hand

(2) $'E_1^{0,q} \cong C^q_{ev}(K(G_1, m), G_2)$
(3) $'E_2^{0,q} \cong H^q_{ev}(K(G_1, m), G_2)$
(4) $'E_1^{p,0} \cong H^p(K(\pi, 1), \varphi_2(\hat{L}))$.

Now taking into account the existence of a cross-section we may conclude from (1), (3) and (4) that

(5) $'E_2^{0,q} \cong {'E}_\infty^{0,q}$
(6) $'E_1^{p,0} \cong {'E}_\infty^{p,0}$.

Now using the cross-section (1), (5) and (6) we conclude

$$H^n(L_\pi(\varphi_1, m), \varphi_2(\hat{L})) \cong {'E}_\infty^{n,0} \oplus {'E}_\infty^{0,n}.$$

This in conjunction with (3) and (4) completes the proof.

We have yet to show that the condition $C^q(K(G_1, m), G_2)$ injective is reasonable. To do this we use the following lemma of Bredon [1].

3.8. LEMMA. *We may choose the space $K(G_1, m)$ in such a way that the group π acts on the cells freely except in dimension 0. Hence we may assume that the chain complex*

$$R = C_0(K(G_1, m), R) \leftarrow C_1(K(G_1, m), R) \leftarrow \cdots \leftarrow C_n(K(G_1, m), R)$$

is a free $R[\pi]$ complex except in dimension 0.

We also will need the following algebraic fact. We assume π is finite.

3.9 LEMMA. *Let M be an $R[\pi]$-module that is free and finite dimensional as an R-module. Let F be a free and finite-dimensional $R[\pi]$-module. Then $\mathrm{Hom}_R[F, M]$ is a free $R[\pi]$-module under the usual action $g(\varphi) = g \circ \varphi \circ g^{-1}$ for $\varphi \in \mathrm{Hom}_R[F, M]$ and $g \in \pi$.*

Proof. Let $e_1, \cdots, e_t$ be a basis for M. Define $\varphi_i \in \mathrm{Hom}_R[R[\pi], M]$ by

$$\varphi_i[r[1]] = re_i, \quad 1 \in \pi,$$

$$\varphi_i[r[g]] = 0, \quad g \neq 1 \in \pi$$

One shows φ_i forms an $R[\pi]$-basis for $\mathrm{Hom}_R[R[\pi], M]$ using the fact that for any $g \in \pi$, $[g]e_1, \cdots, [g]e_t$ also is an R-basis for M. The case of a general F now follows quickly.

We now give applications of the above.

3.10 THEOREM. *Assume G_2 is a self-injective ring. Let φ_2 be trivial, π be finite and G_1 finitely generated. We have*

$$H^n(L_\pi(\varphi_1, m), \varphi_2(\hat{L})) \cong H^n_{ev}(K(G_1, m), G_2) \oplus H^n(K(\pi, 1), G_2)$$

Proof. The triviality of φ_2 combined with 3.9 and 3.8 allows us to conclude that

$$C^0\ (K(G_1, m), G_2) \to C^1(K(G_1, m), G_2) \to \cdots \to C^n(K(G_1, m), G_2) \to \cdots$$

is a free $G_2[\pi]$-module except in dimension 0. We now use 3.3 and 3.4 to conclude that it is injective except in dimension 0. 3.7 completes the proof.

3.11 THEOREM. *Let G_2 be a finite-dimensional vector space over a field k. We have*

$$H^n(L_\pi(\varphi_1, m)\varphi_2(\hat{L})) \cong H^n_{ev}(K(G_1, m), G_2) \oplus H^n(K(\pi, 1), \varphi_2(\hat{L}))$$

Proof. As before.

We now give a direct application of 3.11.

3.12 *Application.* Let P^n_k be a pseudo projective space of dimension n (See [3]; $P^n_2 = P^n$). For $n > 2$,

$$\pi_{n+1}(P^n_k) \cong Z_2[u_1] \oplus \cdots \oplus Z_2[u_{k-1}],$$

$k - 1$ copies of the group Z_2. The action of Z_k on $\pi_{n+1}(P_k^n)$ is

$$[u_1] \to [u_2], \cdots, [u_{k-2}] \to [u_{k-1}], [u_{k-1}] \to [u_1] + \cdots + [u_{k-1}]$$

We state the following theorem which follows at once from 3.11. We refer the reader to [4] where the importance of the theorem in the obstruction theory of pseudo projective spaces is explained.

3.13 THEOREM.
$$H^{n+2}(L_{Z_k}(\pi_n(P_k^n), n), \pi_{n+1}(P_k^n))$$
$$\cong H_{ev}^{n+2}(K(\pi_n(P_k^n), n), \pi_{n+1}(P_k^n)) \oplus H^{n+2}(K(Z_k, 1), \pi_{n+1}(P_k^n)).$$

3.14 *Final Remarks.* It is possible to avoid self injectivity at various places by computing the homology spectral sequence then dualizing to cohomology. This is true for example in 3.10. However, the theorems then take a less pleasing form. More important, when we are forced to study non-trivial systems as in 3.11 there are no general coefficient theorems which allow such dualization.

It should be noted that 3.13 can also be derived by the methods of [4].

One of the points developed in the next section is that this is not always the case.

4. Extendability of cohomology operations

In this last section we discuss the question as to which cohomology operations extend to local coefficient theory [4]. We will present various algebraic and geometric conditions on cohomology operations and discuss their necessity and sufficiency for the extendability of that operation. Finally, we will draw implications from these conditions about the spectral sequences in Sections 2 and 3 above.

We begin by listing the properties of cohomology operations we will use below. Again, [4] offers a more complete guide to the notions involved.

4.1 DEFINITION. Let $\varphi_i : \pi \times G_i \to G_i$ $(i = 1, 2)$ be as before. For each $y \in \pi$ we have $\varphi_{i,y} : G_i \to G_i$ an automorphism of G_i. This induces automorphism $\varphi_{i,y}^* : H^*(X, G_i) \to H^*(X, G_i)$. We call a cohomology operation θ, of type (m, n, G_1, G_2), (φ_1, φ_2)-*equivariant* if $\varphi_{2,y}^* \theta = \theta \varphi_{1,y}^*$ for all $y \in \pi$.

4.2 *Remarks.* This definition can be formulated in a geometric form. Assume φ_1 and φ_2 act in $K(G_1, m)$ and $K(G_2, n)$ as before.

If we assume θ is given as a map $K(G_1, m) \to K(G_2, n)$ then (φ_1, φ_2) equivariant is equivalent to θ being homotopy commutative with respect to φ_1 and φ_2. Implicit in this is the fact that homotopy commutibility does not depend on the model chosen for $K(G_1, m)$ and $K(G_2, n)$. This can be quickly checked.

4.3 DEFINITION. In the setting of 3.5 let

$$i : K(G_1, m) \subseteq L_\pi(\varphi_1, m).$$

We say $\theta \in H^n(K(G_1, m), G_2)$ is (φ_1, φ_2)-*extendable* if $\theta \in$ image i^* where

$$i^* : H^n(L_\pi(\varphi_1, m), \varphi_2(\hat{L})) \to H^n(K(G_1, m), G_2).$$

The form of 4.3 is suitable for our present investigations, but, in fact, the (φ_1, φ_2)-extendable operations are exactly those that extend to L.C. theory [4, 2.3].

4.3 can also be cast into a geometric form [4, 2.5] in which (φ_1, φ_2)-extendable is shown equivalent to the existence of extensions of certain cross-sections. Again, it is simple to show that this property does not depend on the various models chosen.

We now relate 4.1 and 4.3. As before we let $E_r^{p,q}$ be the usual twisted spectral sequence for the bundle

$$K(G_1, m) \subseteq L_\pi(\varphi_1, m) \xrightarrow{p} K(\pi, 1)$$

with coefficients in $\varphi_2(\hat{L})$.

4.4 THEOREM [4, 2.10 and 2.12]. *Identifying $H^n(K(G_1, m), G_2)$ as $E_1^{0,n}$ we have*

(a) *θ is (φ_1, φ_2)-equivariant iff $\delta_1 \theta = 0$*

(b) *θ is (φ_1, φ_2)-extendable iff $\delta_r \theta = 0$ for all r,*

hence

(c) *an operation θ is (φ_1, φ_2)-extendable only if it is (φ_1, φ_2)-equivariant.*

Below we show that the converse does not hold, but first we consider a third condition on an operation. This condition is related to the spectral sequence of Section 3.

4.5 DEFINITION. An operation θ is called (φ_1, φ_2)-*s*(*trongly*)-*equivariant* if there exist models $K(G_1, m)$ and $K(G_2, n)$ and a fixed representation $\theta : K(G_1, m) \to K(G_2, n)$ with $\varphi_{2,y}\theta = \theta\varphi_{1,y}$ $y \in \pi$. (See 4.2.)

Obviously finding a map θ that commutes with the φ_i in the prescribed way will depend on the models chosen for $K(G_1, m)$ and $K(G_2, n)$.

For example, let $\varphi_1 = \varphi_2$ be the trivial action of Z_2 on Z. Let $K(Z, 1)$ be the circle with trivial Z_2 action and $K'(Z, 1)$ be a "free" model (see 3.8). One checks that there is no representative of the identity operation with the required commutivity property. On the other hand, we have the following lemma.

4.6 LEMMA. *Let θ be a (φ_1, φ_2)-s-equivariant operation. We can find a $\theta : K(G_1, m) \to K(G_2, n)$ with $\varphi_{2,y}\theta = \theta\varphi_{1,y}$, $y \in \pi$ and with $K(G_1, m)$ a "free" model (see* 3.8.)

Notice that if we consider equivariant maps and equivariant homotopy classes with $K(G_1, m)$ a free model we get the group $H_{ev}^n(K(G_1, m), G_2)$ considered in Section 3 above.

A complete discussion of the question of choice of model can be found in [1] but is not needed below.

We now relate 4.5, 4.3, and 4.1.

4.7 THEOREM. (a) *A* (φ_1, φ_2)*-s-equivariant operation is* (φ_1, φ_2)*-extendable and* (φ_1, φ_2)*-equivariant.*

(b) *Let* θ *be* (φ_1, φ_2)*-equivariant. Suppose* $H^{n-1}(K(G_1, m), G_2) = 0$. *Then* θ *is* (φ_1, φ_2)*-s-equivariant hence* (φ_1, φ_2)*-extendable.*

Proof. Most of the above is an exercise in homotopy theory. The only point requiring care is part (b) where we must choose a free model for $K(G_1, m)$, then notice that the obstruction for a (φ_1, φ_2)-equivariant operation to be (φ_1, φ_2)-s-equivariant lies in the group which is assumed to be zero.

In a sense 4.7 is a best possible result since we now give an example of an operation that is (φ_1, φ_2)-equivariant but not (φ_1, φ_2)-extendable hence not (φ_1, φ_2)-s-equivariant.

4.8 *Example.* Let $G_1 = Z_2 \oplus Z_2$ and let $G_2 = Z_4$. Let $\pi = Z_2$ act of $Z_2 \oplus Z_2$ by interchange of factor and let Z_2 act on Z_4 non-trivially (i.e. multiplication by -1). We exhibit an element $\theta \in H^3(K(Z_2 \oplus Z_2), 1), Z_4)$ with the property that in the spectral sequence for $L_{Z_2}((Z_2 \oplus Z_2)^\wedge, 1)$ with coefficients in $\hat{Z}_4$ we have $\delta_1 \theta = 0$ but $\delta_2 \theta \neq 0$. It is convenient to exhibit θ on the chain level. Let the cochain complex for $K(Z_2, 1)$ with coefficients in Z_4 be given by

$$Z_4[e_0] \xrightarrow{d^0} Z_4[e_1] \xrightarrow{d^1} \cdots Z_4[e_n] \xrightarrow{d^n} Z_4[e_{n+1}] \longrightarrow \cdots$$

with $d^n = 0$ for n even $d^n =$ multiplication by 2 for n odd. We let $\cdots \longrightarrow Z_4[e_n'] \longrightarrow \cdots$ be a second isomorphic complex. Then the cochain complex $C^*(K(Z_2 \oplus Z_2), 1), Z_4)$ can be considered as the tensor product of these complexes. Let Z_2 act on this complex by $e_i \otimes e_j' \longrightarrow e_j \otimes e_i'$. We let θ be the class of the cocycle $2(e_2 \otimes e_1')$. This cocycle represents a class that is (φ_1, φ_2)-equivariant since $2(e_2 \otimes e_1' - e_1 \otimes e_2') = \delta(e_1 \otimes e_1')$. We know then in light of 4.4, that $\delta_1[\theta] = 0$. On the other hand we can check that $\delta_2 \theta \in E_2^{2,2}$ is not zero. In fact, if $\cdots \longleftarrow Z_4[\bar{e}_n] \longleftarrow \cdots$ is a chain complex for $K(Z_2, 1)$ then $\delta_2 \theta$ is represented by the cochain ψ in

$$C^2(K(Z_2, 1), C^2(K(Z_2 \oplus Z_2, 1), Z_4))$$

given by $\psi(\bar{e}_2) = 2(e_1 \otimes e_1')$.

4.9 *Final Remarks.* 4.8 allows us to draw several conclusions. First, it allows us to separate the notion of (φ_1, φ_2)-equivariance from the other two notions presented above. Second, since Z_4 is a $Z_4[Z_2]$ module that is free as a Z_4 module, the spectral sequence of Section 3 collapses. However, the interest in the above example is that the usual spectral sequence does not collapse.

BIBLIOGRAPHY

1. G. BREDON, *Equivalent cohomology theories*, Lecture Notes in Math., vol. 34, Springer-Verlag, New York.

2. S. Eilenberg and T. Nakayama, *On the dimension of modules and algebras II*, Nagoya Math J., vol. 9 (1955), pp. 1–16.
3. P. Olum, *Self-equivalences of pseudo-projective planes*, Topology, vol. 4 (1967), pp. 109–127.
4. J. Siegel, *Higher order cohomology operations in local coefficient theory*, Amer. J. Math., vol. 89 (1967), pp. 909–931.
5. E. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.
6. N. Steenrod, *The topology of fibre bundles*, Princeton University Press, Princeton, N. J., 1951.
7. N. Steenrod and D. Epstein, *Cohomology operations*, Ann. of Math. Studies, Vol. 50, Princeton, 1962.

Purdue University
Lafayette, Indiana