ODD p GROUPS AS FIXED POINT FREE AUTOMORPHISM GROUPS

BY

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Introduction

In this paper we extend the results of [1] to a large class of odd p groups. With certain prime exceptions then a solvable group G which admits an odd prime power group A as a fixed point free automorphism group (i.e. $C_G(A) = 1$) must have its fitting length bounded by the power of p dividing the order of A. The proof of this result is the same as the weaker theorem of [1]. Since the proof is lengthy it is not given here. The extension is made possible by an examination of representations of p groups. Suppose P is a p group acting on a vector space V over $GF(q)$, $q \neq p$. Then P permutes the vectors of V. Certainly if q is very large, P will have a regular orbit on V. The striking fact is that P "almost always" will have a regular orbit on V. Aside from other applications, this result applies to our fixed point free theorem.

Suppose R is an extra special r group. Suppose PR is an extension of R with $R \Delta PR$ and $[P, D(R)] = 1$. Suppose $\mathfrak{X}$ is a complex irreducible character of PR nontrivial on $D(R)$. It is well known [1, (IV. 9)] that if P is cyclic and faithful on R then $\mathfrak{X}|_P$ "almost always" contains the regular P character. Surprisingly enough this result is almost always true of odd p groups in general.

These two facts along with some minor related results make up the main body of this paper.

We assume a knowledge of the p Sylow subgroups of $GL(n, q)$ [3]. Some of the calculations assume a familiarity with [1].

I. Wreath products

Let $P_n = C_p \wr C_p \cdots C_p \wr C_{p^e}$, n C_p's, where $e \geq 1$ and p is an odd prime. Let r be a prime different from p and t_0 the smallest positive integer with $r^{t_0} \equiv 1 \pmod p$ and $p^e \parallel (r^{t_0} - 1)$. Now P_n is an irreducible p Sylow subgroup of $GL(t_0 p^n, r)$. Let $V = V(t_0 p^n, r)$ be the $t_0 p^n$ dimensional space over $GF(r)$ on which $GL(t_0 p^n, r)$ acts. So V is an irreducible P_n module. Assume that $P_n = \langle \rho_n \rangle \wr P_{n-1}$. Then with $\rho_n = \rho$ we may write

$$P_n^0 = P_{n-1} \times P_{n-1}^{\rho} \times \cdots \times P_{n-1}^{\rho^{p-1}}.$$

We set $P_0 = C_{p^e}$. Now $P_n = \langle \rho \rangle P_n^0$ where ρ permutes the i^{th} component into the $i + 1^{\text{st}}$ component by the conjugation $\nu \to \nu^{\rho} = \rho^{-1}\nu\rho$. Further $V|_{P_n^0} = V_1 \dotplus \cdots \dotplus V_p$ where $\rho^{-1}V_i = V_{i+1}$ and the i^{th} component of P_n^0 is faithful and irreducible on V_i with the $j \neq i$ component trivial.

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We now investigate the characters of P_n. Suppose χ is an irreducible character on P_n. Consider $\chi|_{P_n^0}$. Two cases arise. If $\chi|_{P_n^0} = \psi$ is irreducible then $\psi = \phi\phi^{\rho^{-1}} \cdots \phi^{\rho^{-p+1}}$ where ϕ is an irreducible character on P_{n-1}. Now ϕ is induced from a linear character μ^* on a subgroup A of P_{n-1}. So $\mu^*\mu^{*\rho^{-1}} \cdots \mu^{*\rho^{-p+1}} = \hat{\mu}$ on $A \times A^\rho \times \cdots \times A^{\rho^{p-1}} = A^0$ induces ψ on P_n^0. There is a linear extension μ of $\hat{\mu}$ to $\langle\rho\rangle A^0$ such that $\chi = \mu|^{P_n}$. Clearly $Z(P_n) \leq A^0$. If σ is of order p in $Z(P_n)$ then $\hat{\mu}(\sigma) = 1$. So a character of this first type is not faithful.

If $\chi|_{P_n^0}$ is reducible then $\chi|_{P_n^0} = \phi + \phi^{\rho^{-1}} + \cdots + \phi^{\rho^{-p+1}}$ where ϕ is irreducible on P_n^0. Now

$$\phi = \phi_1\phi_2^{\rho^{-1}} \cdots \phi_p^{\rho^{-p+1}}$$

where ϕ_i is irreducible on P_{n-1}. Since ϕ is not stabilized by ρ we must have $\phi_1 \neq \phi_i$ for some i.

LEMMA I.1. *Suppose χ is an irreducible character of P_n. If $\chi(1) > 1$ then either χ is induced from a linear character μ on a subgroup*

$$\langle\rho\rangle(A \times A^\rho \times \cdots \times A^{\rho^{p-1}})$$

where $A \leq P_{n-1}$ and

$$\mu|_{A\times\cdots\times A^{\rho^{p-1}}} = \mu^*\mu^{*\rho^{-1}} \cdots \mu^{*\rho^{-p+1}}$$

for a linear character μ^* on A. Further $\mu^*|^{P_{n-1}}$ is irreducible and ker $\chi \geq \Omega_1(Z(P_n))$. *Or χ is induced from a character ϕ on P_n^0. If $\phi = \phi_1\phi_2^{\rho^{-1}} \cdots \phi_p^{\rho^{-p+1}}$ where ϕ_i is irreducible on P_{n-1} then for some j, $\phi_1 \neq \phi_j$.*

Next let us consider characters faithful on P_n. We define

$$\hat{P}_1 = P_1^0 = C_{p^e} \times C_{p^e}^{\rho_1} \times \cdots \times C_{p^e}^{\rho_1^{p-1}}, \quad \hat{P}_j = \hat{P}_{j-1} \times P_{j-1}^{\rho_j} \times \cdots \times P_{j-1}^{\rho_j^{p-1}}.$$

We consider a faithful character χ of smallest degree. Then χ is induced from a character $\phi_1\phi_2^{\rho^{-1}} \cdots \phi_p^{\rho^{-p+1}} = \phi$ on P_n^0. By taking an appropriate conjugate we may assume $\phi_1(1)$ is the largest degree of the $\phi_i(1)$. Then $\phi_2, \cdots, \phi_p$ will all be linear and ϕ_1 will be faithful on P_{n-1} of smallest degree. Clearly a character of degree p faithful on P_1 is induced from $\hat{P}_1$. So if we assume ϕ_1 is induced from $\hat{P}_{n-1}$ then χ is induced from a linear character on $\hat{P}_n$. So by induction χ is induced from $\hat{P}_n$. Hence

$$\chi(1) = [P_n:\hat{P}_n] = p^n.$$

LEMMA I.2. *A faithful irreducible character χ of smallest degree on P_n is of degree p^n and is induced from a linear character of order p^e on $\hat{P}_n$.*

Let γ^* be a faithful linear character of C_{p^e}. Set $\gamma_0 = \gamma^* 1^{\rho^{-1}} \cdots 1^{\rho^{-p+1}}$ on $\hat{P}_1$ where γ^*, 1 are characters of $P_0 = C_{p^e}$; $\gamma_j = \gamma_{j-1} 1^{\rho^{-1}} \cdots 1^{\rho^{-p+1}}$ on $\hat{P}_j$ where 1 is the identity on P_{j-1}. Then set $\Gamma_j = \gamma_j|^{P_j}$.

By choosing the representation of the wreath product appropriately we may assume the Brauer character of P_j on $V(t_0 p^j, r)$ is a sum of t_0 algebraic conjugates of Γ_j.

Next we investigate representations of p groups which are not absolutely irreducible. It turns out that absolute irreducibility is not essential to show that irreducibles are induced from cyclic irreducibles. The proof is due to Blichfeldt [2, (50.7)].

LEMMA I.3. *Suppose P is an odd p group, k is a field for which* char $k \nmid p$, *and V is an irreducible $k[P]$ module. Then there is a subgroup $A \leq P$ and an irreducible $k[A]$ module W on which A is cyclic so that $W\,|^P \simeq V$.*

We use induction on $|P|$. If P is abelian or not faithful then we use induction. So we assume P is faithful. Since p is odd, P contains a noncyclic normal abelian subgroup A_0. Hence $A = \Omega_1(A_0)$ is noncyclic normal of exponent p. Since P is irreducible, A is not contained in $Z(P)$.

Consider $V\,|_A = n(W_1 \dotplus \cdots \dotplus W_t)$. We consider the case $t = 1$. In that case W_1 is an irreducible faithful $k[A]$ module. So A would be cyclic. This is not the case so $t > 1$.

Let $P_0 = \text{Stab}\,(W_1, P)$. Then $W^* = nW_1$ is an irreducible $k[P_0]$ module and $W^*\,|^P \simeq V$. Now $\ker[P_0 \to \text{Aut}\, W^*] \geq \ker[A \to \text{Aut}\, W_1] > 1$ so $P_0 < P$. Now by induction there is $B \leq P_0$ and a cyclic irreducible $k[B]$ module W so that $W\,|^{P_0} \simeq W^*$. Hence $W\,|^P \simeq V$ completing the proof.

COROLLARY I.4. *Suppose P is an odd p group with a faithful character χ of degree p^d. Suppose k is a field with* char $k \nmid |P|$ *and V is an irreducible $k[P]$ module whose character is a sum of conjugates of χ. Then V splits in $k[\chi]$.*

There is a subgroup $A \leq P$ and an irreducible cyclic $k[A]$ module W with $W\,|^P \simeq V$. If the character of W is a sum of conjugates of the linear character μ then μ and A may be so chosen that $\mu\,|^P = \chi$. Clearly A splits in $k[\mu]$. The character of V is a sum of t conjugates of χ. That is, $\dim V = \chi(1)t$. Further $\dim V = \dim W \cdot \chi(1)$ since $\chi(1) = [P : A]$. But the character of W must then be a sum of t conjugates of μ. Thus $k[\mu] = k[\chi]$ is a splitting field for P since $\mu\,|^P = \chi$.

We now look again at the wreath product P_n and derive some results on representations of subgroups of P_n. For $P_n^0 = P_{n-1} \times P_{n-1}^{\rho} \times \cdots \times P_{n-1}^{\rho^{p+1}}$ and $(\pi_1, \cdots, \pi_p) = \pi \in P_n^0$ we set $\eta_n(\pi) = \pi_1$. When there is no confusion we write $\eta = \eta_n$. The following results are concerned with extensions of extra special groups.

Now $P_n = (C_p \wr \cdots \wr C_p)A_0^n$, n C_p's, where $A_0^n = A_0$ is an abelian p group of type $(p^e, \cdots, p^e)$, n p^e's.

LEMMA I.5. *If P is a subgroup of P_n irreducible over $GF(r)$ and regular then*

$$P \cap A_0 \leq Z(P_n)[P_n, A_0].$$

Let $(\pi_1, \cdots, \pi_p) \in P_n^0 \cap A_0 \cap P = P \cap A_0$. Take $\rho(\tau_1, \cdots, \tau_p) \in P - P_n^0$.

Then using $(ab)^p = a^p b^p S^p$ for $S \in \langle a, b\rangle'$ we have

$$\{\rho(\tau_1 \pi_1, \cdots, \tau_p \pi_p)\}^p$$

$$= (\tau_2 \pi_2 \cdots \tau_p \pi_p \tau_1 \pi_1, \tau_3 \pi_3 \cdots \tau_p \pi_p \tau_1 \pi_1 \tau_2 \pi_2, \cdots, \tau_1 \pi_1 \cdots \tau_p \pi_p)$$

$$= (\tau_2 \cdots \tau_p \tau_1 \pi_1^p, \cdots, \tau_1 \cdots \tau_p \pi_p^p) S^p$$

Let $M = \Omega^{p^{e-1}}(A_0)$. Then working mod M we have

$$\tau_{i+1} \pi_{i+1} \cdots \tau_p \pi_p \cdots \tau_i \pi_i \equiv \tau_{i+1} \cdots \tau_p \cdots \tau_i \pi_i^p.$$

So $\pi_1 \pi_2^{\tau_3 \cdots \tau_1} \pi_{\pi_3}^{\tau_4 \cdots \tau_1} \cdots \tau_p^{\tau_1} \equiv 1$. Therefore $(\pi_1, \pi_2^{\tau_3 \cdots \tau_1}, \cdots, \pi_p^{\tau_1}) \in [P_n, A_0]M$. But $[P_n, A_0]M$ char P_n hence

$$(\pi_1, \cdots, \pi_p) \in [P_n, A_0]M = [P_n, A_0]Z(P_n).$$

LEMMA I.6. *If P is a regular subgroup of P_n irreducible over $GF(r)$ then $P \cap P_n^0$ contains no elements of the form $(1, \cdots, 1, \pi, 1, \cdots, 1) = \alpha$.*

Suppose otherwise. Then by conjugating we may assume $P \cap P_n^0$ contains $(\pi, 1, \cdots, 1) = \alpha$. Let $\beta = \rho(\tau_1, \cdots, \tau_p) \in P - P_n^0$. Set $T = \langle \alpha^{\beta^i} \mid i \geq 0 \rangle$, $T_0 = \langle [T, \beta^p] \rangle$. Let M be the inverse image in T of $D(T/T_0)$. By factoring M out we assume $M = 1$. Now T is abelian of type $(p, \cdots, p)$, p p's, and β operates regularly as $\langle \beta \rangle / \langle \beta^p \rangle$. Hence $\langle T, \beta \rangle$ is an irregular section of P. This contradiction proves the lemma.

THEOREM I.7. *Suppose P is an irreducible subgroup of P_n. Then P has a regular orbit on V unless P is irregular and $p^e = r^{t_0} - 1$.*

Let $\mathfrak{O}_1, \cdots, \mathfrak{O}_t$ be the regular orbits of P_{n-1} on V_1. Choose $v_1, \cdots, v_t$ as orbit representatives. We form vectors $v_{i_1} + \rho^{-1} v_{i_2} + \cdots + \rho^{-p+1} v_{i_p}$ where some $i_j \neq i_k$. Each of these vectors generates a regular orbit on P_n. Hence we get a regular orbit by restricting to P. The only problem arises when $t = 1$. We see that if $t \geq 2$ then we get at least $(t^p - t)/p$ regular orbits on P_n, $n \geq 0$. For odd p this is ≥ 2. Hence we only need worry about $n = 0$. In this case $t \geq 2$ if $p^e \neq r^{t_0} - 1$. We are reduced to considering a regular p group P.

If $p^e = r^{t_0} - 1$ then we get a regular P orbit as follows: For $(\pi_1, \cdots, \pi_p) \in P_n^0$ set $\eta((\pi_1, \cdots, \pi_p)) = \pi_1$. Then $\eta(P \cap P_n^0) = P^* \leq P_{n-1}$ has a regular orbit on V_1. Choose $v_i \in V_i$ so that v_i generates a regular $P^{*\rho^i}$ orbit on V_i. Set $u = v_1 + \cdots + v_{p-1}$. For $\alpha \in P$ to centralize u, $\alpha = (1, 1, \cdots, \pi)$. By the previous lemma, no such α exists in P. Hence P always has a regular orbit on V.

COROLLARY I.8. *Suppose R is an elementary abelian r group and P is faithful and irreducible on R. Then $1_P \,|^{PR}|_P$ contains the regular P character unless $p^e = r^{t_0} - 1$ and P is irregular where $p^e \leq$ exp P, $r^{t_0} \mid |R|$.*

We look at $1_P \mid^{PR}\mid_P = \sum_{P \times P} 1_P^x \mid_{P^x \cap P}\mid^P$. We may choose $x \in R$ so that $P^x \cap P = 1$ by the theorem. This proves the corollary.

Now we turn our attention to representations of P on extra special r groups.

LEMMA I.9. *Suppose P is a p group and $P^* = C_p \wr P$. Let $C_p = \langle\rho\rangle$ and $P^0 = P \times P^\rho \times \cdots \times P^{\rho^{p-1}}$. If $|P| = p^a$ then there are $p^{a(p-1)}$ elements $(\tau_1, \cdots, \tau_p) \in P^0$ with $\tau_1 \tau_2 \cdots \tau_p = \nu_0 \in P$. Further if $\nu = (\sigma_1, \cdots, \sigma_p) \in P^0$ and $\sigma_1 \sigma_2 \cdots \sigma_p = \nu_0 \in P$ then*

$$\dim C_V(\rho\nu) = \dim C_{V_1}(\nu_0).$$

Choose $\tau_1, \cdots, \tau_{p-1}$ arbitrarily in P. Then solve $\tau_1 \cdots \tau_{p-1} x = \nu_0$ for x. This shows $(\tau_1, \cdots, \tau_p) \in P^0$ may be chosen in $p^{a(p-1)}$ ways.

Next we show that $v = v_1 + \cdots + v_p \in C_V(\rho\nu)$ where $v_i \in V_i$ if and only if $v_1 \in C_{V_1}(\nu_0)$ which completes the proof of the lemma.

So computing

$$\begin{aligned} v = \rho\nu u &= \rho(\sigma_1, \cdots, \sigma_p)v_1 + \cdots + \rho(\sigma_1, \cdots, \sigma_p)v_p \\ &= (\sigma_2, \sigma_3, \cdots, \sigma_p, \sigma_1)\rho v_1 + \cdots + (\sigma_2, \cdots, \sigma_p, \sigma_1)\rho v_p . \end{aligned}$$

Therefore $(\sigma_2, \cdots, \sigma_p, \sigma_1)\rho v_i = v_{i-1}$ or $(\sigma_1 \cdots \sigma_p, 1, \cdots, 1)v_1 = \nu_0 v_1 = v_1$.

Next let P_n be faithful and irreducible on $R/D(R)$ and trivial on $D(R)$ where R is extra special of order r^{2m+1}. Then $2m = t_0 p^n$. Let $\mathfrak{X}_\lambda \mid_{P_n} = \mathfrak{X}_n$ where $\mathfrak{X}_\lambda$ is the character of $P_n R$ given by [1, (IV. 15)]. Suppose $\sigma \in P_n$. Then $R/D(R) = V$ as a $\langle\sigma\rangle$ module splits as in [1, (IV. 3)]. So $C_V(\sigma)$ has even dimension $2m(\sigma)$. Also $2m = \sum_i n_i t_o p^{ei} + 2m(\sigma)$ where there are n_i irreducible $\langle\sigma\rangle$ modules on V of dimension $t_0 p^{ei}$. So $l(\sigma) = (2m - 2m(\sigma)/t_0 = \sum_i n_i p^{ei} \equiv \sum n_i = n(S) \pmod 2$ since p is odd. Hence

$$\mathfrak{X}_n(\sigma) = r^{m(\sigma)}(-1)^{l(\sigma)} = r^{m(\sigma)}(-1)^{2(m-m(\sigma))/t_0}.$$

LEMMA I.10. *Assume $A \leq P_{n-1}$ and $A^0 = A \times A^p \times A^{\rho^{p-1}} \leq P_n^0$. Let μ be a linear character of $T = \langle\rho\rangle A^0 \leq P_n$. Now $\mu\mid_A = \mu^*$. So*

$$(\mathfrak{X}_n \mid_T, \mu)_T = (1/p)[(\mathfrak{X}_{n-1} \mid_A, \mu^*)_A^p + (\delta p - 1)(\mathfrak{X}_{n-1} \mid_A, \mu^*)_A]$$

where $\delta = 0$ if $\mu(\rho) \neq 1$ and $\delta = 1$ otherwise.

$$|T|\ (\mathfrak{X}_n \mid_T, \mu)_T = \sum_{\nu \in A^0} \mathfrak{X}_n(\nu)\bar{\mu}(\nu) + \sum_{i=1}^{p-1} \sum_{\nu \in A^0} \mathfrak{X}_n(\rho^i \nu)\bar{\mu}(\rho^i \nu).$$

For each $\nu \in A^0$ there is a unique $\hat{\nu} \in A^0$ such that $\rho^i \nu = (\rho\hat{\nu})^i$. Further $\nu \to \hat{\nu}$ is a one-one map of A^0 onto A^0. Also, μ being a linear character is multiplicative. So

$$\begin{aligned} \sum_{i=1}^{p-1} \sum_{\nu \in A^0} \mathfrak{X}_n(\rho^i \nu)\bar{\mu}(\rho^i \nu) &= \sum_{i=1}^{p-1} \sum_{\hat{\nu} \in A^0} \mathfrak{X}_n(\{\rho\hat{\nu}\}^i)\bar{\mu}(\rho^i)\bar{\mu}(\hat{\nu}^i) \\ &= \sum_{i=1}^{p-1} \bar{\mu}(\rho^i) \sum_{\hat{\nu} \in A^0} \mathfrak{X}_n(\rho\hat{\nu})\bar{\mu}^i(\hat{\nu}) \end{aligned}$$

We have

$$\begin{aligned} \mathfrak{X}_n(\rho\hat{\nu}) &= \mathfrak{X}_n(\{\rho\hat{\nu}\}^i) \qquad \text{(since } (i, p) = 1) \\ &= \sum_{i=1}^{p-1} \bar{\mu}^i(\rho) p^{a(p-1)} \sum_{\nu_0 \in A} \mathfrak{X}_{n-1}(\hat{\nu}_0)\bar{\mu}^{*i}(\hat{\nu}_0). \end{aligned}$$

Since μ is multiplicative and $\mu(\mathfrak{p}) = \mu^*(\mathfrak{p}_0)$, by (I.9),

$$\begin{aligned} \mathfrak{X}_n(\rho\mathfrak{p}) &= \textstyle\sum_{i=1}^{p-1} \bar{\mu}^i(\rho) \mid A^0 \mid (\mathfrak{X}_{n-1} \mid_A, \mu^{*i})_A \\ &= \mid A^0 \mid (\mathfrak{X}_{n-1} \mid_A, \mu^*)_A \textstyle\sum_{i=1}^{p-1} \bar{\mu}^i(\rho) \\ &= \mid A^0 \mid (\mathfrak{X}_{n-1} \mid_A, \mu^*)_A (\delta p - 1). \end{aligned}$$

Next

$$\textstyle\sum_{\nu \epsilon A^0} \mathfrak{X}_n(\nu)\bar{\mu}(\nu) = \mid A^0 \mid (\mathfrak{X}_n \mid_{A^0}, \mu \mid_{A^0})_{A^0}$$

Now $\mathfrak{X}_n \mid_{A^0} = \mathfrak{X}_{n-1} \mid_A (\mathfrak{X}_{n-1} \mid_A)^{\rho^{-1}} \cdots (\mathfrak{X}_{n-1} \mid_A)^{\rho^{-p+1}}$. So

$$\mathfrak{X}_n(\rho\mathfrak{p}) = \mid A^0 \mid (\mathfrak{X}_{n-1} \mid_A, \mu^*)_A^p .$$

Since $\mid T \mid / \mid A^0 \mid = p$ the lemma follows.

Suppose χ is an irreducible character on P_n. We call χ a *regular character* if

(i) $\chi \mid_{P_n^0} = \phi\phi^{\rho^{-1}} \cdots \phi^{\rho^{-p+1}}$ is irreducible and

  (a) $n = 1$ and ϕ is nontrivial on P_0, or

  (b) $n > 1$ and ϕ is regular on P_{n-1}.

(ii) $\chi = (\phi_1\phi_2^{\rho^{-1}} \cdots \phi_p^{\rho^{-p+1}}) \mid^{P_n}$ where $\phi_1, \cdots, \phi_p$ are irreducible on P_{n-1} and

  (a) $n = 1$ and at most one $\phi_j = 1$, or

  (b) $n > 1$ and $\phi_1, \cdots, \phi_p$ are all regular.

LEMMA I.11. *Suppose P is a regular irreducible subgroup of P_n and χ is irreducible on P. Then $\chi \mid^{P_n}$ contains a regular character.*

We use induction on n. Suppose first that $n = 1$. We set $T = Z(P_1)[P_1, A_0]$. Since $n = 1$ we have $A_0 = P_0 \times P_n^0 \times \cdots \times P_0^{\rho^{i-1}}$. Writing A_0 additively over Z_{p^e}, then letting v_i be a generator for $P_0^{\rho^{i-1}}$, we have $\{v_1, \cdots, v_p\}$, a Z_{p^e} basis for A_0. A character $f : A_0 \to Z_{p^e}^+$ is a linear functional into the additive group of Z_{p^e}.

Now $\{u_1 = v_1 + \cdots + v_p, u_i = v_1 - v_i, i = 2, \cdots, p\}$ is a set of generators over Z_{p^e} for T. Let $g : T \to Z_{p^e}^+$ be a character on T. Then

$$g(u_i) = \alpha_i, \quad i = 1, \cdots, p.$$

Now $g(pv_1) = g(\sum u_i) = \sum g(u_i) = \sum \alpha_i$ is in pZ_{p^e} so we may choose $\beta \epsilon Z_{p^e}$ so that $p\beta = \sum \alpha_i$. Now consider the character $f_\varepsilon : A_0 \to Z_{p^e}^+$ defined by

$$f_\varepsilon(v_1) = \beta + \varepsilon, \qquad f_\varepsilon(v_i) = \beta + \varepsilon - \alpha_i, \quad i = 2, \cdots, p,$$

where $\varepsilon \epsilon p^{e-1}Z_{p^e}$. Then clearly

$$f_\varepsilon \mid_T = g.$$

Next $p^{e-1}Z_{p^e} - \{-(\beta - \alpha_i); i = 2, \cdots, p\}$ is not empty since $p^{e-1}Z_{p^e}$ contains p elements. So we choose ε from this set. Then

$$\beta + \varepsilon - \alpha_i \neq 0 \quad \text{for } i = 2, \cdots, p.$$

In other words $f_\varepsilon(v_i) \neq 0$ for $i = 2, \cdots, p$. Therefore $f_\varepsilon \mid^{P_1}$ contains only regular characters. We have proved that if μ is an arbitrary character of T then $\mu \mid^{P_1^0}$ contains a character ν such that $\nu \mid^{P_1}$ is a sum of regular characters.

Now suppose $n > 1$. Set $\eta(P \cap P^+) = P^*$ and

$$P^+ = P^* \times P^{*\rho} \times \cdots \times P^{*\rho^{p-1}}.$$

Now P^* is a regular irreducible subgroup of P_{n-1}. Let $\hat{\phi}_1, \cdots, \hat{\phi}_p$ be irreducible characters on P^*. By induction on n, $\hat{\phi}_i |^{P_{n-1}}$ contains a regular character ϕ_i on P_{n-1}. But now $(\phi_1 \phi_2^{\rho^{-1}} \cdots \phi_p^{\rho^{-p+1}}) |^{P_n}$ must be a sum of regular characters on P_n. In other words, if μ is any irreducible character of P^+ then $\mu |^{P_n^0}$ contains a character ν with $\nu |^{P_n}$ a sum of regular characters.

Now let χ be irreducible on P. Since $P \cap P_n^0 \leq P^+$ or $P \cap P_1^0 \leq T$, for every character μ contained in $\chi |_{P \cap P_n^0}$, $\mu |^{P_n^0}$ contains a character ν with $\nu |^{P_n}$ a sum of regular characters. Finally, since $PP_n^0 = P_n$, we have

$$\chi |^{P_n} |_{P_n^0} = \chi |_{P \cap P_n^0} |^{P_n^0}$$

and

$$(\chi |^{P_n}, \nu |^{P_n}) = (\chi |^{P_n} |_{P_n^0}, \nu) = (\chi |_{P \cap P_n^0} |^{P_n^0}, \nu)$$
$$= (\chi |_{P \cap P_n^0}, \nu |_{P \cap P_n^0}) \geq (\chi |_{P \cap P_n^0}, \mu) > 0.$$

Therefore $\chi |^{P_n}$ contains a regular character.

Recall that P_n is faithful and irreducible on $R/D(R)$ where $|R| = r^{2m+1}$, $2m = t_0 p^n$, and $P_n = C_p \wr \cdots \wr C_{p^e}$.

LEMMA I.12. (a) *Suppose χ is a regular irreducible character of P_n. Assume that $p^e \neq r^{t_0/2} + 1$. Then $(\mathfrak{X}_n, \chi) \geq 2$.*

(b) *If, both $2p^e \neq r^{t_0/2} + 1$ and $p^e \neq r^{t_0/2} + 1$, then for any irreducible χ on P_n,*

$$(\mathfrak{X}_n, \chi) \geq 2\chi(1).$$

Proof is by induction on n. For any irreducible character χ of P_0, we see that by [1, (IV.9)],

$$(\mathfrak{X}_0, \chi) \geq 2 \quad \text{in case (b) and case (a) if } \chi \neq 1$$
$$\geq 1 \quad \text{in case (a) if } \chi = 1.$$

Suppose $n > 0$. First, assume that $\chi = (\phi_1 \phi_2^{\rho^{-1}} \cdots \phi_p^{\rho^{-p+1}}) |^{P_n}$ where $\phi_1, \cdots, \phi_p$ are irreducible on P_{n-1}. Then $(\mathfrak{X}_n, \chi) = \prod_{i=1}^p (\mathfrak{X}_{n-1}, \phi_i)$. In case (a), χ is regular so $(\mathfrak{X}_{n-1}, \phi_i) \geq 2^{p-1} \geq 2$. In case (b), $(\mathfrak{X}_{n-1}, \phi_i) \geq 2\phi_i(1)$ (even if $n = 1$). So

$$\prod_{i=1}^p (\mathfrak{X}_{n-1}, \phi_i) \geq 2^p \phi_1(1) \cdots \phi_p(1) = 2(2^{p-1}/p)\chi(1) \geq 2\chi(1).$$

Second, assume that $\chi |_{P_n^0} = \phi\phi^{\rho^{-1}} \cdots \phi^{\rho^{-p+1}}$ where ϕ is irreducible on P_{n-1}. Then

$$(\mathfrak{X}_n, \chi) = (1/p)[(\mathfrak{X}_{n-1}, \phi)^p + (\delta p - 1)(\mathfrak{X}_{n-1}, \phi)]$$
$$\geq [(\mathfrak{X}_{n-1}, \phi)/p][(\mathfrak{X}_{n-1}, \phi)^{p-1} - 1].$$

In case (a), $(\mathfrak{X}_{n-1}, \phi) \geq 2$ so $(\mathfrak{X}_n, \chi) \geq (2/p)[2^{p-1} - 1] \geq 2$. In case (b), $(\mathfrak{X}_{n-1}, \phi) \geq 2\phi(1)$ so $(\mathfrak{X}_n, \chi) \geq (2^p\phi(1)^p - 2\phi(1))/p \geq 2\chi(1)$.

THEOREM I.13. *Suppose PR is a group with normal extra special r subgroup R and odd p group P. Suppose P is faithful and irreducible on $R/D(R)$ and trivial on $D(R)$. Suppose X is any irreducible character of PR which is non-trivial on $D(R)$. Then for any irreducible character χ of P,*

$$(X|_P, \chi) \geq 2n_\chi$$

where

(1) $p^c \neq r^d + 1$, $n_\chi = 1$, *and P is regular, or*
(2) $\gamma p^c \neq r^d + 1$, *and* $n_\chi = \chi(1)$ *for any* $\gamma = 1, 2$; $p^c \leq \exp P$; *and* $r^{2d+1} \mid |R|$.

Now $P \leq P_n$ for some n where P_n is irreducible on $R/D(R) = V$ and is a p Sylow subgroup of $S_p(V)$, the symplectic group on V. By [1, (II.2), (IV.15)], we know that $X = \mu X_\lambda$ for some μ on PR/R and for some λ. But also $\mathfrak{X}_\lambda = \mathfrak{X}_n|_P$. Let χ be any irreducible character of P. So

$$(X, \chi) = (\mu\mathfrak{X}_\lambda, \chi) = (\mathfrak{X}_\lambda, \bar{\mu}\chi) = (\mathfrak{X}_n|_P, \bar{\mu}\chi) = (\mathfrak{X}_n, [\bar{\mu}\chi]|^{P_n}).$$

If P is regular, then we may select ψ a regular character in $[\bar{\mu}\chi]|^{P_n}$ by (I.11). So by (I.12) (a),

$$(X, \chi) = (\mathfrak{X}_n, [\bar{\mu}\chi]|^{P_n}) \geq (\mathfrak{X}_n, \psi) \geq 2.$$

In case (b), $[\bar{\mu}\chi]|^{P_n} = \sum a_\psi \psi$ so

$(X, \chi) = (\mathfrak{X}_n, [\bar{\mu}\chi]|^{P_n}) = \sum a_\psi (\mathfrak{X}_n, \psi) \geq \sum a_\psi 2n_\psi = 2[\bar{\mu}\chi]|^{P_n}(1) \geq 2\chi(1)$.
This completes the proof.

Remark. If $p > 3$, then the result in (I.12) a) is much too small. A much better lower estimate would be $(\mathfrak{X}_n, \chi) \geq ((2^p - 2)/p)^{2^{n-1}}$ if $p > 3$. For $p = 3$ there is actually a linear regular character μ of P_n for which $(\mathfrak{X}_n, \mu) = 2$ for the choice $2p^3 = r^{t_0/2} + 1$.

II. Applications

Suppose A is an odd p group for a prime p. Let AG be a solvable group with normal subgroup G where $(p, |G|) = 1$. We assume that $p^e \neq r^c + 1$ for any $p^e \leq \exp A$ and $r^{2c+1} \mid |G|$. Further, if A is an irregular p group we assume $p^e \neq r^d - 1$ and $2p^3 \neq r^c + 1$ for $r^d \mid |G|$. These prime assumptions are made to handle the prime exceptions of (I.7) and (I.13).

Suppose that V is a direct sum of equivalent irreducible $k[AG]$ modules where k is a field of characteristic unequal to p. We also assume that G is represented faithfully on V.

Under the conditions outlined above we have

THEOREM II.1. (1) $C_V(A) \neq (0)$ *or*
(2) $C_V(A') = (0)$ *or*
(3) $C_V(A') \neq (0)$ *and there is cyclic* $D \leq A$ *with*

$$\text{(a) } C_V(A'D) = (0), \qquad \text{(b) } C_G(A'D) \geq C_G(A').$$

In [1] this theorem (VI.1) was proved under the assumption that A was of class ≤ 2. The proof given there is quite general. The step (VI.10) may be made using (I.7) of the previous section. The step (VI.11) follows from the lemma below. Finally (VI.15) follows from (I.13) of the previous section. Aside from this the proof follows verbatin.

The representation theorem just cited has application to groups with fixed point free automorphism groups. Suppose the derived series of A is

$$A = A^{(0)} > A^{(1)} > \cdots > A^{(n)} = 1.$$

Then $\prod_{i=0}^{n-1} [A^{(i)} : C_{A^{(i)}} C_G(A^{(i+1)})] = p^f$. We may set $\psi(G) = f$. If $|A| = p^d$ then clearly $f \leq d$. We then have

THEOREM (II.2). *If A is fixed point free on G, that is, $C_G(A) = 1$, then the Fitting length of G is bounded by $\psi(G)$. In particular, the Fitting length is bounded by d.*

The proof here is again exactly as (VII.1) of [1]. Except now we use (II.1) the previous theorem.

LEMMA II.3. *If* (II.1) *holds for* (A, G, V) *and* $V|_{A_0G}$ *is homogeneous for all* $A_0 \trianglelefteq A$ *then there is a subgroup* $D \leq A$ *so that* $D \leq C_A(G)$ *and* $[V, D] = V$ *unless* (1) *of* (II.1) *is true.*

We use induction on $|A|$. By the theorem there is a $D_0 \leq A$ so that $C_G(A'D_0) \geq C_G(A')$. Further $[A'D_0, V] = V$. Since D_0 is cyclic we must have $A'D_0 < A$ or A is abelian. In the former case we use induction on $(A'D_0, G, V)$. In the latter case $D_0 \leq C_G(A') = G$ so we are done.

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