NOTE ON QUASIFIBRATIONS AND FIBRE BUNDLES

BY
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1. Introduction

In a previous paper of the authors' [4], the problem of classifying rank 2 H-spaces up to homotopy type was considered. The methods used in that paper led to the study of CW-complexes of the form $X = S^q \cup_\alpha e^n \cup_\beta e^{n+q}$ which are quasifibrations over S^n in the sense of Dold-Thom [1], i.e. there exists a map of pairs

$$p : (X, S^q) \to (S^n, \text{point})$$

inducing isomorphisms $p_* : \pi_i(X, S^q) \to \pi_i(S^n)$ for all i.

In [4], our major interest was in the cases $q = 3$ and $n = 5$ or 7; as a by-product of our main results, the following theorem was obtained.

THEOREM 1.1. *If $X = S^q \cup_\alpha e^n \cup_\beta e^{n+q}$ quasifibres over S^n with $q = 3$ and $n = 5$ or 7, then X has the homotopy type of an orthogonal S^q-bundle over S^n.*

This theorem does not generalize to other values of q and n; in fact, a specific counterexample has been given by Sutherland [11] with $q = 3$, $n = 8$. Our purpose here is to provide a whole family of examples of S^q-quasifibrations over S^n which are not homotopy equivalent to orthogonal S^q-bundles over S^n. In all of our examples, we will have $q = 2$ and the first attaching map $\alpha = 0$; the latter implies the existence of a "cross-section" for the quasifibration.

Sutherland constructs his example so that the total space has the homotopy type of a closed, smooth manifold. He then shows that his example does not even have the homotopy type of a differentiable S^3-bundle over S^8, i.e. a fibre bundle over S^8 with fiber S^3 and structural group Diff (S^3), the group of diffeomorphisms of S^3. In fact, Sutherland observes that results from Cerf's thesis can be used to show that any differentiable S^q-bundle over S^n must be fibre homotopy equivalent to an orthogonal S^q-bundle over S^n.

Many of the examples we construct also turn out to be homotopy equivalent to closed, smooth manifolds so that the above remark applies to these examples. However, we can even go one step further and assert that none of our examples has the homotopy type of an S^2-fibre bundle over S^n, with structural group the full group of homeomorphisms of the fibre. This is due to the classical fact that the full group of homeomorphisms of S^2 has the homotopy type of the 3-dimensional orthogonal group.

One might enquire whether the manifolds we construct here have a reasonable "geometric" description. We do not attempt to give such a description for all our examples, but we do succeed in doing this for our lowest dimensional

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example yielding a smooth manifold, a certain S^2-quasifibration over S^4. To carry this out, we base ourselves on Wall's work on 6-manifolds [12].

The rest of the paper is organized as follows. In §2, we construct our examples and show that they have the purported properties. We also discuss the relevance of these examples to the phenomena discussed in [3]. Finally, in §3, we discuss in detail our 6-dimensional example.

2. Construction of the examples

We begin by establishing notation and recalling some needed results. Let

$$X = X_{\alpha\beta} = S^q \cup_\alpha e^n \cup_\beta e^{n+q} = C_\alpha \cup_\beta e^{n+q}, \qquad 2 \leq q \leq n - 2.$$

Let $\sigma \in \pi_n(C_\alpha, S^q)$ be the generator which satisfies $\partial(\sigma) = \alpha$, let $\iota_q \in \pi_q(S^q)$ be the generator, and let $i : S^q \to C_\alpha$ and $j : C_\alpha \to (C_\alpha, S^q)$ be the respective inclusions. We shall later only be concerned with elements $\beta \in \pi_{n+q-1}(C_\alpha)$ for which $j_* \beta = [\sigma, \iota_q]$, the relative Whitehead product of σ and ι_q (see [4]). Under these circumstances, we have the following result, which is a special case of a theorem of Sasao [8].

THEOREM 2.1. *If $j_* \beta = [\sigma, \iota_q]$, the natural collapsing map $C_\alpha \to S^n$ extends to a map $p : (X, S^q) \to (S^n, \text{point})$ and p is a quasifibration.*

As to the homotopy type of $X_{\alpha\beta}$, we have the following elementary result.

THEOREM 2.2. *If $X_{\alpha\beta} \simeq X_{\alpha'\beta'}$, then $\pm\alpha' = (\pm 1) \circ \alpha$.*

Proof. Let $h : X_{\alpha\beta} \simeq X_{\alpha'\beta'}$. Since $2 \leq q \leq n - 2$, it follows by cellular approximation that we may suppose

$$h : C_\alpha \simeq C_{\alpha'}, \quad h : S^q \simeq S^q.$$

It then follows by applying Corollary 7.4′ of [2] that there is a homotopy-commutative diagram

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\alpha} & S^q \\ {\scriptstyle \pm 1}\downarrow & & \downarrow{\scriptstyle \pm 1} \\ S^{n-1} & \xrightarrow{\alpha'} & S^q \end{array}$$

from which the result follows.

Now, if E is the total space of an S^q-bundle over S^n, then certainly $E = X_{\alpha\beta}$ for some α, β. Moreover, $\alpha \in \pi_{n-1}(S^q)$ is the characteristic class of the bundle, that is, the obstruction to a cross-section. Thus the bundle admits a cross-section if and only if $\alpha = 0$. We thus have the following corollary to Theorem 2.2.

COROLLARY 2.3. *If $X = X_{\alpha\beta}$ with $\alpha = 0$ and if X has the homotopy type of the total space of an S^q-bundle over S^n, then that bundle has a cross-section.*

We wish to apply Theorem 2.1 in the case $q = 2$, $\alpha = 0$. For $\alpha = 0$, $C_\alpha = S^q \vee S^n$ and we note that the condition $j_* \beta = [\sigma, \iota_q]$ is equivalent to having β of the form

$$\beta = [\iota_n, \iota_q] + \theta, \theta \in \pi_{n+q-1}(S^q); \tag{2.4}$$

we have here identified $\pi_{n+q-1}(S^q)$ with its isomorphic image

$$i_*(\pi_{n+q-1}(S^q)) \subset \pi_{n+q-1}(S^q \vee S^n).$$

We may now state our key technical lemma.

Lemma 2.5. *Let $X_\theta = (S^2 \vee S^n) \cup_\beta e^{n+2}$ with β as in (2.4), $n \geq 3$. Then X_θ is homotopy equivalent to $X_{\theta'}$ iff $\theta = \pm\theta'$. In particular, X_θ is homotopy equivalent to $S^2 \times S^n$ iff $\theta = 0$.*

As a corollary of Lemma 2.5, we have the main theorem.

Theorem 2.6. *If $\theta \neq 0$, $n \geq 4$, then X_θ is an S^2-quasifibration over S^n but is not homotopy equivalent to an S^2-bundle over S^n with structural group* Top (S^2), *the group of all self-homeomorphisms of S^2.*

Proof of Theorem 2.6. The first clause of the theorem follows from Theorem 2.1, since $n \geq 4$. We next show that X_θ cannot be homotopy equivalent to (the total space of) an orthogonal S^2-bundle over S^n. For let E be such a space. Then, since E has a cross-section, by Corollary 2.3, it follows that the structural group of E can be reduced from O_3 to O_2, O_i being the i-dimensional orthogonal group. But, since $n \geq 3$, $\pi_{n-1}(O_2) = 0$, so that E must be trivial, i.e. $E = S^2 \times S^n$. It follows that $X_\theta \simeq S^2 \times S^n$ and this contradicts Lemma 2.5, θ being nonzero. Finally, we recall that, by a classical theorem of H. Kneser [6], the natural inclusion $O_3 \to$ Top (S^2) is a homotopy equivalence. This clearly completes the proof of the theorem.

We turn now to Lemma 2.5.

Proof of Lemma 2.5. We prove that if $X_\theta \simeq X_{\theta'}$, then $\theta = \pm\theta'$, the proof of the converse being quite elementary. Let $h : X_\theta \simeq X_{\theta'}$. By cellular approximation, we may suppose $h : S^2 \vee S^n \simeq S^2 \vee S^n$ and, as in the proof of Theorem 2.2, we apply Corollary 7.4′ of [2] to infer a homotopy-commutative diagram

$$\begin{array}{ccc} S^{n+1} & \xrightarrow{\beta} & S^2 \vee S^n \\ \varepsilon = \pm 1 \downarrow & & \downarrow h \\ S^{n+1} & \xrightarrow{\beta'} & S^2 \vee S^n \end{array} \qquad \beta' = [\iota_n, \iota_2] + \theta'. \tag{2.7}$$

Now h must belong to the homotopy class $\{\pm\iota_2, \pm\iota_n + \omega\}$ where ω is an arbitrary element of $\pi_n(S^2)$. Assume first that $h = \{\iota_2, \iota_n + \omega\}$. Then,

$$\begin{aligned} h_* \beta &= \{\iota_2, \iota_n + \omega\} \circ ([\iota_n, \iota_2] + \theta) = [\iota_n + \omega, \iota_2] + \theta \\ &= \beta + [\omega, \iota_2] = \beta, \end{aligned}$$

since we have $n \geq 3$ so that $[\omega, \iota_2] = 0$ by a result of Hilton-Whitehead [5]. We are now forced to take $\varepsilon = +1$ in (2.7) and obtain $\beta' = \beta, \theta' = \theta$.

Similarly, if $h = \{\iota_2, -\iota_n + \omega\}$, we find $h_* \beta = [-\iota_n, \iota_2] + \theta$, so we must take $\varepsilon = -1$ and obtain $-\theta' = \theta$. If $h = \{-\iota_2, \iota_n + \omega\}$, we find

$$h_* \beta = -[\iota_n, \iota_2] + (-\iota_2) \circ \theta.$$

However, $(-\iota_2) \circ \theta = \theta$ since $(-\iota_2) \circ \gamma = \gamma$ for the Hopf map $\gamma \in \pi_3(S^2)$. Thus again we must take $\varepsilon = -1$ and obtain $-\theta' = \theta$. Finally, with $h = \{-\iota_2, -\iota_n + \omega\}$ we find $h_* \beta = [\iota_n, \iota_2] + \theta, \varepsilon = +1, \theta' = \theta$, and the lemma is proved.

We remark before proceeding that this argument is very special to the case $q = 2$. It would apply with minor modifications to the cases $q = 3$, $q = 7$, but would fail entirely for $q = 4$.

We now discuss circumstances in which the spaces X_θ are homotopy equivalent to closed, smooth manifolds. Obviously, the X_θ are Poincaré Duality spaces, and are 1-connected. In order to put a manifold structure on X_θ, by the Browder-Novikov Theorem, we must study the Spivak normal spherical fibration $\nu = \nu(X_\theta)$ of X_θ (cf. [10]) and determine whether ν can be lifted to a vector bundle over X_θ. Now the easiest sufficient condition for ν to come from a vector bundle is simply that ν be (fibre homotopy) trivial. This condition, in turn, is equivalent to having X_θ stably reducible, i.e. $([\iota_2, \iota_n] + \theta) \in \ker \Sigma^N$,

$$\Sigma^N : \pi_{n+1}(S^2 \vee S^n) \to \pi_{N+n+1}(S^{N+2} \vee S^{N+n})$$

the N-fold iterated suspension, N large. As $\Sigma([\iota_2, \iota_n]) = 0$, we see that the stable reducibility of X_θ is equivalent to the stable triviality of θ. We thus have

THEOREM 2.8. *Let* $\theta \in \ker \Sigma^N : \pi_{n+1}(S^2) \to \pi_{N+n+1}(S^{N+2})$, $n \geq 3$, *and suppose further that* $n \neq 4k, k \neq 1, 3$. *Then* X_θ *has the homotopy type of a closed, smooth* π-*manifold of dimension* $n + 2$.

Proof. The above discussion shows that the Thom complex of the trivial N-dimensional vector bundle over X_θ is reducible so that we may apply the Browder-Novikov procedure. The restrictions on n insure that there are no middle-dimensional obstructions to surgery. In particular, since the signature of X_θ is always 0, the case $n = 4k + 2$ presents no difficulty.

Examples where the hypotheses of Theorem 2.8 are satisfied are, of course, numerous. The lowest dimensional example where Theorem 2.8 applies occurs for $n = 5$, θ any nonzero element of $\pi_6(S^2) = \mathbf{Z}_{12}$; it is well known that

$\Sigma(\pi_6(S^2)) = 0$. There is one S^2-quasifibration over S^4 (the smallest possible dimension to which Theorem 2.6 applies) with $\theta \neq 0$, to which Theorem 2.8 does not apply, but which nevertheless has the homotopy type of a closed, smooth manifold. We state this as

THEOREM 2.9. *Let* $X = X_\mu = (S^2 \vee S^4) \cup_{[\iota_2, \iota_4]+\mu} e^6$, μ *the generator of* $\pi_5(S^2) = \mathbf{Z}_2$. *Then X has the homotopy type of a closed, smooth* 6-*manifold.*

Of course, such a manifold cannot be stably parallelizable since μ stably suspends to a nonzero element of the 3-stem. More precise information concerning manifolds realizing X will be given in the next section.

Proof of Theorem 2.9. It suffices, as above, to lift the Spivak fibration $\nu(X)$ to a vector bundle. More generally, one can show, using known facts about the homotopy structure of F/O in low dimensions, that such a lifting exists for any 1-connected, 6-dimensional Poincaré Duality space Y with $H^3(Y; \mathbf{Z}_2) = 0$. For details, see [12].

To conclude this section, we show that the quasifibrations $S^2 \to X_\theta \to S^n$ furnish further examples of the phenomena discussed in [3].

THEOREM 2.10. X_θ *and* $S^2 \times S^n$ *have isomorphic homotopy groups and integral cohomology rings, but are of different homotopy types if* $\theta \neq 0$.

Proof. For the homotopy groups, we look at the exact sequence of the quasifibration $S^2 \to X_\theta \to S^n$ and use the existence of a cross-section to split the sequence. For the cohomology structure, we simply compare the cellular structures of X_θ and $S^2 \times S^n$.

Of course, a similar statement holds for sphere bundles over spheres with cross-section. The smallest dimension for which such an example exists is 5; namely, take the nontrivial S^3-bundle over S^2. Note that the nontrivial S^2-bundle over S^2 has a different cohomology *ring* from $S^2 \times S^2$ although its cohomology *groups* (as well as its homotopy groups) are isomorphic to those of $S^2 \times S^2$. (Recall that for 1-connected 4-manifolds, the homotopy type is completely determined by the integral cohomology *ring* structure; see Milnor [7].)

Remarks. (1) Observe that the nontrivial S^q-bundle over $S^2 (q \geq 2)$, in particular, the 5-dimensional example above, fails to be a spin manifold—this is simply because the nonzero element of $\pi_2(BO_{q+1})$ is detected by the second Stiefel-Whitney class w_2. For 1-connected, 5-dimensional spin manifolds, it is known that the diffeomorphism class is completely determined by the second homology group; cf. Smale [9].

(2) We shall show in §3 that our 6-dimensional examples (see Theorem 2.9) are all spin manifolds; hence 6 is the minimal dimension for examples exhibiting the phenomena of Theorem 2.10 in the class of 1-connected spin manifolds.

(3) The 5-dimensional complex $X_\lambda = (S^2 \vee S^3) \cup_{[\iota_2, \iota_3]+\lambda} e^5$, λ the generator of $\pi_4(S^2) = \mathbf{Z}_2$, is of the type appearing in Theorem 2.1 except that the condi-

tion $n - q \geq 2$ is not met. It is true, nevertheless, that, the conclusion of Theorem 2.1 holds for this space, i.e. that X_λ is an S^2-quasifibration over S^3.[2] It should be noted that X_λ cannot be realized by a smooth manifold. Indeed, the same reasoning as in Lemma 3.1 below would show that any such manifold would have to be a spin manifold and it would then follow, again using Smale's classification [9], that $X_\lambda \simeq S^2 \times S^3$, violating Lemma 2.5.[3]

3. Geometric description of the complex X_μ

In [12], Wall gives a classification of closed, 1-connected, torsion-free, 6-dimensional spin manifolds. If $M = M^6$ is a closed manifold of the homotopy type of the complex $X = X_\mu$ of Theorem 2.9, then M is certainly 1-connected and torsion-free. As we shall now show, M must also be a spin manifold, so that we may apply Wall's results in studying M.

LEMMA 3.1. *The second Stiefel-Whitney class $w_2(X)$ vanishes. Hence, any manifold M having the homotopy type of X is a spin manifold.*

Proof. We recall that the i^{th} Wu class of X, $v_i = v_i(X)$, is defined by the formula

$$v_i \cup u = Sq^i(u), \quad u \in H^{6-i}(X; \mathbf{Z}_2).$$

Since $Sq^2 : H^4(X; \mathbf{Z}_2) \to H^6(X; \mathbf{Z}_2)$ is obviously the zero map, it follows that $v_2 = 0$. The lemma now follows from the fact that $w_2(X) = v_2(X)$.

In order to state the main result of this section, we must first recall a few facts; see [12]. Let C_3^3 be the group of isotopy classes of Haefliger knots, i.e. embeddings of S^3 in S^6, and let FC_3^3 be the group of isotopy classes of framed Haefliger knots, i.e. embeddings of $S^3 \times D^3$ in S^6. There is an obvious exact sequence

$$0 \to \pi_3(SO_3) \to FC_3^3 \to C_3^3 \to 0;$$

since, by Haefliger's work, $C_3^3 = \mathbf{Z}$, the sequence splits. Moreover, there is a preferred splitting, induced by a geometrically defined map $FC_3^3 \to \pi_3(SO_3) = \mathbf{Z}$. Hence, by means of this splitting $FC_3^3 = \pi_3(SO_3) \oplus C_3^3$, a framed Haefliger knot is characterized by a pair (m, n) of integers. We now have

THEOREM 3.2. *If M is a closed manifold having the homotopy type of X, then M can be obtained from S^6 by performing surgery on a framed Haefliger knot; the*

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[2] The proof follows the lines of Sasao's proof of Theorem 2.1, utilizing the Wang sequence to show that the "fibre" F of the evident map $p : X_\lambda \to S^3$ is homotopy equivalent to S^2. The fact that $\alpha = 0$ is used essentially. Indeed, this argument provides an elementary proof of Theorem 2.1, with $2 \leq q \leq n - 1$, without appeal to Sasao's more general theorem in the case $\alpha = 0$.

[3] This example is due originally to Gitler-Stasheff (*The first exotic class of BF*, Topology, vol. 4 (1965), pp. 257–266). The proof given by Gitler-Stasheff that X_λ does not admit a manifold structure is quite different from ours, and consists of showing that the first exotic mod 2 class of the Spivak fibration of X_λ is non-zero.

pair (m, n) characterizing the knot satisfies: (1) n is odd, (2) $m + 6n = 0$. Conversely, any manifold N obtained by surgering out such a knot is homotopy equivalent to X.

Remark. If we replace the condition "n is odd" by "n is even", we obtain all closed manifolds realizing $S^2 \times S^4$.

Proof of Theorem 3.2. The fact that M comes from surgering out a framed Haefliger knot follows from Theorem 2 of [12]. The equation $m + 6n = 0$ follows from Theorem 4 of [12]. (We use here the fact that the map

$$H^2(X) \otimes H^2(X) \otimes H^2(X) \to \mathbf{Z}$$

given by cup product is trivial.) The condition that n is odd will be a consequence of the fact that $t(M)$, the stable tangent bundle of M, is not fibre homotopy trivial. (See the first sentence following Theorem 2.9). This requires a closer examination of $t(M)$. By looking at the $\widetilde{KO}$-exact sequence associated to the cofibration

$$S^2 \vee S^4 \to M \to S^6,$$

we see that $\widetilde{KO}(M)$ is isomorphic to $\widetilde{KO}(S^2) \oplus \widetilde{KO}(S^4)$. But the $\widetilde{KO}(S^2)$-component of $t(M) \in \widetilde{KO}(M)$ is 0 by Lemma 3.1 so that we may identify $t(M)$ with some multiple, $l\beta$, of the (Bott) generator $\beta \in \widetilde{KO}(S^4)$. Now, by Theorem 4 of [12], we have

$$p_1(M) = 4m,$$

$p_1(M) \in H^4(M) = \mathbf{Z}$ being the first (integral) Pontryagin class. Hence, since $p_1(l\beta) = 2l$, we conclude that

$$l = 2m = -12n.$$

If now n is even, then l would be divisible by 24 and this would imply that $t(M) = l\beta$ is fibre homotopy trivial ($\pi_3^S = \mathbf{Z}_{24}$).

The converse follows from Theorem 7 of [12].

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