CLOSED l-IDEALS IN A CLASS OF LATTICE-ORDERED ALGEBRAS

BY

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Introduction

The Φ-algebras of Henriksen and Johnson come equipped with a natural topology called the uniform topology. In this topology, the closure of an l-ideal need not be an l-ideal; in particular, maximal l-ideals which are not closed may exist. Nevertheless, closed l-ideals are of some interest. We show here that, if I is an l-ideal in a uniformly closed Φ-algebra A, then A/I is a (necessarily uniformly closed) Φ-algebra iff I is closed. As a special case, a maximal l-ideal in a uniformly closed Φ-algebra is real iff it is closed.

A Φ-algebra A is called normal if every proper closed l-ideal in A is contained in a closed maximal l-ideal in A. Using normality, we are able to modify a result of Henriksen and Johnson which gives an l-algebra characterization of $C(X)$, for X Lindelöf. Our modification includes an earlier result of Brainerd.

1. Preliminaries

We begin by summarizing some of the definitions and results of [2] and [12]. A familiarity with the terminology and basic results of the Gillman and Jerison text [11] will be assumed. We adopt the convention that X *will always denote a completely regular Hausdorff space.*

An *l-algebra* is an algebra A over an ordered field K which, under a partial ordering $\geq$, is a lattice which satisfies

(i) $a \geq b$ implies $a + c \geq b + c$,
(ii) $a \geq 0$ and $b \geq 0$ implies $ab \geq 0$, and
(iii) $\alpha \geq 0$ and $a \geq 0$ implies $\alpha a \geq 0$,

for $a, b, c \in A$ and $\alpha \in K$. An l-algebra A is called an *f-algebra* if it satisfies

(iv) $a \wedge b = 0$ and $c \geq 0$ implies $ca \wedge b = ac \wedge b = 0$.

An l-algebra A is said to be *archimedean* if, for $a, b \in A$, $a = 0$ whenever $na \leq b$ for all integers n. A real archimedean f-algebra with an identity is known as a *Φ-algebra*; a Φ-algebra is necessarily commutative [2, Theorem 13]. Simple examples of Φ-algebras are the *trivial* Φ-algebra $\{0\}$ and the Φ-algebra $\mathbf{R}$ of real numbers, each with the obvious operations. Another example is the Φ-algebra $C(X)$ of all continuous real-valued functions on X under the pointwise operations.

An algebra and lattice homomorphism $\psi : A \to B$ between the real l-algebras A and B is called an *l-homomorphism*; similarly we define *l-monomorphism*, *l-epimorphism* and *l-isomorphism*.

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Let A be a nontrivial real f-algebra with identity 1. The mapping $r \to r.1$ is an l-monomorphism of the Φ-algebra $\mathbf{R}$ into A; thus we shall consider $\mathbf{R}$ as a sub-Φ-algebra of A by identifying r with $r.1$. For $a, b \in A$, define

$$\rho(a, b) = \inf \{r \in \mathbf{R} : | a - b | \wedge 1 \leq r\}$$

where $| c | = c \vee (-c)$ for each $c \in A$. Then ρ is easily seen to be a pseudometric on A. Define ρ in the obvious way on the trivial Φ-algebra.

If A is a real f-algebra with identity, then ρ is called the *uniform pseudometric* on A and the topology induced by ρ is called the *uniform topology.* If A is complete in the pseudometric ρ, then we say that A is *uniformly closed.* The Φ-algebras $\{0\}$, $\mathbf{R}$ and $C(X)$ are uniformly closed.

If A and B are real f-algebras with identity and $\psi : A \to B$ is an l-homomorphism with $\psi(1) = 1$, then ψ is uniformly continuous with respect to the uniform pseudometrics on A and B.

If A is a Φ-algebra, then ρ is a metric. Even in the case of $A = C(X)$, the uniform topology need not make A into a topological algebra.

In the following, *all topological properties of a real f-algebra with identity will refer to the uniform topology.*

By an *ideal* in an l-algebra, we shall mean an algebra ideal. An ideal I in an l-algebra A is said to be an *l-ideal* if $a \in I$ whenever $a \in A$, $b \in I$ and $| a | \leq | b |$.

Let I be an l-ideal in the l-algebra A, and let $I : A \to A/I$ denote also the natural algebra epimorphism. By defining $I(a) \geq 0$ iff $a \wedge 0 \in I$, A/I becomes an l-algebra and I an l-epimorphism. If A is a real f-algebra with identity, then so is A/I; furthermore, $I(1) = 1$, so that the l-epimorphism I is continuous.

Let A be a Φ-algebra. The collection of all maximal l-ideals in A is denoted by $\mathfrak{M}(A)$. An l-ideal I in A is called a *real l-ideal* if A/I is l-isomorphic to $\mathbf{R}$. The collection of all real l-ideals in A is denoted by $\mathcal{R}(A)$; clearly $\mathcal{R}(A) \subset \mathfrak{M}(A)$. For $a \in A$, let $\mathfrak{M}(a) = \{M \in \mathfrak{M}(A) : a \in M\}$. If we take the collection of all sets $\mathfrak{M}(a)$ as a base for the closed sets, then we obtain a compact Hausdorff topology on $\mathfrak{M}(A)$. In the sequel, *we shall consider $\mathfrak{M}(A)$ as a topological space with this topology.*

Let $\gamma\mathbf{R}$ denote the two-point compactification of $\mathbf{R}$. For a compact space X, let $D(X)$ denote the set of all continuous $f : X \to \gamma\mathbf{R}$ for which

$$\mathcal{R}(f) = \{x \in X : f(x) \in \mathbf{R}\}$$

is dense. If, for $f, g \in D(X)$, there is some (necessarily unique) $h \in D(X)$ such that $h(x) = f(x) + g(x)$ for all $x \in \mathcal{R}(f) \cap \mathcal{R}(g)$, then we write $h = f + g$. Similarly, we define $fg, f \vee g, f \wedge g$ and rf for $r \in \mathbf{R}$. While $f + g$ and fg need not exist in general, $f \vee g, f \wedge g$ and rf always exist in $D(X)$. A subset of $D(X)$ which becomes a Φ-algebra under these operations will be called a sub-Φ-algebra of $D(X)$.

Let A be a Φ-algebra. It is shown in [12, Theorem 2.3] that A is l-isomorphic to a sub-Φ-algebra A' of $D(\mathfrak{M}(A))$ via an isomorphism which carries each element of the copy of $\mathbf{R}$ contained in A onto the corresponding constant function on $\mathfrak{M}(A)$. Furthermore, $\mathfrak{R}(A) = \bigcap \{\mathfrak{R}(f) : f \in A'\}$ and, if F and K are disjoint nonempty closed subsets of $\mathfrak{M}(A)$, then there exists $f \in A'$ such that $f[F] = \{0\}$ and $f[K] = \{1\}$. We shall identify A and A'; hence each $a \in A$ will be considered as a continuous function on $\mathfrak{M}(A)$ into $\gamma\mathbf{R}$. For $M \in \mathfrak{M}(A)$, $a \in M$ iff $(ab)(M) = 0$ for all $b \in A$ [12, Theorem 2.5]; thus, for $M \in \mathfrak{R}(A)$, $a \in M$ iff $a(M) = 0$. The Φ-algebra A is uniformly closed iff $C(\mathfrak{M}(A)) \subset A$ [12, 3.2].

We shall let $\mathbf{N}$ denote the set $\{1, 2, 3, \cdots\}$ of positive integers considered both as a countable discrete space and as an index set.

2. Closed l-ideals

Let A denote a nontrivial Φ-algebra in this section. If I is an l-ideal in A, then A/I is a commutative f-algebra with identity, but may fail to be archimedean.

Let I be an l-ideal in A. Following [7], we call an element $I(a) \in A/I$ an *infinitesimal* if $|I(na)| \leq 1$ for all $n \in \mathbf{N}$. If A/I is archimedean, then clearly 0 is its only infinitesimal; the converse fails, as is shown in 2.4.

2.1. Lemma. *Let $a \in A$, and let I be an l-ideal in A. Then a is in the closure of I iff $I(a)$ is an infinitesimal in A/I.*

Proof. We may assume that I is proper.

Suppose that a is in the closure of I. Then, for each $n \in \mathbf{N}$, there exists $b_n \in I$ such that $|a - b_n| \leq 1/n$. Hence

$$|I(a)| \leq |I(a - b_n)| + |I(b_n)| = I(|a - b_n|) \leq 1/n,$$

for each $n \in \mathbf{N}$.

Now suppose that $I(a)$ is an infinitesimal in A/I. For $n \in \mathbf{N}$, define

$$b_n = (-1/n) \vee (a \wedge 1/n),$$

whence $|b_n| \leq 1/n$ and $I(b_n) = I(a)$. If we define $c_n = a - b_n$, then each $c_n \in I$, and $c_n \to a$ uniformly.

This gives a rather nice characterization of closed l-ideals.

2.2. Theorem. *The following conditions are equivalent for an l-ideal I in A.*

(a) *I is closed.*

(b) *A/I contains no infinitesimals except* 0.

(c) *A/I is Hausdorff.*

Proof. The equivalence of (a) and (b) is immediate from 2.1. Condition (b) is easily seen to be equivalent to the T_1-property for A/I; since A/I is a pseudometric space, this in turn is equivalent to (c).

It is clear by 2.2 that any $M \in \mathcal{R}(A)$ is closed. More specifically, we can prove the following.

2.3. THEOREM. *Let* $M \in \mathfrak{M}(A)$. *Then* $M \in \mathcal{R}(A)$ *iff* A/M *is archimedean.*

Proof. For $M \in \mathfrak{M}(A)$, A/M is a totally ordered integral domain [12, 1.6 (ii)]. If A/M is archimedean, then it is l-isomorphic to a subring of $\mathbf{R}$ [8, Chap. VIII, Theorem 1], so by [11, Theorem 0.22], $M \in \mathcal{R}(A)$.

If $M \in \mathfrak{M}(A)$ and M is closed, then it need not follow that $M \in \mathcal{R}(A)$, as the following example of Henriksen and Johnson shows.

2.4. *Example* [12, Example 3.6]. Let $\mathbf{R}^+ = \{y \in \mathbf{R} : y \geq 0\}$, and let A denote the Φ-algebra of all $f \in C(\mathbf{R}^+)$ for which there exists a point $x \in \mathbf{R}^+$ and a polynomial p such that $f(y) = p(y)$ for all $y \geq x$. Now let M denote the set of all $f \in A$ such that, for some $x \in \mathbf{R}^+$, $f(y) = 0$ for all $y \geq x$. Then $M \in \mathfrak{M}(A)$, and M is closed. But $M \notin \mathcal{R}(A)$, since M is not a maximal ring ideal.

This Φ-algebra, however, is not uniformly closed. There is no uniformly closed example. This will be a simple corollary of the next result, which seems to have some independent interest.

2.5. THEOREM. *Let A be uniformly closed, and let I be an l-ideal in A. Then A/I is a (necessarily uniformly closed) Φ-algebra iff I is closed.*

Proof. If A/I is a Φ-algebra, then I is closed, by 2.2.

Suppose that I is closed. Clearly A/I is a commutative real f-algebra with identity. We must show that A/I is (a) uniformly closed and (b) archimedean. We may assume that I is proper.

(a) Suppose that $(I(a_n))$ is a Cauchy sequence in A/I. By going to a subsequence, if necessary, we may assume that

$$| I(a_{n+1}) - I(a_n) | \leq 2^{-n} \quad \text{for all } n \in \mathbf{N}.$$

For each $n \in \mathbf{N}$, define $b_n = -2^{-n} \vee ((a_{n+1} - a_n) \wedge 2^{-n})$, whence

$$| b_n | \leq 2^{-n} \quad \text{and} \quad I(b_n) = I(a_{n+1} - a_n).$$

If we define $c_1 = a_1$ and $c_n = a_1 + b_1 + b_2 + \cdots + b_{n-1}$ for $n \geq 2$, then (c_n) is clearly a Cauchy sequence in A. So there is some $a \in A$ such that $c_n \to a$ uniformly. Then $I(c_n) \to I(a)$ uniformly in A/I, and $I(c_n) = I(a_n)$ for all $n \in \mathbf{N}$.

(b) Suppose $nI(a) \leq I(b)$ for all integers n. Since A is uniformly closed, $1 + | b |$ has an inverse $c \in A$ [12, 3.3]. But then $nI(a)I(c) \leq I(bc) \leq 1$ for all integers n, whence $| I(a)I(c) | \leq 1/n$ for all $n \in \mathbf{N}$. Since I is closed, we must have $I(a)I(c) = 0$ by 2.2. Clearly then $I(a) = 0$.

2.6. COROLLARY. *If A is uniformly closed and $M \in \mathfrak{M}(A)$, then M is closed iff $M \in \mathcal{R}(A)$.*

Proof. By 2.5, M is closed iff A/M is archimedean. Now apply 2.3.

2.7. *Example.* Let $\mathcal{L}$ denote the uniformly closed Φ-algebra of Lebesgue measurable functions on $\mathbf{R}$. If $I = \{f \in \mathcal{L} : \int_{-\infty}^{\infty} |f| \, d\mu = 0\}$, then I is a closed l-ideal in $\mathcal{L}$, and $\mathcal{L}_0 = \mathcal{L}/I$ is a Φ-algebra, the algebra of *Lebesgue measurable functions modulo null functions.* Similarly, if we replace $\mathcal{L}$ by its subalgebra $\mathcal{B}$ of Baire functions on $\mathbf{R}$, then $\mathcal{B}_0 = \mathcal{B}/I$ is a Φ-algebra, the Φ-algebra of *Baire functions modulo null functions.* (See [12] for a discussion of $\mathcal{L}_0$ and $\mathcal{B}_0$.)

3. Normal Φ-algebras

Let A denote a nontrivial Φ-algebra in this section. We present now a variation of a theorem of Henriksen and Johnson which characterizes $C(X)$, for X Lindelöf. The result hinges on the following observation.

3.1. THEOREM. *If every proper closed l-ideal in A is contained in some real l-ideal, then $\mathcal{R}(A)$ is a dense Lindelöf subspace of $\mathfrak{M}(A)$.*

Proof. Suppose that $\mathcal{R}(A)$ is not dense in $\mathfrak{M}(A)$. Let U be a nonempty open subset of $\mathfrak{M}(A)$ whose closure is disjoint from the closure of $\mathcal{R}(A)$. Define

$$I = \{a \in A : a[U] = \{0\}\}.$$

Since each $b \in A$ is real-valued on a dense subset of $\mathfrak{M}(A)$, I is an l-ideal; clearly I is closed. By [12, 2.3 (ii)], there is some $a \in A$ such that $a[U] = \{0\}$ and $a[\mathcal{R}(A)] = \{1\}$. So $a \in I$ while $a \notin M$ for all $M \in \mathcal{R}(A)$.

Suppose that every proper closed l-ideal in A is contained in some real l-ideal, and let $\mathcal{U}$ be an open cover of $\mathcal{R}(A)$. Let I denote the set of all $a \in A$ for which there is a countable subfamily $\mathcal{U}_a$ of $\mathcal{U}$ such that $\bigcup \mathcal{U}_a$ contains all $M \in \mathcal{R}(A)$ with $a \notin M$.

I is an l-ideal. Clearly $0 \in I$, so that I is not empty. If $a, b \in I$, then $a - b \in I$ (take $\mathcal{U}_{a-b} = \mathcal{U}_a \cup \mathcal{U}_b$). If $a \in I, b \in A$, then $ab \in I$ (take $\mathcal{U}_{ab} = \mathcal{U}_a$). If $a \in I, b \in A, |b| \leq |a|$, then $b \in I$ (take $\mathcal{U}_b = \mathcal{U}_a$).

I is closed in A. Suppose $a_n \in I$ for $n \in \mathbf{N}$ and $a_n \to a \in A$ uniformly. Define $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{U}_{a_n}$, a countable subfamily of $\mathcal{U}$. To see that $a \in I$, it suffices to take $\mathcal{U}_a = \mathcal{V}$. To show this, suppose $a \notin M \in \mathcal{R}(A)$. Then $M(|a|) = \varepsilon$ for some $\varepsilon \in \mathbf{R}$, $\varepsilon > 0$. Let (ε_n) be a decreasing sequence of positive real numbers such that $|a - a_n| \leq \varepsilon_n$ for each $n \in \mathbf{N}$ and $\lim \varepsilon_n = 0$. Now

$$\varepsilon_n \geq M(|a - a_n|) \geq \varepsilon - M(|a_n|)$$

for all $n \in \mathbf{N}$, so that $M(|a_m|) > 0$ for some $m \in \mathbf{N}$. Hence $a_m \notin M$ and $M \in \bigcup \mathcal{U}_{a_m} \subset \bigcup \mathcal{V}$.

Suppose that I is a proper l-ideal. Then, by hypothesis, there is some $M_0 \in \mathcal{R}(A)$ such that $I \subset M_0$. Choose $U_0 \in \mathcal{U}$ such that $M_0 \in U_0$, and choose $a \in A$ such that $M_0 \in \mathcal{R}(A) \setminus \mathfrak{M}(a) \subset U_0$. Then letting $\mathcal{U}_a = \{U_0\}$, we have $a \in I \subset M_0$, and this contradicts our choice of $a \notin M_0$.

Thus, I is not proper, so we must have $1 \in I$. So there is a countable subfamily $\mathcal{U}_1$ of $\mathcal{U}$ such that $\bigcup \mathcal{U}_1$ contains all $M \in \mathcal{R}(A)$ with $1 \notin M$. That is, $\mathcal{R}(A) = \bigcup \mathcal{U}_1$.

For $a \in A$, let $\langle a \rangle$ denote the smallest l-ideal in A containing a. If A is uniformly closed, then $\langle a \rangle = A$ iff a has an inverse $a^{-1} \in A$ [12, 3.3 and 3.4]. We say that A is *closed under inversion* if $\langle a \rangle = A$ whenever $a \in A$, a is not a zero-divisor of A, and $a \notin M$ for all $M \in \mathcal{R}(A)$.

Following Michael [15, Definition B.1], we make the following definition.

3.2. DEFINITION. A Φ-algebra A will be called *normal* if every proper closed l-ideal in A is contained in a closed maximal l-ideal in A.

3.3. THEOREM. *Let A be uniformly closed.*

(a) *If A is normal, then $\mathcal{R}(A)$ is a dense Lindelöf subspace of $\mathfrak{M}(A)$.*

(b) *If A is closed under inversion and $\mathcal{R}(A)$ is a dense Lindelöf subspace of $\mathfrak{M}(A)$, then A is normal.*

Proof. (a) This follows immediately from 2.6 and 3.1.

(b) Let A be closed under inversion, and let $\mathcal{R}(A)$ be a dense Lindelöf subspace of $\mathfrak{M}(A)$. Suppose that I is a closed l-ideal which is not contained in any $M \in \mathcal{R}(A)$; we will show that I is not proper. Clearly then $\mathcal{R}(A)$ is a union of open sets of the form

$$U(a) = \{M \in \mathcal{R}(A) : a \notin M\}$$

where $a \in I$. Since $\mathcal{R}(A)$ is Lindelöf, there is a sequence (a_n) in I such that $\mathcal{R}(A) = \bigcup_{n=1}^{\infty} U(a_n)$. For each $n \in \mathbf{N}$, a_n and $a_n(1 + | a_n |)^{-1}$ are associates in A by [12, 3.3], so we may assume $| a_n | \leq 1$. Now let $a = \sum_{n=1}^{\infty} 2^{-n} | a_n |$; $a \in I$, since I is closed. But each $M \in \mathcal{R}(A)$ is in some $U(a_n)$, whence

$$| M(a) | \geq 2^{-n} | M(a_n) | > 0 \quad \text{and} \quad a \notin M.$$

Since $\mathcal{R}(A)$ is dense in $\mathfrak{M}(A)$, a is not a zero-divisor of A, and therefore $\langle a \rangle = A$; that is, $I = A$.

3.4. COROLLARY. *The following conditions are equivalent for a uniformly closed Φ-algebra A.*

(a) *A is normal.*

(b) *A/I is normal for each proper closed l-ideal I in A.*

(c) *Every proper closed l-ideal in A is an intersection of closed maximal (i.e., real) l-ideals.*

Proof. (a) implies (b). Let A be normal, and let I be a proper closed l-ideal in A. Any l-ideal in A/I is of the form J/I where J is an l-ideal in A containing I; furthermore, since $(A/I)/(J/I) \cong A/J$, J/I is closed iff J is closed, by 2.5. It follows that A/I is normal.

(b) implies (c). Let I be a proper closed l-ideal in A, and suppose that A/I is normal. By 3.3, $\mathcal{R}(A/I)$ is dense in $\mathfrak{M}(A/I)$ which means that $\bigcap \mathcal{R}(A/I) = (0)$. Since, for an l-ideal J/I in A/I, $J/I \in \mathcal{R}(A/I)$ iff $J \in \mathcal{R}(A)$, we have $I = \bigcap \{M \in \mathcal{R}(A) : I \subset M\}$.

(c) implies (a). Obvious.

The converse of 3.3 (a) is false if we do not assume that A is closed under inversion; the following example illustrates this.

3.5. *Example.* Let X be any (non-normal) realcompact space with disjoint nonempty closed subsets X_0 and X_1 which are not completely separated and such that X_0 is Lindelöf. One such example would be the Sorgenfrey plane $X = \mathbf{R} \times \mathbf{R}$ described in [14, 1 L and 4 I], where a basic neighborhood of the point $(a, b) \in X$ is of the form $[a, c) \times [b, d)$; define

$$X_0 = \{(x, 1 - x) : x \text{ rational}\} \quad \text{and} \quad X_1 = \{(x, 1 - x) : x \text{ irrational}\}.$$

Let $A = C(X) \mid X_0$, the uniformly closed Φ-algebra of restrictions $f \mid X_0$, where $f \in C(X)$. One easily checks that $\mathfrak{M}(A) = \text{cl}_{\beta X} X_0$ and $\mathfrak{R}(A) = \text{cl}_{\upsilon X} X_0 = X_0$, so that $\mathfrak{R}(A)$ is a dense Lindelöf subspace of $\mathfrak{M}(A)$. But A is not normal; for let

$$I = \{g \in A : g = f \mid X_0 \quad \text{for some} \quad f \in C(X) \quad \text{with} \quad X_1 \subset Z(f)\}.$$

Then I is a closed l-ideal in A; I is proper, since X_0 and X_1 are not completely separated. For each $x \in X_0$, there exists $f \in C(X)$ such that $X_1 \subset Z(f)$ and $f(x) = 1$, so that I is contained in no real l-ideal.

In [12], Henriksen and Johnson give an l-algebra characterization of $C(X)$, for a Lindelöf X. We can now prove a modification of this result which has a slightly more algebraic flavor.

3.6. THEOREM. *A nontrivial Φ-algebra A is l-isomorphic to $C(X)$, for some Lindelöf space X, iff*

(a) *A is uniformly closed,*
(b) *A is closed under inversion, and*
(c) *A is normal.*

Proof. In [12, Theorem 5.4], it is shown that A is l-isomorphic to $C(X)$, for some Lindelöf X, iff (a) and (b) hold together with the condition that $\mathfrak{R}(A)$ be a dense Lindelöf subspace of $\mathfrak{M}(A)$. The theorem now follows from 3.3.

An equivalent version of 3.6 is obtained if we replace "l-ideal" by "ideal" when defining $\mathfrak{M}(A)$, $\mathfrak{R}(A)$ and "normal". This is clear by the following result.

3.7. THEOREM. *If I is either a closed or a maximal ideal in the uniformly closed Φ-algebra A, then I is an l-ideal.*

Proof. Let A be uniformly closed, let I be either closed or maximal, and suppose that $a \in A$, $b \in I$ satisfy $|a| \leq |b|$.

Since $C(\mathfrak{M}(A)) \subset A$, $I^* = I \cap C(\mathfrak{M}(A))$ is an ideal in $C(\mathfrak{M}(A))$. If I is maximal, then I^* is prime, hence is an l-ideal by [11, Theorem 5.5]. If I is closed, then I^* is closed, and by [11, 4O.4] is therefore an l-ideal.

Let $c = (1 + |b|)^{-1} \in C(\mathfrak{M}(A))$; then $|ac| \leq |bc|$ and $bc \in I^*$. Thus, $ac \in I^*$, and therefore $a \in I$.

4. F-rings

An l-algebra A is said to be *σ-complete* if every countable subset of A which is bounded above in A has a supremum in A. A topological space is *basically disconnected* if the closure of every cozero-set is open. Nakano [16] and Stone [17] have proved that $C(X)$ is σ-complete iff X is basically disconnected.

A ring A is said to be *regular* if, for each $a \in A$, there exists $b \in A$ such that $aba = a$. We call a topological space a *P-space* if every cozero-set is closed. A theorem of Gillman and Henriksen says that $C(X)$ is regular iff X is a P-space [10, Theorem 5.3].

In [4], Brainerd defines an *F-ring* to be a σ-complete commutative l-algebra with a positive identity which is a weak order unit. By [2, Theorem 19], an F-ring is precisely a σ-complete Φ-algebra, and as such, is uniformly closed [6, Theorem 2.1].

In [3], [4] and [5], special attention is paid to regular F-rings. Since every P-space is basically disconnected, $C(X)$ is σ-complete whenever it is regular. More generally, any regular, uniformly closed Φ-algebra is an F-ring [6, Theorem 2.1]. In a regular F-ring, every ideal is an l-ideal [13, Corollary 1.10].

An ideal I in an F-ring A will be called *σ-closed* if, whenever a countable subset of I has a supremum in A, this supremum is in I.

4.1. Theorem. *Let A be an F-ring.*

(a) *Every σ-closed ideal in A is closed.*

(b) *If A is regular, then every closed ideal in A is σ-closed.*

Proof. (a) Suppose that I is a σ-closed ideal in A, and (a_n) is a sequence in I which converges uniformly to $a \in A$. Then $b_m = \bigvee_{n=m}^{\infty} a_n$ exists in I, for $m \in \mathbf{N}$, and likewise we can define $c = \bigwedge_{m=1}^{\infty} b_m \in I$. We will show that $a = c$. Let $\varepsilon \in \mathbf{R}$ with $\varepsilon > 0$; then for some $n \in \mathbf{N}$, we have $a - \varepsilon \leq \bigwedge_{m=n}^{\infty} b_m \leq a + \varepsilon$. But (b_m) is a decreasing sequence, so that $c = \bigwedge_{m=n}^{\infty} b_m$, and $| c - a | \leq \varepsilon$. Since ε is arbitrary, $a = c \in I$.

(b) Let A be a regular F-ring, and let I be a closed ideal in A; we may assume that I is proper. Suppose that (a_n) is a sequence in I with $a = \bigvee_{n=1}^{\infty} a_n \in A$. Define $b = \sum_{n=1}^{\infty} 2^{-n}(| a_n | \wedge 1) \in I$. Then

$$\emptyset \neq \mathfrak{M}(b) \subset \bigcap_{n=1}^{\infty} \mathfrak{M}(a_n),$$

and $\mathfrak{M}(b)$ is open, since A is regular [9, Theorem 3.13]. If f is the characteristic function of $\mathfrak{M}(A) \setminus \mathfrak{M}(b)$, then $f \in C(\mathfrak{M}(A)) \subset A$. For each $c \in A$, $\mathfrak{R}(c)$ is dense, so that $(\jmath c)(M) = 0$ for all $M \in \mathfrak{M}(b)$; that is, $f \in M$ for all $M \in \mathfrak{M}(b)$ [12, Theorem 2.5]. Since in a regular commutative ring with identity any proper ideal is an intersection of maximal ideals (see [1, p. 459]), $f \in I$. But $a_n \leq af \leq a$ for all $n \in \mathbf{N}$, and therefore $a = af \in I$.

Combining 2.6 and 4.1, we see that a maximal ideal in a regular F-ring is real iff it is σ-closed; this has been observed by Brainerd [5 p. 83]. Regularity is needed here (hence in 4.1 (b)), for let $A = C(\beta\mathbf{N})$. Then A is an F-ring which is not regular. If, for $n \in \mathbf{N}$, we let a_n be the characteristic function of $\{n\}$, then for any $p \in \beta\mathbf{N}\backslash\mathbf{N}$, each a_n is in the real maximal ideal $M_p = \{f \in A : f(p) = 0\}$. But $\bigvee_{n=1}^{\infty} a_n = 1 \notin M_p$.

In [5, Theorem 6], Brainerd gives an l-algebra characterization of $C(X)$, for X a Lindelöf P-space, using the equivalent of our normality with "closed" replaced by "σ-closed". Since, in a regular F-ring, closed ideals and σ-closed ideals are the same (4.1), the following corollary to 3.6 is merely a restatement of Brainerd's result.

4.2. THEOREM (Brainerd). *A nontrivial Φ-algebra A is l-isomorphic to $C(X)$, for some Lindelöf P-space X, iff*

(a) *A is σ-complete (i.e., A is an F-ring),*

(b) *A is regular, and*

(c) *A is normal.*

Proof. Clearly (a) (b) and (c) are satisfied for $C(X)$, where X is a Lindelöf P-space.

Suppose that A is a Φ-algebra satisfying conditions (a), (b) and (c). By (a), A is uniformly closed. Since, in a regular commutative ring with identity, every non-zero-divisor is a unit, A is closed under inversion, by (b).

Thus, by 3.6, A is l-isomorphic to $C(X)$ for some Lindelöf space X. Since A is regular, X is a P-space.

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