CLOSED ONE-SIDED IDEALS IN CERTAIN B^*-ALGEBRAS

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1. Introduction

Throughout this paper we work in a B^*-algebra B with a special property we call Property A (Definition 2.4). Essentially this property assures that B has enough projections for our purposes. AW^*-algebras have Property A. We relate the closed left ideals of B to subsets of a certain ordered set of sequences of projections in B (Theorem 3.8). Then this relationship between closed left ideals of B and sets of projections in B is used to characterize the maximal left ideals of B. When B is commutative, a proper closed ideal M of B is maximal if and only if whenever E is a projection on B such that $E \notin M$, then $(I - E) \in M$. This can be verified for AW^*-algebras using the results of (7). We generalize this result to the case where B is non-commutative (and say an AW^*-algebra) as follows. When E and F are projections in B such that $E \cap F = 0$ and $E + F$ is invertible in B then we call F a strong complement of E. Then a proper closed left ideal M of B is maximal if and only if whenever $E \notin M$, then E has a strong complement in M (Theorem 4.5).

In the last two sections of the paper we apply the results relating closed left ideals and sets of projections in B. First we give a new proof (and a slight generalization) of the known theorem that E is a central projection of B if and only if E has a unique complement in B (Theorem 5.1). Then in the last section we characterize the null space of a pure state of B and use this result to give a necessary and sufficient condition that a pure state of a closed *-subalgebra of B with property A have a unique extension to a pure state of B.

2. Preliminaries

Throughout this paper we assume that B is a B^*-algebra with an identity I. $E \in B$ is a projection if $E = E^2 = E^*$. If $\{E_n\}$ is a sequence of projections in B with the property that $\lim_{n\to\infty} (I - E_m)E_n = 0$ for every fixed m, then $\{E_n\}$ is called an admissible sequence. In particular any decreasing sequence of projections is admissible. We denote the set of all admissible sequences of projections in B as $\mathcal{S}$. If $\{E_n\}$ and $\{G_n\}$ are in $\mathcal{S}$, we define $\{E_n\} \leq \{G_n\}$ if $\lim_{n\to\infty} (I - G_m)E_n = 0$ for every m.

PROPOSITION 2.1. *$\leq$ is reflexive and transitive on $\mathcal{S}$.*

Proof. Reflexivity is immediate since every sequence in $\mathcal{S}$ is admissible. Now assume that $\{G_n\}, \{F_n\}, \{E_n\} \in \mathcal{S}$, and $\{G_n\} \leq \{F_n\}$ and $\{F_n\} \leq \{E_n\}$. Fix

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m and assume that $\varepsilon > 0$. Choose k so large that $\| (I - E_m)F_k \| < \varepsilon/3$. Then choose N so large that $n \geq N$ implies $\| (I - F_k)G_n \| < \varepsilon/3$.

$$(I - E_m)G_n = (G_n - F_kG_n) + (F_kG_n - E_mF_kG_n) + (E_mF_kG_n - E_mG_n)$$
$$= (I - F_k)G_n + (I - E_m)F_kG_n + E_m(F_k - I)G_n .$$

Therefore when $n \geq N$,

$$\| (I - E_m)G_n \| \leq \| (I - F_k)G_n \| + \| (I - E_m)F_k \| + \| (I - F_k)G_n \| < \varepsilon.$$

This proves that $\lim_{n\to\infty} (I - E_m)G_n = 0$. Therefore $\{G_n\} \leq \{E_n\}$.

If $\{E_n\}$ and $\{F_n\}$ are in $\mathcal{S}$, and $\{E_n\} \leq \{F_n\}$ and $\{F_n\} \leq \{E_n\}$, we call $\{E_n\}$ and $\{F_n\}$ equivalent and we write $\{E_n\} \sim \{F_n\}$. It follows from Proposition 2.1 that $\sim$ is an equivalence relation on $\mathcal{S}$. Let $\mathcal{K}$ denote the set of equivalence classes of $\mathcal{S}$ determined by $\sim$. When $\{E_n\} \in \mathcal{S}$, we denote the equivalence class in $\mathcal{K}$ containing $\{E_n\}$ by $[E_n]$. We extend the ordering from $\mathcal{S}$ to $\mathcal{K}$ in the usual way: If $a, b \in \mathcal{K}$, then $a \leq b$ if there exists $\{E_n\} \in a$ and $\{F_n\} \in b$ such that $\{E_n\} \leq \{F_n\}$.

If E is a projection in B we identify the sequence $\{E, E, E, \cdots\}$ in $\mathcal{S}$ with E. Furthermore we again identify E with the equivalence class containing $\{E, E, E, \cdots\}$. It is not difficult to verify that $\{E_n\} \sim \{E, E, E, \cdots\}$ if and only if there exists an integer N such that $E_n = E$ for all $n \geq N$. Also $[\{E, E, E, \cdots\}] \leq [\{F, F, F, \cdots\}]$ in $\mathcal{K}$ if and only if $E < F$ in the usual ordering of projections in B ($E < F$ means $EF = E$). Thus from now on we consider the lattice of projections of B as embedded in $\mathcal{S}$ and $\mathcal{K}$, and we write without confusion, $E \in \mathcal{S}$ or $E \in \mathcal{K}$.

DEFINITION 2.2. Given $T \in B$, we call $\{E_n\} \in \mathcal{S}$ an annihilating sequence of T if

(1) $E_n \neq 0$ all n,

(2) $\lim_{n\to\infty} TF_n = 0$,

(3) for every m, there exists $T_m \in B$ such that $T_mT = I - E_m$.

PROPOSITION 2.3. *Assume* $T \in B$ *and* $\{E_n\}, \{F_n\} \in \mathcal{S}$. Then:

(1) *If* $\{F_n\} \leq \{E_n\}$ *and* $\lim_{n\to\infty} TE_n = 0$, *then* $\lim_{n\to\infty} TF_n = 0$.

(2) *If* $\lim_{n\to\infty} TF_n = 0$ *and* $\{E_n\}$ *is an annihilating sequence of* T, *then* $\{F_n\} \leq \{E_n\}$.

(3) *If* $\{F_n\}$ *and* $\{E_n\}$ *are annihilating sequences of* T, *then* $\{F_n\} \sim \{E_n\}$.

Proof. Assume that $\{F_n\}$ and $\{E_n\}$ satisfy the hypotheses given in (1). Then $TF_n = TE_mF_n + T(I - E_m)F_n$ for all n, m. Given $\varepsilon > 0$, choose m_0 so large that $\| TE_{m_0} \| < \varepsilon/2$. Since $\{F_n\} \leq \{E_n\}$, there exists an integer N such that whenever $n \geq N$, then $\| T \| \, \|(I - E_{m_0})F_n \| < \varepsilon/2$. Therefore when $n \geq N$, then $\| TF_n \| < \varepsilon$. This proves that $\lim_{n\to\infty} TF_n = 0$.

Now assume that $\{E_n\}$ and $\{F_n\}$ are as given in (2). Let $T_m \in B$ be such that $T_mT = I - E_m$ for every m. Then

$$\lim_{n\to\infty} (I - E_m)F_n = \lim_{n\to\infty} (T_mTF_n) = 0$$

for each m. Therefore $\{F_n\} \leq \{E_n\}$. This proves (2). (3) follows immediately from (2) and Definition 2.2.

The theorems that we prove in this paper hold when B is an AW^*-algebra. However the results are true for more general algebras B. Therefore we introduce a property which is sufficient for our purposes. An additional hypothesis concerning B will be assumed in Section 5 and part of Section 4.

DEFINITION 2.4. B has property A if whenever T is a noninvertible positive element in B, then there is an annihilating sequence of T in $\mathcal{S}$.

B will have property A if every maximal commutative *-subalgebra of B is generated by projections. We shall not prove this. Particular examples are AW^*-algebras (see [3, p. 236]), and the $B_p{}^*$-algebras introduced by C. Rickart (see [5, pp. 534–536]; Lemma 2.9, p. 535 is especially relevant). For the remainder of this section we shall be concerned with the proof that when B has property A, then every two elements of $\mathcal{K}$ have a greatest lower bound. The formal statement of this result is given in Theorem 2.8. Now we prove several technical lemmas.

LEMMA 2.5. *Assume that* $\{E_n\}, \{F_n\} \in \mathcal{S}$ *and that for each* $m \geq 1$, *there exists* $\{G_n{}^{(m)}\} \in \mathcal{S}$ *such that* $G_n{}^{(m)} \neq 0$ *for all* n, m,

$$\lim_{n\to\infty} (I - E_m)G_n^m = 0 \quad \textit{for all } m,$$

and

$$\lim_{n\to\infty} (I - F_m)G_n^{(m)} = 0 \quad \textit{for all } m.$$

Then the operator,

$$T = \textstyle\sum_{k=1}^{+\infty} (\frac{1}{2})^k((I - E_k) + (I - F_k))$$

is not invertible.

Proof. Assume $\varepsilon > 0$. Take N so large that $\sum_{k=N+1}^{+\infty} (\frac{1}{2})^k < \varepsilon/6$. Choose m so large that $\| (I - E_k)E_m \| < \varepsilon/6$ and $\| (I - F_k)F_m \| < \varepsilon/6$ for all k such that $1 \leq k \leq N$.

$$\begin{aligned} TG_n^{(m)} = {} & \textstyle\sum_{k=1}^{N} (\frac{1}{2})^k((I - E_k)E_m + (I - F_k)F_m)G_n^{(m)} \\ & + \textstyle\sum_{k=N+1}^{+\infty} (\frac{1}{2})^k((I - E_k) + (I - F_k))G_n^{(m)} \\ & + \textstyle\sum_{k=1}^{N} (\frac{1}{2})^k((I - E_k)(I - E_m)G_n^{(m)} + (I - F_k)(I - F_m)G_n^{(m)}). \end{aligned}$$

Therefore,

$$\begin{aligned} \| TG_n^{(m)} \| \leq {} & \textstyle\sum_{k=1}^{N} (\frac{1}{2})^k(\varepsilon/3) + \sum_{k=N+1}^{+\infty} (\frac{1}{2})^k(2) \\ & + \textstyle\sum_{k=1}^{N} (\frac{1}{2})^k(\| (I - E_m)G_n^{(m)} \| + \| (I - F_m)G_n^{(m)} \|). \end{aligned}$$

We can choose n so large that this last term is less than $\varepsilon/3$. Then $\| TG_n^{(m)} \| < \varepsilon$. This proves that T can not be invertible.

LEMMA 2.6. (1) *If T and S are positive elements in B and*

$$\lim_{n\to\infty} (T + S)G_n = 0$$

where $\{G_n\} \in \mathcal{S}$, then $\lim_{n\to\infty} TG_n = 0$ and $\lim_{n\to\infty} SG_n = 0$.

(2) *Assume that $\{T_n\}$ is a bounded sequence of positive elements in B, and let $T = \sum_{n=1}^{+\infty} (\frac{1}{2})^n T_n$. If $\lim_{n\to\infty} TG_n = 0$ where $\{G_n\} \in \mathcal{S}$, then $\lim_{n\to\infty} T_m G_n = 0$ for all m.*

Proof. First we note the following results concerning sums of positive elements of a B^*-algebra. Any finite sum of positive elements is positive by [6, Lemma (4.7.10), p. 234]. Also a limit of a sequence of positive elements is again positive by the remarks on p. 37 in [6]. We assume these results in the proof of (1) and (2).

Assume that (1) holds and T is defined as in (2). We can write T as the sum of two positive elements:

$$T = (\tfrac{1}{2})^m T_m + \sum_{n=1, n\neq m}^{+\infty} (\tfrac{1}{2})^n T_n .$$

Then if $\lim_{n\to\infty} TG_n = 0$, $\lim_{n\to\infty} T_m G_n = 0$ by (1).

Now we prove (1). Assume that T, S and $\{G_n\}$ satisfy the hypotheses of (1). Then $\| (T + S) G_n \| = \varepsilon_n$ and $\varepsilon_n \to 0$. By [6, Theorem (4.8.11), p. 244], we may assume that T, S and G_n, $n \geq 1$, are operators on a Hilbert space $\mathcal{H}$, and that $\| \cdot \|$ is the operator norm. For any h in the unit ball of $\mathcal{H}$,

$$((T + S)G_n h, G_n h) \leq \varepsilon_n .$$

Then $(TG_n h, G_n h) + (SG_n h, G_n h) \leq \varepsilon_n$, and therefore

$$(TG_n h, G_n h) \leq \varepsilon_n \quad \text{and} \quad (SG_n h, G_n h) \leq \varepsilon_n .$$

It follows that $\| G_n TG_n \| \to 0$ and $\| G_n SG_n \| \to 0$. $\| G_n TG_n \| = \| T^{1/2}G_n \|^2$, so that

$$\| TG_n \| \leq \| T^{1/2} \| \, \| T^{1/2}G_n \| \to 0.$$

Similarly $\| SG_n \| \to 0$.

LEMMA 2.7. *Assume that B has property* A. *Suppose that $\{E_n\}$ and $\{F_n\} \in \mathcal{S}$ have the property that $(I - E_n) + (I - F_n)$ is not invertible for all $n \geq 1$. Then there exists $\{J_n\} \in \mathcal{S}$ with the following properties:*

(1) *$\{J_n\}$ is not equivalent to* 0.

(2) *$\{J_n\} \leq \{E_n\}$ and $\{J_n\} \leq \{F_n\}$.*

(3) *If $\{G_n\} \in \mathcal{S}$ and $\{G_n\} \leq \{E_n\}$ and $\{G_n\} \leq \{F_n\}$, then $\{G_n\} \leq \{J_n\}$.*

Proof. Let

$$T = \sum_{k=1}^{+\infty} (\tfrac{1}{2})^k((I - E_k) + (I - F_k)).$$

Since $(I - E_k) + (I - F_k)$ is not invertible for any k, there exist for each k, an annihilating sequence for $(I - E_k) + (I - F_k)$, $\{G_n^{(k)}\}$. Then

$$\lim_{n\to\infty} (I - E_k)G_n^{(k)} = 0 \quad \text{and} \quad \lim_{n\to\infty} (I - F_k)G_n^{(k)} = 0$$

by Lemma 2.6 (1). By Lemma 2.5, T is not invertible. Let $\{J_n\} \in \mathcal{S}$ be an annihilating sequence of T in B. By Lemma 2.6, $\lim_{n\to\infty} (I - E_m)J_n = 0$ and $\lim_{n\to\infty} (I - F_m)J_n = 0$ for every m. It follows that $\{J_n\} \leq \{E_n\}$ and $\{J_n\} \leq \{F_n\}$. Now assume that $\{G_n\} \in \mathcal{S}$, $\{G_n\} \leq \{E_n\}$, and $\{G_n\} \leq \{F_n\}$. Then for each m,

$$\lim_{n\to\infty} ((I - E_m) + (I - F_m))G_n = 0.$$

It is easy to verify that this implies $\lim_{n\to\infty} TG_n = 0$. Then by Proposition 2.3 (2), $\{G_n\} \leq \{J_n\}$. This completes the proof.

Now we are in a position to prove that any two elements in $\mathcal{K}$ have a greatest lower bound in $\mathcal{K}$.

THEOREM 2.8. *Assume that B has property* A. *If $a, b \in \mathcal{K}$, then a and b have a greatest lower bound in $\mathcal{K}$ which we denote $a \wedge b$. Furthermore $[E_n] \wedge [F_n] \neq 0$ if and only if $(I - E_n) + (I - F_n)$ is not invertible for all n.*

Proof. Given $[E_n]$ and $[F_n] \in \mathcal{K}$. If $(I - E_n) + (I - F_n)$ is not invertible for all n, then we can choose $\{J_n\} \in \mathcal{S}$ with the properties listed in Lemma 2.7. Then clearly $[J_n]$ is a greatest lower bound of $[E_n]$ and $[F_n]$. Now assume that there exists m such that $(I - E_m) + (I - F_m)$ is invertible. Assume $\{G_n\} \leq \{E_n\}$ and $\{G_n\} \leq \{F_n\}$. Then

$$\lim_{n\to\infty} [(I - E_m) + (I - F_m)]G_n = 0.$$

It follows that $G_n = 0$ for all but a finite number of n. Therefore $[G_n] = 0$. This proves that 0 is the greatest lower bound of $[E_n]$ and $[F_n]$.

3. The closed left or right ideals of B

Throughout this section we assume that B has property A.

DEFINITION 3.1. $\mathfrak{M}$ is a proper ideal of $\mathcal{K}$ if
(1) $a \in \mathfrak{M}$ implies $a \neq 0$,
(2) a and $b \in \mathfrak{M}$ implies $a \wedge b \in \mathfrak{M}$,
(3) $a \in \mathfrak{M}$, $b \in \mathcal{K}$, and $a \leq b$, implies $b \in \mathfrak{M}$.

Assume $\mathfrak{M}$ is a proper ideal of $\mathcal{K}$. We define $L(\mathfrak{M})$ to be the set of all $T \in B$ with the property that there exists $[E_n] \in \mathfrak{M}$ such that $\lim_{n\to\infty} TE_n = 0$. Similarly we define $R(\mathfrak{M})$ to be the set of all $T \in B$ with the property that there exists $[E_n] \in \mathfrak{M}$ such that $\lim_{n\to\infty} E_n T = 0$. We restrict our attention to the sets $L(\mathfrak{M})$. Results concerning $L(\mathfrak{M})$ are easily extended to $R(\mathfrak{M})$ using the fact that $R(\mathfrak{M}) = (L(\mathfrak{M}))^*$.

LEMMA 3.2. *If $\mathfrak{M}$ is a proper ideal of $\mathcal{K}$, then $L(\mathfrak{M})$ is a proper left ideal of B.*

Proof. Assume $T \in L(\mathfrak{M})$ and $S \in B$. Then there exists $[E_n] \in \mathfrak{M}$ such that $\lim_{n\to\infty} TE_n = 0$. Then clearly $\lim_{n\to\infty} (STE_n) = 0$. Now assume T, $S \in L(\mathfrak{M})$. There exist $[E_n]$, $[F_n] \in \mathfrak{M}$ such that $\lim_{n\to\infty} TE_n = 0$ and $\lim_{n\to\infty} SF_n = 0$. Assume $[G_n] = [E_n] \wedge [F_n]$. Then $\{G_n\} \leq \{E_n\}$ and

$\{G_n\} \leq \{F_n\}$ so by Proposition 2.3 (1), $\lim_{n\to\infty} (T + S)G_n = 0$. Since $[G_n] \in \mathfrak{M}$, $T + S \in L(\mathfrak{M})$. If $I \in L(\mathfrak{M})$, then for some $[E_n] \in \mathfrak{M}$, $\lim_{n\to\infty} I(E_n) = 0$. This contradicts the hypothesis that $\mathfrak{M}$ is proper. Therefore $L(\mathfrak{M})$ is a proper left ideal of B.

LEMMA 3.3. *Assume T and S are positive elements in B such that $T + S$ is not invertible. Let $\{E_n\}$, $\{F_n\}$, and $\{G_n\}$ be annihilating sequences of T, S, and $T + S$, respectively. Then*

$$[E_n] \wedge [F_n] = [G_n].$$

Proof. $\lim_{n\to\infty} (T + S)G_n = 0$. Then by Lemma 2.6 (1), $\lim_{n\to\infty} TG_n = 0$ and $\lim_{n\to\infty} SG_n = 0$. By Proposition 2.3 (2), $\{G_n\} \leq \{E_n\}$ and $\{G_n\} \leq \{F_n\}$. Therefore

$$[G_n] \leq [E_n] \wedge [F_n].$$

Conversely assume $\{J_n\} \in [E_n] \wedge [F_n]$. Then by Proposition 2.3 (1), $\lim_{n\to\infty} TJ_n = 0$ and $\lim_{n\to\infty} SJ_n = 0$. Thus $\lim_{n\to\infty} (T + S)J_n = 0$ which implies by Proposition 2.3 (2) that $\{J_n\} \leq \{G_n\}$. Thus

$$[E_n] \wedge [F_n] \leq [G_n].$$

This proves the lemma.

Assume that N is a proper left ideal of B. Define $\mathfrak{M}(N)$ to be the set of all $a \in \mathcal{K}$ with the property that there exists a positive element $T \in N$ with annihilating sequence $\{E_n\}$ such that $[E_n] \leq a$.

LEMMA 3.4. *If N is a proper left ideal of B, then $\mathfrak{M}(N)$ is a proper ideal of $\mathcal{K}$.*

Proof. Assume that $a \in \mathfrak{M}(N)$, $b \in \mathcal{K}$, and $a \leq b$. By definition there exists a positive element $T \in N$ with annihilating sequence $\{E_n\}$ such that $[E_n] \leq a$. Then $[E_n] \leq b$, so $b \in \mathfrak{M}(N)$. Next assume $a, b \in \mathfrak{M}(N)$. Let T and S be positive elements in N with annihilating sequence $\{E_n\}$ and $\{F_n\}$ respectively such that $[E_n] \leq a$ and $[F_n] \leq b$. Let $\{G_n\}$ be an annihilating sequence of $T + S$. Then by Lemma 3.3,

$$[G_n] = [E_n] \wedge [F_n] \leq a \wedge b,$$

and since $T + S \in N$, $a \wedge b \in \mathfrak{M}(N)$. Finally assume $0 \in \mathfrak{M}(N)$. Then there exists a positive element $T \in N$ and an annihilating sequence $\{E_n\}$ of T such that $[E_n] = 0$. But this is impossible by the definition of annihilating sequence. Thus $\mathfrak{M}(N)$ is proper.

The purpose of this section is to describe precisely the relationship between the closed left ideals of B and the ideals in $\mathcal{K}$. Lemmas 3.2 and 3.4 are the beginning of this program. The full results are stated in Theorems 3.7 and 3.8. We now prove a technical lemma.

LEMMA 3.5. *Assume that N is a proper closed left ideal of B. Assume that $\{E_n\} \in \mathcal{S}$ and $E_n \in \mathfrak{M}(N)$ for all n. Then $I - E_n \in N$ for all n.*

Proof. For each m there is a positive element $T_m \in N$ and an annihilating sequence $\{G_n^{(m)}\}$ of T_m such that $\{G_n^{(m)}\} \leq E_m$. Therefore for each m, $\lim_{n\to\infty} (I - E_m)G_n^{(m)} = 0$. Also for each m, $n \geq 1$, there exists $S_{n,m} \in B$ such that $S_{n,m} T_m = I - G_n^{(m)}$. Since N is a left ideal $(I - G_n^{(m)}) \in N$ for all m, n. Then

$$\| (I - E_m) - (I - E_m)(I - G_n^{(m)}) \| \to 0$$

as $n \to \infty$, and since N is a closed left ideal, $(I - E_m) \in N$ for all $m \geq 1$.

In order to relate closed left ideals in B to ideals in $\mathcal{K}$, we need the concept of a closed ideal in $\mathcal{K}$.

DEFINITION 3.6. An ideal $\mathfrak{M}$ in $\mathcal{K}$ is closed if whenever $\{E_n\} \in \mathcal{S}$ and $E_n \in \mathfrak{M}$ for all n, then $[E_n] \in \mathfrak{M}$.

THEOREM 3.7. *If $\mathfrak{M}$ is a closed proper ideal in $\mathcal{K}$, then $L(\mathfrak{M})$ is a closed proper left ideal of B. If N is a closed proper left ideal in B, then $\mathfrak{M}(N)$ is a closed proper ideal in $\mathcal{K}$.*

Proof. Let $\mathfrak{M}$ and N be as in the statement of the theorem. Then by Lemma 3.2, $L(\mathfrak{M})$ is a proper left ideal of B, and by Lemma 3.4, $\mathfrak{M}(N)$ is a proper ideal in $\mathcal{K}$. It remains to be shown that $L(\mathfrak{M})$ and $\mathfrak{M}(N)$ are closed.

Assume that $\{T_m\}$ is a sequence in $L(\mathfrak{M})$ and that $T_m \to T$. Then $T_m^* T_m \in L(\mathfrak{M})$ for all m and $T_m^* T_m \to T^*T$. We choose a projection $E_1 \in \mathfrak{M}$ such that $\| T_1^*T_1 E_1 \| < 1$. Assume we have chosen projections $E_k \in \mathfrak{M}$, $1 \leq k \leq n$, with the properties that

$$\| T_k^* T_k E_k \| < 1/k \quad \text{and} \quad \| (I - E_j)E_k \| < 1/k$$

whenever $1 \leq j \leq k$. Let $S_{n+1} = T_{n+1}^* T_{n+1} + \sum_{k=1}^{n} (I - E_k)$. Then $S_{n+1} \in L(\mathfrak{M})$, and therefore there exists $[F_n] \in \mathfrak{M}$ such that $\lim_{m\to\infty} S_{n+1} F_m = 0$. By Lemma 2.6 (1),

$$\lim_{m\to\infty} (T_{n+1}^* T_{n+1})F_m = 0 \quad \text{and} \quad \lim_{m\to\infty} (I - E_k)F_m = 0$$

for $1 \leq k \leq n$. Therefore we can choose a projection $E_{n+1} \in \mathfrak{M}$ with the properties

$$\| T_{n+1}^* T_{n+1} E_{n+1} \| < 1/n + 1 \quad \text{and} \quad \| (I - E_k)E_{n+1} \| < 1/n + 1, 1 \leq k \leq n.$$

By induction we define a sequence of projections $\{E_n\}$ which is admissible by the construction. Since $E_n \in \mathfrak{M}$ for all n and $\mathfrak{M}$ is closed, $[E_n] \in \mathfrak{M}$. Furthermore,

$$\begin{aligned}\| TE_n \|^2 &= \| E_n T^*TE_n \| \\ &\leq \| E_n(T^*T - T_n^* T_n)E_n \| + \| E_n T_n^* T_n E_n \| \\ &\leq \| T^*T - T_n^* T_n \| + 1/n \to 0\end{aligned}$$

as $n \to \infty$. This proves that $T \in L(\mathfrak{M})$.

Now assume that N is a proper closed left ideal of B. Assume $\{E_n\} \in \mathcal{S}$ and $E_n \in \mathfrak{M}(N)$ for all n. By Lemma 3.5, $I - E_n \in N$ for all n. Let

$$T = \sum_{n=1}^{+\infty} (\tfrac{1}{2})^n(I - E_n).$$

Since N is closed, T is a positive element in N. Let $\{F_n\} \in \mathcal{S}$ be an annihilating sequence of T. Then by Lemma 2.6 (2), $\lim_{n\to\infty} (I - E_m)F_n = 0$ for all m. Therefore $\{F_n\} \leq \{E_n\}$. It follows by definition that $[E_n] \in \mathfrak{M}(N)$. Therefore $\mathfrak{M}(N)$ is closed.

THEOREM 3.8. *If N is a proper closed left ideal of B, then $N = L(\mathfrak{M}(N))$. If $\mathfrak{N}$ is a proper closed ideal of $\mathcal{K}$, then $\mathfrak{N} = \mathfrak{M}(L(\mathfrak{N}))$.*

Proof. Assume N is a proper closed left ideal of B. First assume $T \in L(\mathfrak{M}(N))$. Then there exists $[E_n] \in \mathfrak{M}(N)$ such that $\lim_{n\to\infty} TE_n = 0$. By Lemma 3.5, $I - E_n \in N$ for all n. Then $\| T - T(I - E_n) \| \to 0$ as $n \to \infty$, which implies $T \in N$. Conversely assume $T \in N$. Then $T^*T \in N$. Let $\{E_n\}$ be an annihilating sequence of T^*T. By definition, $[E_n] \in \mathfrak{M}(N)$. Then $\| T^*TE_n \| \to 0$ as $n \to \infty$, and $\| TE_n \|^2 = \| E_n T^*TE_n \| \to 0$ as $n \to \infty$. Therefore $T \in L(\mathfrak{M}(N))$. This completes the proof that $N = L(\mathfrak{M}(N))$.

Now assume $\mathfrak{N}$ is a proper closed ideal of $\mathcal{K}$. If $[E_n] \in \mathfrak{N}$, then $(I - E_n) \in L(\mathfrak{N})$ all n. Let $T = \sum_{k=1}^{+\infty} (\frac{1}{2})^k(I - E_k)$. By Theorem 3.7, $L(\mathfrak{N})$ is closed, so that $T \in L(\mathfrak{N})$. Let $\{F_n\}$ be an annihilating sequence of T. Then by Lemma 2.6 (2), $\lim_{n\to\infty} (I - E_m)F_n = 0$ for all m. Therefore $\{F_n\} \leq \{E_n\}$. By definition $[E_n] \in \mathfrak{M}(L(\mathfrak{N}))$. Conversely assume $[E_n] \in \mathfrak{M}(L(\mathfrak{N}))$. Then there exists a positive element $T \in L(\mathfrak{N})$ and an annihilating sequence $\{F_n\}$ of T such that $\{F_n\} \leq \{E_n\}$. Since $T \in L(\mathfrak{N})$, there exists $[G_n] \in \mathfrak{N}$ such that $\lim_{n\to\infty} TG_n = 0$. By Proposition 2.3 (2), $\{G_n\} \leq \{F_n\}$. Therefore $\{G_n\} \leq \{E_n\}$, so that $[E_n] \in \mathfrak{N}$.

COROLLARY 3.9. *$\mathfrak{M} \to L(\mathfrak{M})$ is a one-to-one order preserving map from the set of all proper closed ideals of $\mathcal{K}$ onto the set of all proper closed left ideals of B.*

COROLLARY 3.10. *If N and M are two closed left ideals of B which contain the same projections, then $N = M$.*

Proof. Assume $N = L(\mathfrak{N})$ and $M = L(\mathfrak{M})$ where $\mathfrak{N}$ and $\mathfrak{M}$ are ideals in $\mathcal{K}$. Assume $T \in N$. Then there exists $[E_n] \in \mathfrak{N}$ such that $\lim_{n\to\infty} TE_n = 0$. Also $I - E_n \in N$ for all n. Then by hypothesis $I - E_n \in M$ for all n. Then since $\| T - T(I - E_n) \| \to 0$, $T \in M$. Thus $N \subset M$. By symmetry $M \subset N$.

4. The maximal left ideals of B

We assume throughout the remainder of the paper that B has property A. It is well known that N is a maximal closed left ideal of B if and only if N is a maximal left ideal of B. Using this we prove that $\mathfrak{M}$ is a maximal closed ideal of $\mathcal{K}$ if and only if $\mathfrak{M}$ is a maximal ideal of $\mathcal{K}$. First assume that $\mathfrak{M}$ is a maximal closed ideal of $\mathcal{K}$, and suppose that $\mathfrak{M} \subset \mathcal{J}$ where $\mathcal{J}$ is a proper ideal of $\mathcal{K}$. Suppose that $\mathfrak{M} \neq \mathcal{J}$. Then $\mathcal{J}$ is not closed, and therefore there exists $\{E_n\} \in \mathcal{S}$, $E_n \in \mathcal{J}$ for all n, and $[E_n] \notin \mathcal{J}$. It follows that $\mathcal{J}$ contains a projection E such that $E \notin \mathfrak{M}$. But $L(\mathfrak{M})$ is a maximal left ideal of B by Corollary 3.9, $L(\mathcal{J})$ is a proper left ideal of B by Lemma 3.2, $L(\mathfrak{M}) \subset L(\mathcal{J})$, and $I - E \in$

$L(\mathcal{J})$, $I - E \notin L(\mathfrak{M})$. This is a contradiction which proves that $\mathfrak{M} = \mathcal{J}$. Conversely assume that $\mathfrak{M}$ is a maximal ideal of $\mathcal{K}$. Then $L(\mathfrak{M})$ is a proper left ideal of B. Let $N = \overline{L(\mathfrak{M})}$. $\mathfrak{M}(N) \supset \mathfrak{M}$, and therefore $\mathfrak{M}(N) = \mathfrak{M}$. Finally $\mathfrak{M}$ is closed by Theorem 3.7. By this result and Corollary 3.9 we have that N is a maximal left ideal of B if and only if $N = L(\mathfrak{M})$ where $\mathfrak{M}$ is a maximal ideal of $\mathcal{K}$. Now we characterize the maximal left ideals of B in another fashion. We prove the following result.

THEOREM 4.1. *Assume that N is a closed left ideal of B. Then the following are equivalent:*

(1) *N is maximal.*

(2) *$N = L(\mathfrak{M})$ where $\mathfrak{M}$ is a maximal ideal of $\mathcal{K}$.*

(3) *If E is a projection in B and $E \notin N$, then there exists a projection $F \in N$ such that $E + F$ is invertible in B.*

We have already noted the equivalence of (1) and (2). Before completing the proof of the theorem, we establish a lemma.

LEMMA 4.2. *Assume that $\mathfrak{M}$ is a proper ideal of $\mathcal{K}$ and that $a \in \mathcal{K}$ has the property that $a \wedge b \neq 0$ for all $b \in \mathfrak{M}$. Then there is a proper ideal $\mathcal{J}$ of $\mathcal{K}$ such that $a \in \mathcal{J}$ and $\mathfrak{M} \subset \mathcal{J}$.*

Proof. Let $\mathcal{J}$ be the set of all $c \in \mathcal{K}$ such that there exists $b \in \mathfrak{M}$ with $a \wedge b \leq c$. Clearly $a \in \mathcal{J}$ and $\mathfrak{M} \subset \mathcal{J}$. We verify that $\mathcal{J}$ is a proper ideal of $\mathcal{K}$. First if $c \in \mathcal{J}$ and $c \leq d$, then it is obvious that $d \in \mathcal{J}$. Assume $c, d \in \mathcal{J}$. Then there exists $e, f \in \mathfrak{M}$ such that $a \wedge e \leq c$ and $a \wedge f \leq d$. Then

$$a \wedge (e \wedge f) = (a \wedge e) \wedge (a \wedge f) \leq c \wedge d.$$

By the definition of $\mathcal{J}$, $c \wedge d \in \mathcal{J}$. $0 \notin \mathcal{J}$ since by hypothesis it is not true that $a \wedge b \leq 0$ for any $b \in \mathfrak{M}$. This completes the proof.

Now we complete the proof of Theorem 4.1. Assume (3) holds. N is contained in some maximal left ideal M of B. Assume that E is a projection in M. Then $E \in N$; for if not, there exists a projection $F \in N$ such that $E + F$ is invertible. Thus N and M contain the same projections. By Corollary 3.10, $N = M$. Conversely assume that $N = L(\mathfrak{M})$ where $\mathfrak{M}$ is a maximal ideal of $\mathcal{K}$. Assume $E \notin N$. Suppose that whenever $[E_n] \in \mathfrak{M}$, $E + (I - E_n)$ is not invertible for all n. Then $(I - E) \wedge [E_n] \neq 0$ by Theorem 2.8. By Lemma 4.2, there is a proper ideal $\mathcal{J}$ in $\mathcal{K}$ such that $(I - E) \in \mathcal{J}$ and $\mathfrak{M} \subset \mathcal{J}$. But this is impossible since $\mathfrak{M}$ is maximal and $(I - E) \notin \mathfrak{M}$. Therefore there exists an idempotent $(I - F) \in \mathfrak{M}$ such that $E + F$ is invertible. This proves (3).

If B has an additional property that we now describe, then we can sharpen the result in Theorem 4.1. We assume for the remainder of this section that whenever E and F are projections in B, then E and F have a greatest lower bound in B with respect to the usual ordering of projections ($E < F$ means $EF = E$). We denote this glb as $E \cap F$. Any AW^*-algebra has this additional property.

DEFINITION 4.3. Let E and F be projections in B. Then F is a strong complement of E if $E \cap F = 0$ and $E + F$ is invertible in B.

If F is a strong complement of E in B, then F is a complement of E in the usual sense that $E \cap F = 0$ and $E \cup F = I$. However it is not difficult to find examples of complements which are not strong complements.

LEMMA 4.4. *Assume that E and F are projections and that $E + F$ is invertible in B. Let $G = E \cap F$. Then $(F - G)$ is a strong complement of E.*

Proof. First we verify that $E \cap (F - G) = 0$. For let $J = E \cap (F - G)$. $J < E$, $J < F$ and therefore $J < E \cap F = G$. Then $J = J(F - G) = JF - JG = J - JG$. Thus $JG = 0$. Therefore $J = JG = 0$. Now there exists $K \in B$ such that $K(E + F) = I$. Then

$$(K + KG)(E + (F - G)) = I - KG + KG + KG - KG = I.$$

Therefore $(F - G)$ is a strong complement of E.

Now we have the following result.

THEOREM 4.5. *Assume that whenever E and F are projections in B, then $E \cap F$ exists in B. Assume that N is a proper closed left ideal of B. Then N is a maximal left ideal of B if and only if whenever E is a projection in B and $E \notin N$, then E has a strong complement in N.*

Proof. Assume that N is a maximal left ideal of B and E is a projection in B such that $E \notin N$. Then by Theorem 4.1 there exists $F \in N$ such that $E + F$ is invertible in B. Let $G = E \cap F$. Since $F \in N$ and $GF = G$, then $G \in N$. Therefore $F - G \in N$. Finally $(F - G)$ is a strong complement of E by Lemma 4.4.

5. Central projections

We assume throughout this section that whenever E and F are projections in B, then E and F have a greatest lower bound in B. A linear functional α on B is a state of B if $\alpha(T) \geq 0$ for all positive elements T in B and $\alpha(I) = 1$. If α is an extreme point of the convex set of all states of B, then α is a pure state. Given a state α, let

$$K_\alpha = \{T \in B \mid \alpha(T^*T) = 0\}.$$

K_α is a closed left ideal of B and when α is a pure state, then K_α is a maximal left ideal of B by [1, Théorème 2.9.5, p. 48].

It is a well-known theorem that when B is an AW^*-algebra, then a projection E in B is central if and only if E has a unique complement; see [4, Theorem 70, p. 119]. We prove a slightly more general form of this theorem.

THEOREM 5.1. *A projection $E \in B$ is a central projection if and only if E has a unique strong complement in B.*

Proof. We prove the "if" direction of the theorem. By hypothesis the unique strong complement of E is $I - E$. Assume that α is any pure state. K_α is a maximal left ideal of B, and therefore by Theorem 4.5 either $E \in K_\alpha$ or $I - E \in K_\alpha$. The generalized Cauchy-Schwartz inequality, [6, p. 213], states that when $R, S \in B$,

$$| \alpha(R^*S) |^2 \leq \alpha(R^*R)\alpha(S^*S).$$

Therefore given any $T \in B$ we have,

$$| \alpha(ET(I - E)) |^2 \leq \alpha(E)\alpha((T(I - E))^*T(I - E))$$

and

$$| \alpha(ET(I - E)) |^2 \leq \alpha((ET)(ET)^*)\alpha(I - E).$$

But by the previous part of the proof either $\alpha(E) = 0$ or $\alpha(I - E) = 0$. In either case $\alpha(ET(I - E)) = 0$. This proves that for an arbitrary pure state α of B, $\alpha(ET(I - E)) = 0$. Since the pure states of B separate the elements of B by the remarks in [2, p. 112], then $ET(I - E) = 0$. A similar proof shows that $(I - E)TE = 0$. Therefore $ET = ETE = TE$ which proves the theorem.

6. The null space of a pure state and an application

Assume that α is a pure state of B and let $\mathfrak{M}$ be the unique maximal ideal of $\mathcal{K}$ such that $K_\alpha = L(\mathfrak{M})$. We define $N(\mathfrak{M})$ to be the set of all $T \in B$ with the property that there exists $[E_n] \in \mathfrak{M}$ such that $\| E_n TE_n \| \to 0$. It is not difficult to verify that $N(\mathfrak{M})$ is a proper subspace of B. Note that $L(\mathfrak{M}) + (L(\mathfrak{M}))^* \subset N(\mathfrak{M})$. It is a result of R. V. Kadison [1, Proposition 2.9.1, p. 46] that $\alpha^{-1}(0) = K_\alpha + (K_\alpha)^*$ for α a pure state. Therefore $\alpha^{-1}(0) = N(\mathfrak{M})$. If $T \in B$, then $T - \alpha(T)I \in N(\mathfrak{M})$, and therefore there exists $[E_n] \in \mathfrak{M}$ such that $\| E_n TE_n - \alpha(T)E_n \| \to 0$. We state these results as a lemma.

LEMMA 6.1. *Assume that α is a pure state of B and $K_\alpha = L(\mathfrak{M})$, $\mathfrak{M}$ a maximal ideal of $\mathcal{K}$. Then $\alpha^{-1}(0) = N(\mathfrak{M})$ and for any $T \in B$, there exists $[E_n] \in \mathfrak{M}$ such that $\| E_n TE_n - \alpha(T)E_n \| \to 0$.*

We apply this result to the question of when a pure state of a subalgebra of B has a unique extension to a pure state of B. Let B_0 be a closed *-subalgebra of B which contains I and such that B_0 has property A. Let $\mathcal{K}_0$ be the set of all equivalence classes of admissible sequences of projections in B_0. Assume that α_0 is a pure state of B_0, and let $\mathfrak{M}_0$ be the unique maximal ideal of $\mathcal{K}_0$ such that $L(\mathfrak{M}_0) = K_{\alpha_0}$.

THEOREM 6.2. *α_0 has a unique extension to a pure state of B if and only if given any $T \in B$, there exists a scalar λ and $[E_n] \in \mathfrak{M}_0$ such that*

$$\| E_n TE_n - \lambda E_n \| \to 0.$$

Proof. Assume that given any $T \in B$ there exists a scalar λ and $[E_n] \in \mathfrak{M}_0$ such that $\| E_n TE_n - \lambda E_n \| \to 0$. Let α be any state of B which extends α_0. Let $T \in B$, and assume λ and $[E_n]$ are as given in the previous hypothesis.

Since $E_n \in \mathfrak{M}_0$ for all n, then $\alpha(I - E_n) = \alpha_0(I - E_n) = 0$ for all n. We write T as

$$T = E_n TE_n + E_n T(I - E_n) + (I - E_n)T.$$

By the general Cauchy-Schwarz inequality,

$$\alpha(E_n T(I - E_n)) = \alpha((I - E_n)T) = 0.$$

Therefore $\alpha(T) = \alpha(E_n TE_n)$ for all n. Then

$$| \alpha(T) - \lambda | = | \alpha(E_n TE_n - \lambda E_n) | \leq \| E_n TE_n - \lambda E_n \| \to 0.$$

This proves that any state α of B which extends α_0 takes the values λ at T. It follows that α_0 has a unique extension to a state α of B. α must be a pure state of B by [1, Lemma 2.10.1, p. 50].

Conversely assume that α_0 has a unique extension to a pure state α of B. Let L_0 be the set of all $T \in B$ with the property that there exists $[E_n] \in \mathfrak{M}_0$ such that $\| TE_n \| \to 0$. L_0 is a closed left ideal of B by the proof of Theorem 3.7. Suppose L_0 were not a maximal left ideal of B. Then by [1, Théorème 2.9.5, p. 48] there exist maximal left ideals of B, L_1 and L_2, such that $L_0 \subset L_1$, $L_0 \subset L_2$, and $L_1 \neq L_2$. By this same Theorem there exist corresponding pure states α_1 and α_2 of B such that $K_{\alpha_1} = L_1$ and $K_{\alpha_2} = L_2$. Assume $T \in B_0$. Then there exists $[E_n] \in \mathfrak{M}_0$ such that

$$\| E_n TE_n - \alpha_0(T)E_n \| \to 0 \qquad \text{(Lemma 6.1).}$$

Since $L_0 \subset L_1$ and $L_0 \subset L_2$, then $\alpha_1(E_n) = \alpha_2(E_n) = 1$ for all n. By the same argument as used in the first paragraph of the proof it follows that $\alpha_1(T) = \alpha_0(T)$ and $\alpha_2(T) = \alpha_0(T)$. Therefore α_1 and α_2 extend α_0 which is a contradiction. It follows that L_0 is a maximal left ideal and $K_\alpha = L_0$. Therefore $\alpha^{-1}(0) = L_0 + (L_0)^*$. Then by the definition of L_0, given any $T \in B$ there exists $[E_n] \in \mathfrak{M}_0$ such that

$$\| E_n TE_n - \alpha(T)E_n \| \to 0.$$

This completes the proof of the theorem.

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