GLOBALLY FRAMED f-MANIFOLDS

BY

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1. Introduction

Let M be a $(2n + 1)$-dimensional almost contact manifold with *fundamental affine collineation* ϕ, *fundamental vector field* E and *contact form* η. Consider a $2n$-dimensional manifold P imbedded in M by $i : P \rightarrow M$. In a recent paper [4] the authors studied invariant and noninvariant hypersurfaces of M which arise when for each $p \in P$, the vector $E_{i(p)}$ does not belong to the tangent hyperplane of the hypersurface $i(P)$. These hypersurfaces admit almost complex structures, and when M is quasi-Sasakian (see [1] for the definition), e.g., when M is cosymplectic or a normal contact space, and $i(P)$ is noninvariant, then P carries a Kaehler structure. On the other hand, an invariant hypersurface of a cosymplectic manifold is totally geodesic and Kaehlerian.

We now study the case where the fundamental vector field is always tangent to the hypersurface. The structure induced on P turns out to be an f-structure (see [3], [6], [10]) which is neither almost complex nor almost contact. It gives rise to the notion of a quasi-symplectic manifold which has nice properties. This structure on P is determined by a $(1, 1)$ tensor field f and the metric induced on P by the metric of the quasi-Sasakian manifold.

Following [5], the f-manifold P is said to be normal if the almost complex structure tensor J of a certain subbundle $V(P)$ of the tangent bundle $T(P)$ of P is integrable and the connection γ of $V(P)$ in terms of which J is defined is flat. For example, a totally umbilical hypersurface of a cosymplectic manifold gives rise to a normal framed f-structure. Necessary and sufficient conditions are given for the integrability of a quasi-symplectic structure when the ambient space is either a cosymplectic or a normal contact manifold.

An almost complex structure may also be defined on P in terms of the structure tensors of the framed f-structure. Moreover, if the ambient space is a cosymplectic manifold and P is a totally geodesic hypersurface, P carries a Kaehlerian structure. By considering more general framed f-structures the same conclusions are obtained under weaker conditions. Other examples of Kaehler manifolds are obtained by considering complete and simply connected quasi-symplectic normal framed f-manifolds P. Indeed, if P is a totally geodesic hypersurface and the ambient space is cosymplectic, then P is a product with one factor Kaehlerian. The same conclusion prevails under more general conditions on the second fundamental form of the immersion.

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The direct product $P_1 \times P_2$ of two framed f-structures P_1 and P_2 has a naturally induced almost complex structure other than the one arising from the underlying almost complex structures of P_1 and P_2. If $i(P_1)$ and $i(P_2)$ are totally geodesic hypersurfaces and the ambient space is cosymplectic, then $P_1 \times P_2$ is Kaehlerian.

In §8, we study the topology of compact even-dimensional globally framed f-manifolds P whose structure tensors are parallel fields. Since P has an associated Kaehlerian structure, its topology may be studied by means of the theory of harmonic integrals. We choose, however, to introduce an analogous theory with the fundamental form F of the f-structure playing the role of the Kaehler 2-form. If the rank of f is maximal, F is the Kaehler 2-form. Hence, if only F is assumed to be parallel, this yields a generalization of Kaehler geometry.

2. Framed f-structures

It is assumed throughout that the vector field E is tangent to the hypersurface $i(P)$. As examples, we may consider R^{2n} imbedded in R^{2n+1} or the torus T^{2n} imbedded in T^{2n+1}. Since i is a regular map, there is a vector field E' on P such that

$$E = i_*E'. \tag{2.1}$$

Hence

$$\phi(i_*E') = 0, \qquad \eta(i_*E') = 1. \tag{2.2}$$

Putting

$$\eta' = i^*\eta, \tag{2.3}$$

we obtain

$$\eta'(E') = 1. \tag{2.4}$$

PROPOSITION 1. *There exist vector fields N on $i(P)$ and A on P such that $N_{i(p)} \notin i(P)_{i(p)}$ for all $p \in P$ and*

$$\phi N = -i_*A, \qquad \eta(N) = 0. \tag{2.5}$$

Proof. It is well known that a metric G may be defined on M with the properties

$$G(\phi x, y) = -G(x, \phi y), \tag{2.6}$$

for all vector fields x, y on M, that is, ϕ is skew-symmetric with respect to G, and

$$\eta = G(E, \cdot). \tag{2.7}$$

Let N be the unit normal to $i(P)$ with respect to G. Then, since ϕN is orthogonal to N with respect to G, it is tangent to the hypersurface and consequently can be expressed as

$$\phi N = -i_* A \tag{2.8}$$

for some vector field A on P. Moreover, N is orthogonal to the vector field E since $E_{i(p)} \in i(P)_{i(p)}$, $p \in P$, that is,

$$\eta(N) = 0. \tag{2.9}$$

PROPOSITION 2. *Let P be a $2n$-dimensional manifold imbedded in the almost contact manifold M with imbedding i. Then, there exist tensor fields f, E', η', A and α on P satisfying the relations*

$$f^2 = -I + \eta' \otimes E' + \alpha \otimes A, \tag{2.10}$$

$$fE' = 0, \qquad fA = 0, \tag{2.11}$$

$$\eta' \circ f = 0, \qquad \alpha \circ f = 0, \tag{2.12}$$

$$\eta'(E') = 1, \qquad \eta'(A) = 0, \tag{2.13}$$

$$\alpha(E') = 0, \qquad \alpha(A) = 1, \tag{2.14}$$

where I is the identity transformation of P_p.

Proof. Put

$$\phi i_* X = i_* fX + \alpha(X)N. \tag{2.15}$$

Then, from (2.1), $0 = \phi i_* E' = i_* fE' + \alpha(E')N$. Since $\eta \circ \phi = 0$ and $\eta(N) = 0$, $\eta(i_* fX) = 0$. From (2.8), $0 = \eta(\phi N) = -\eta(i_* A)$. Applying ϕ to both sides of (2.15) yields (2.10) and the second half of (2.12) by virtue of (1.1), (2.1), (2.3) and (2.5). Applying ϕ to both sides of (2.5), we obtain the second halves of (2.11) and (2.14).

COROLLARY. *The hypersurface $i(P)$ is not invariant with respect to ϕ (see* [4]).

This is an immediate consequence of (2.14) and (2.15).

We shall occasionally refer to P rather than $i(P)$ as the hypersurface.

3. Normal framed f-structures

If the tensor field f of type (1, 1) on P has the property

$$f^3 + f = 0$$

and f is of rank r everywhere, it is said to define an *f-structure of rank r* on P. As examples, we have the almost complex and almost contact structures, the former being of maximal rank and the latter having rank one less than the maximum.

THEOREM 3. *The structure on P given in Proposition 2 is an f-structure of rank $2n - 2$.*

Proof. By (2.10)–(2.14), $f^3 + f = 0$. Let X be a vector field on P satisfying $fX = 0$. Then, by (2.10), $f^2X = -X + \eta'(X)E' + \alpha(X)A$, so that

$$X = \eta'(X)E' + \alpha(X)A.$$

This structure on P is called a *framed f-structure* of rank $2n - 2$ or, simply, a *framed f-structure.* We shall occasionally refer to it as a *globally framed f-structure.* When the vector bundle $V(P)$ over P formed by the set of all tangent vectors $v = v^a E_a$, $a = 2n + 1, 2n + 2$, $E_{2n+1} = E'$, $E_{2n+2} = A$ is endowed with an affine connection γ, it admits a natural almost complex structure (see [5]). If it is integrable, the framed f-structure is said to be *normal.*

PROPOSITION 4 (Ishihara [5]). *In order that the framed f-structure be normal, it is necessary and sufficient that the tensor field S of type* (1, 2) *given by*

$$S(X, Y) = [f, f](X, Y) + \{(\nabla'\eta')(X, Y) - (\nabla'\eta')(Y, X)\}E' + \{(\nabla'\alpha)(X, Y) - (\nabla'\alpha)(Y, X)\}A$$

vanish and the connection γ of $V(P)$ have zero curvature, where

$$[f, f](X, Y) = [fX, fY] - f[fX, Y] - f[X, fY] + f^2[X, Y]$$

and ∇' denotes covariant differentiation with respect to γ.

By defining γ in such a way that E' and A are parallel (absolute parallelism), γ has zero curvature. By Proposition 4, a framed f-structure is then normal if $S = [f, f] + d\eta' \otimes E' + d\alpha \otimes A$ vanishes.

Putting $Y = E'$ in Proposition 4, we find after taking the interior product by E' that $L_{E'}\,\eta' = 0$ if the structure is normal. Similarly, $L_{E'}\,\alpha = 0$, $L_A\,\eta' = 0$ and $L_A\,\alpha = 0$. These relations together with (2.12) imply that $L_{E'}\,f = 0$. Moreover, $L_A f$ vanishes. These formulae are required in the proof of Theorem 19.

4. Quasi-symplectic framed f-structures

Let $M(\phi, \eta, G)$ be an almost contact manifold. We give to the framed f-structure P the metric g induced by G, that is, $g = i^*G$. Then, by (2.6) and (2.15), since the vector field N is normal to $i(P)$, f is skew-symmetric with respect to the metric g. We put

$$F(X, Y) = g(fX, Y)$$

and call F the *fundamental* 2*-form* of the framed f-structure.

Assume now that $M(\phi, \eta, G)$ is quasi-Sasakian (e.g., assume that it is either cosymplectic or a normal contact manifold). Let Φ be the fundamental 2-form of M, that is, $\Phi(x, y) = G(\phi x, y)$. Then, since $F = i^*\Phi$, $dF = di^*\Phi = i^*d\Phi = 0$. In this case, P is called a *quasi-symplectic framed f-manifold of rank* $2n - 2$. (Observe that a framed f-structure cannot be symplectic (of rank $2n$).) If $M(\phi, \eta, G)$ is a contact manifold, F is an exact 2-form. For, Φ is an exact 2-form, that is, $\Phi = d\eta$, so $F = i^*\Phi = i^*d\eta = di^*\eta = d\eta'$.

Note that on a quasi-symplectic manifold

$$\iota(E')F^{n-1} = 0 \quad \text{and} \quad \iota(A)F^{n-1} = 0$$

(where $\iota(X)$ denotes the interior product by X), so

$$\iota(E')(\eta' \wedge F^{n-1}) = F^{n-1} \quad \text{and} \quad \iota(A)(\alpha \wedge F^{n-1}) = F^{n-1},$$

from which $\eta' \wedge F^{n-1} \neq 0$ and $\alpha \wedge F^{n-1} \neq 0$. If the ambient space is a contact manifold, $F = d\eta'$, and in this case, $\eta' \wedge (d\eta')^{n-1} \neq 0$. Moreover, $(d\eta')^n = 0$, for, $\iota(E')(\eta' \wedge F^n) = F^n = 0$.

From (2.7) and (2.8), we find $\eta' = g(E', \cdot)$, $\alpha = g(A, \cdot)$. Thus, E' and A are orthonormal vectors.

If $M(\phi, \eta, G)$ is a cosymplectic manifold, that is, if η is closed, then $\nabla\phi = 0$, $\nabla\eta = 0$, where ∇ denotes covariant differentiation with respect to the Riemannian connection of M (see [1]). Denote by D the induced connection on P. Then, the equations of Gauss and Weingarten are

$$\nabla_{i_*X} i_*Y = i_*D_XY + h(X, Y)N \quad \text{and} \quad \nabla_{i_*X} N = -i_*HX,$$

respectively, where

$$h(X, Y) = g(HX, Y),$$

the tensor h being symmetric. The tensor fields h and H are the second fundamental tensors of P (with respect to N) of types (0, 2) and (1, 1), respectively.

Covariant differentiation of both sides of (2.1), (2.3), (2.5) and (2.15) along the hypersurface yields after again taking account of (2.5), (2.9) and (2.15):

PROPOSITION 5. *Let $P(f, E', A, g)$ be a quasi-symplectic globally framed hypersurface of a cosymplectic manifold. Then,*

$$\begin{aligned} (D_X f)Y &= \alpha(Y)HX - h(X, Y)A, \\ D_X E' = 0, &\qquad D_X A = fHX, \\ D_X \eta' = 0, &\qquad (D_X \alpha)(Y) = -h(X, fY), \\ h(X, E') = 0, &\qquad h(X, A) = \alpha(HX), \\ \eta'(HX) &= 0. \end{aligned} \tag{4.1}$$

The last relation follows from (2.9) by virtue of Weingarten's equation.

For any vector field X, it is easily checked that $D_X f$ is a skew-symmetric linear transformation with respect to g and $(D_X F)(Y, Z) = g((D_X f)Y, Z)$. This is also true when the ambient space is a Sasakian manifold.

From the equations (4.1), we obtain

PROPOSITION 6. *Let M be a cosymplectic manifold. If the quasi-symplectic framed f-structure on P is totally umbilical, then it is totally geodesic and normal.*

Proof. Since P is totally umbilical, $h = \lambda g$ and $H = \lambda I$, where I is the identity transformation of P_p. Hence, $\lambda = 0$, that is, P is totally geodesic. One then observes that

$$[f, f](X, Y) = (D_{fX} f)Y - (D_{fY} f)X - f(D_X f)Y + f(D_Y f)X.$$

Moreover, the curvature of the connection γ of $V(P)$ is zero. For,

$$t(D_X E') = 0, \qquad t(D_X A) = 0,$$

where $t = f^2 + I$ is the projection operator on $V(P)$, (see §6), that is, both E' and A are parallel with respect to the connection induced on $V(P)$ by the connection of the ambient space. The result is now a consequence of Proposition 4.

From (4.1), it is seen that the vector field A on P is a Killing field if and only if $h(X, fY) = -h(Y, fX)$, that is, if and only if, H commutes with f (see Theorem 9).

From (4.1),

$$\begin{aligned}(D_X F)(Y, Z) &= \alpha(Y)g(HX, Z) - h(X, Y)g(A, Z)\\ &= \alpha(Y)h(X, Z) - \alpha(Z)h(X, Y).\end{aligned}$$

The 2-form F has vanishing covariant derivative, if and only if, $h = \mu\alpha \otimes \alpha$.

If P is totally geodesic, then f is a parallel tensor field. Conversely, if f is a parallel field, DF vanishes. Thus, $\alpha(HX) = h(X, A) = \mu\alpha(X)$. This implies that $H = \mu I + lf + \nu \otimes E'$ for some function l and 1-form ν. On the other hand, since $h(X, fY) = \mu\alpha(X)\alpha(fY) = 0$, we obtain $D\alpha = 0$ and $DA = 0$. To see the latter, note that

$$g(D_X A, Y) = g(fHX, Y) = -g(HX, fY) = -h(X, fY) = 0.$$

Hence, fHX vanishes, from which $\mu fX + lf^2X$ also vanishes. Thus,

$$\mu^2 f^2 = -l\mu f^3 = l\mu f = -l^2 f^2,$$

so that if $n \geqq 2$, $\mu = l = 0$, that is, P is totally geodesic.

PROPOSITION 7. *Let M be a cosymplectic manifold whose dimension is at least 5. Then, if the linear transformation field f of the framed f-structure on P is parallel, P is a totally geodesic hypersurface, and conversely.*

COROLLARY. *Let M be a cosymplectic manifold whose dimension is at least 5. Then, if f is a parallel field, so are E' and A. Moreover, E' and A are infinitesimal automorphisms of the quasi-symplectic structure on P.*

Let $M(\phi, \eta, G)$ be a Sasakian manifold, that is, M is a contact manifold and the almost contact structure of M is normal. Then, since

$$\nabla E = \phi, \qquad (\nabla_x \phi)y = -G(x, y)E + \eta(y)x,$$

the following relations are obtained in a manner entirely analogous to those of (4.1).

$$(4.2)\quad \begin{gathered} (D_X f)Y = -g(X, Y)E' + \eta'(Y)X + \alpha(Y)HX - h(X, Y)A, \\ DE' = f, \qquad D_X A = fHX, \\ D\eta' = F, \qquad (D_X \alpha)(Y) = -h(X, fY), \\ h(X, E') = \alpha(X), \qquad h(X, A) = \alpha(HX), \\ \eta'(HX) = \alpha(X). \end{gathered}$$

These equations then yield

PROPOSITION 8. *Let $M(\phi, \eta, G)$ be a normal contact manifold and $P(f, E', A, g)$ a quasi-symplectic framed f-structure. Then, the connection of the vector bundle $V(P)$ over P is locally flat.*

Proof. By (4.2) and (2.12),

$$t(D_X E') = (\eta' \otimes E' + \alpha \otimes A)fX = 0$$

and

$$t(D_X A) = (\eta' \otimes E' + \alpha \otimes A)fHX = 0,$$

that is, both E' and A are parallel with respect to the connection induced on $V(P)$ by the connection of the ambient space.

(Observe that there are no totally umbilical framed f-hypersurfaces of a normal contact manifold. Moreover, the covariant derivative of F is different from zero.)

From (4.2), it follows that A is a Killing vector field, if and only if H commutes with f. Moreover, the vector field E' is a Killing field (see Theorem 10).

5. Quasi-symplectic normal globally framed f-structures

In this section, we seek necessary and sufficient conditions for the normality of the structure induced on P when the ambient space M is either a cosymplectic or a normal contact manifold. To this end, we apply Proposition 4; hence, since the connection of the vector bundle $V(P)$ is locally flat in the cases considered, we seek only necessary and sufficient conditions for the vanishing of the tensor field S.

We compute $\tilde{S}(i_*X, i_*Y)$, where $\tilde{S}$ is the tensor field of type (1, 2) on $M(\phi, E, \eta)$ whose vanishing means that M is normal:

$$\begin{aligned} \tilde{S}(x, y) &= [\phi, \phi](x, y) + d\eta(x, y)E \\ &= (\nabla_{\phi x}\phi)y - (\nabla_{\phi y}\phi)x - \phi\{(\nabla_x\phi)y - (\nabla_y\phi)x\} \\ &\quad + \{(\nabla_x\eta)(y) - (\nabla_y\eta)(x)\}E, \end{aligned}$$

where x and y are vector fields on M. Thus, putting $x = i_*X$ and $y = i_*Y$, then applying (2.5), (2.15) and the Gauss and Weingarten equations judiciously, we obtain after a rather long computation

$$\begin{aligned}\tilde{S}(i_*X, i_*Y) = i_*\{[f, f](X, Y) &+ d\eta'(X, Y)E' + d\alpha(X, Y)A \\ &- \alpha(Y)(Hf - fH)X + \alpha(X)(Hf - fH)Y\} \\ &+ \{(D_{fX}\alpha)(Y) - (D_{fY}\alpha)(X) \\ &- \alpha(D_Y(fY)) + \alpha(D_Y(fX)) \\ &+ \alpha(HX)\alpha(Y) - \alpha(HY)\alpha(X)\}N \\ &+ (\nabla_N\phi)(\alpha(X)i_*Y - \alpha(Y)i_*X).\end{aligned}$$

Case 1. $M(\phi, \eta, G)$ is cosymplectic. Then, since ϕ has vanishing covariant derivative with respect to the Riemannian connection,

$$([f, f] + d\eta' \otimes E' + d\alpha \otimes A)(X, Y) - \alpha(Y)(Hf - fH)X + \alpha(X)(Hf - fH)Y = 0.$$

THEOREM 9. *Let M be a cosymplectic manifold. Then, a necessary and sufficient condition that the framed f-structure P be normal is that $Hf - fH = \alpha \otimes v$, where v is the vector field $-D_A A$. If the structure on P is normal and the integral curves of the vector field A are geodesics, then H commutes with f. Conversely, if the structure on P is normal and H commutes with f, then the integral curves of A are geodesics.*

Case 2. $M(\phi, \eta, G)$ is Sasakian. Then,

$$(\nabla_N\phi)(\alpha(X)i_*Y - \alpha(Y)i_*X) = \{\alpha(X)\eta'(Y) - \alpha(Y)\eta'(X)\}N.$$

Consequently,

$$([f, f] + d\eta' \otimes E' + d\alpha \otimes A)(X, Y) - \alpha(Y)(Hf - fH)X + \alpha(X)(Hf - fH)Y = 0.$$

THEOREM 10. *Let M be a Sasakian manifold. Then, a necessary and sufficient condition that the framed f-structure P be normal is that $Hf - fH = \alpha \otimes v$, where v is the vector field $-D_A A$. If the structure on P is normal and the integral curves of the vector field A are geodesics, then H commutes with f. Conversely, if the structure on P is normal and H commutes with f, then the integral curves of A are geodesics.*

Theorems 9 and 10 may also be obtained by computing S directly from the relations (4.1) and (4.2).

In both cases, the f-structures are normal if A is a Killing field. For, then H commutes with f and $-v = D_A A = fHA = HfA = 0$.

6. Examples of Kaehler manifolds

In this section, we show that an almost complex structure may be defined on the globally framed hypersurface P. Moreover, if the ambient space is cosymplectic and P is totally geodesic, it is Kaehlerian. We also show that the direct product of two globally framed totally geodesic hypersurfaces of a cosymplectic manifold is Kaehlerian. Other examples are provided by the following theorem.

THEOREM 11. *Let M be a cosymplectic manifold and let P be a hypersurface of M with the induced quasi-symplectic globally framed f-structure. If P is complete and simply connected, and if it is totally geodesic, it is a product with one factor Kaehlerian.*

Proof. By (4.1), $Df = 0$, so DF also vanishes. Thus,

$$P'_p = \{X \in P_p \mid F(X, P_p) = 0\}$$

defines a parallel distribution. Therefore, the orthogonal complement P''_p (with respect to g) also gives a parallel distribution. Note that E'_p and A_p do not belong to P''_p. By the de Rham decomposition theorem $P = P' \times P''$ where $F = 0$ on P' and F has maximal rank on P''. By Proposition 6, P is normal, so by Proposition 4 and (4.1), $[f, f] = 0$. The almost complex structure on P'' obtained by restricting f to P'' is therefore integrable. Since F is closed, P'' is symplectic; in fact, since F has vanishing covariant derivative, P'' is a Kaehler manifold.

A generalization of Theorem 11 is obtained by generalizing Proposition 6.

PROPOSITION 12. *A sufficient condition that a hypersurface P of a cosymplectic manifold M have a quasi-symplectic normal globally framed f-structure is that its second fundamental form h be proportional to $\alpha \otimes \alpha$, that is $h = \mu\alpha \otimes \alpha$ where $\mu = h(A, A)$.*

The generalization of Proposition 6 is obtained by replacing $h = \lambda g$ by $h = \lambda g + \mu\alpha \otimes \alpha$.

It is now shown that a hypersurface of an almost contact manifold with the induced framed metric f-structure carries an almost hermitian structure. This is accomplished by 'twisting' the frames consisting of the vector fields E' and A. In fact, the following stronger statement is established.

THEOREM 13. *A hypersurface P of a cosymplectic manifold with the induced globally framed f-structure is Kaehlerian if its second fundamental form is proportional to $\alpha \otimes \alpha$.*

Proof. Put $\tilde{f} = f + \eta' \otimes A - \alpha \otimes E'$. Then, from (2.10)–(2.14), rank $\tilde{f} = 2n$. Moreover, $\tilde{f}$ defines an almost complex structure on P. Putting $\tilde{F}(X, Y) = g(\tilde{f}X, Y)$, we obtain

$$\tilde{F} = F + 2\,\eta' \wedge \alpha.$$

Thus, the hypersurface P has an almost hermitian structure $(\tilde{f}, g)$. Since the ambient space is a cosymplectic manifold, F and η' are closed forms. In addition by (4.1), since $h = \mu\alpha \otimes \alpha$, α is also closed, so $P(\tilde{f}, g)$ is almost

Kaehlerian. That it is a Kaehler manifold is a consequence of the fact that $[\tilde{f}, \tilde{f}]$ is zero. For, f, η' and α are covariant constant with respect to the Riemannian connection of P.

COROLLARY 1. *Let P be a hypersurface of a cosymplectic manifold with the induced globally framed f-structure. Then, if f has vanishing covariant derivative with respect to the Riemannian connection of the induced metric, P is Kaehlerian.*

Proof. By (4.1), $\alpha(Y)HX = h(X, Y)A$. Hence, $\alpha(Y)\alpha(HX) = h(X, Y)$, from which $h(X, fY) = 0$ by (2.12). Thus, $D\alpha = 0$.

COROLLARY 2. *A totally geodesic hypersurface P of a cosymplectic manifold with the induced globally framed f-structure is Kaehlerian.*

Observe that

$$(f - \eta' \otimes A + \alpha \otimes E', g)$$

is also a Kaehler structure on P as are

$$(-f + \eta' \otimes A - \alpha \otimes E', g) \quad \text{and} \quad (-f - \eta' \otimes A + \alpha \otimes E', g).$$

Moreover, the vector fields E and A generate one-parameter groups of automorphisms of the various Kaehler structures.

Let P be a manifold (not necessarily a hypersurface) with an f-structure of rank r. If we put $s = -f^2$ and $t = f^2 + I$, where I is the identity transformation field, $s + t = I$, $s^2 = s$, $t^2 = t$, $f^2 s = -s$ and $ft = 0$. Thus, the operators s and t acting in the tangent space at each point of P are complementary projection operators defining two distributions S and T in P corresponding to s and t, respectively. The distribution S is r-dimensional and $\dim T = m - r$, $m = \dim P$. The set of all tangent vectors belonging to the distribution T has a bundle structure, denoted by $V(P)$, which is a subbundle of the tangent bundle of P of dimension $2m - r$.

If there are $m - r$ vector fields E_a spanning the distribution T at each point of P, and $m - r$ linear differential forms η^a satisfying

$$\eta^a(E_b) = \delta^a_b, \tag{6.1}$$

$$f^2 = -I + \eta^a \otimes E_a, \tag{6.2}$$

where a and b range over the set $\{1, \ldots, m - r\}$, P is said to have a *framed f-structure* and P is then called a *framed f-manifold* or is said to be *globally framed*. From (6.1) and (6.2) one may easily obtain

$$fE_a = 0, \qquad \eta^a \circ f = 0. \tag{6.3}$$

(If the rank is maximal, that is, if $r = m$ when $m = 2n$ and $r = m - 1$ when $m = 2n - 1$, then $f^2 = -I$ in the former case, and $f^2 = -I + \eta^1 \otimes E_1$ when P is odd dimensional.)

For a globally framed f-manifold P, if we put

$$\tilde{f} = f + \eta^{2i} \otimes E_{2i-1} - \eta^{2i-1} \otimes E_{2i}, \qquad i = 1, \ldots, [(m-r)/2] \tag{6.4}$$

an almost complex structure tensor $\tilde{f}$ is defined if dim $P = 2n$, and an almost contact structure $(\tilde{f}, E_{2n-r-1}, \eta^{2n-r-1})$ if dim $P = 2n - 1$.

The framed f-manifold $P(f, E_a, \eta^a)$, $a = 1, \ldots, m - r$, is called a *framed metric f-manifold* if a Riemannian metric g on M is distinguished such that (i) $\eta^a = g(E_a, \cdot)$, $a = 1, \ldots, m - r$ and (ii) f is skew-symmetric with respect to g. It can be shown that a framed f-manifold carries a metric with these properties. We put $F(X, Y) = g(fX, Y)$ and call it the *fundamental* 2-*form* of the framed *for f-manifold.*

A framed metric f-manifold $P(f, \eta^a, g)$ is said to be *covariant constant* if the covariant derivatives (with respect to the Riemannian connection) of its structure tensors are zero.

An examination of the proof of Theorem 11 yields the following generalization.

THEOREM 14. *Let P be a complete covariant constant even dimensional globally framed f-manifold. Then, if P is simply connected there is a Kaehlerian submanifold whose dimension is rank f.*

Note that the vector fields E_a, $a = 1, \ldots,$ dim $P - r$, are orthogonal to the Kaehlerian factor.

Let P be a framed metric f-manifold of dimension $m = 2n$. Then, an almost complex structure $\tilde{f} = f + \eta^{2i} \otimes E_{2i-1} - \eta^{2i-1} \otimes E_{2i}$ is defined on P in terms of which the metric g is hermitian. Setting $\tilde{F}(X, Y) = g(\tilde{f}X, Y)$, we obtain

$$\tilde{F} = F + 2 \textstyle\sum_i \eta^{2i} \wedge \eta^{2i-1}. \tag{6.5}$$

If the fundamental 2-form F and the η^a are closed forms, the almost hermitian structure on P is almost Kaehlerian. It is Kaehlerian if $\tilde{f}$ has vanishing covariant derivative with respect to g, that is, if the structure tensors f and E_a are covariant constant with respect to the metric g. (In this case, the vector fields E_a, $a = 1, \ldots, 2n - r$, are infinitesimal automorphisms of the Kaehlerian structure.).

THEOREM 15. *A covariant constant even dimensional globally framed f-manifold carries a Kaehlerian structure.*

In the odd dimensional case the framed metric f-structure $P(f, \eta^a, g)$ gives rise to the almost contact metric structure $P(\tilde{f}, \eta^{2n-r-1}, g)$. For,

$$g(\tilde{f}X, \tilde{f}Y) = g(X, Y) - \eta^{2n-r-1}(X)\, \eta^{2n-r-1}(Y).$$

Set $\phi = \tilde{f}$, $E = E_{2n-r-1}$, $\eta = \eta^{2n-r-1}$ and $\Phi(X, Y) = g(\phi X, Y)$. Then

$$\Phi = F + 2 \textstyle\sum_i \eta^{2i} \wedge \eta^{2i-1}.$$

If the fundamental 2-form Φ and the 1-form η are closed, the almost contact structure on P is almost cosymplectic [4]. It is cosymplectic, if and only if, the almost contact structure is normal. Clearly, P cannot be a contact manifold.

THEOREM 16. *A covariant constant odd dimensional globally framed f-manifold carries a cosymplectic structure.*

Proof. Since f and the η^a have vanishing covariant derivatives with respect to the Riemannian connection of g, so does ϕ. Hence, the torsion

$$(D_{\phi X}\,\phi)Y - (D_{\phi Y}\,\phi)X + \phi(D_Y\,\phi)X - \phi(D_X\,\phi)Y + \{(D_X\,\eta)(Y) - (D_Y\,\eta)(X)\}E$$

vanishes, where D denotes covariant differentiation with respect to the Riemannian connection of g. Thus, $P(\phi, E, \eta)$ is normal.

Besides the twisted structures of Theorem 13, there are the almost complex manifolds provided by taking direct products.

THEOREM 17. *The direct product of the framed metric f-structures $P_i(f_i, \eta_i, \alpha_i, g_i)$, $i = 1, 2$, has a naturally induced almost complex structure J. If the f-structures are normal, then J is integrable, and conversely. If the P_i are totally geodesic hypersurfaces and the ambient space $M(\phi, \eta, G)$ is a cosymplectic manifold, then $P_1 \times P_2$ is Kaehlerian.*

Proof. For $X_i \in P_{ip_i}$ $i = 1, 2$, we put

$$J_{(p_1,p_2)}(X_1, X_2) = (f_1X_1 - \eta_2'(X_2)E_1' - \alpha_2(X_2)A_1,\, f_2X_2 + \eta_1'(X_1)E_2' + \alpha_1(X_1)A_2).$$

Then, it is easily checked that $J^2 = -I$ where $I(X_1, X_2) = (X_1, X_2)$. If the f-structures on P_i are normal, then the almost complex structure on $P_1 \times P_2$ is integrable (see [8]). The converse is obtained by employing the relations

$$[J, J](X_1 \times 0, Y_1 \times 0) = 0, \quad [J, J](0 \times X_2, 0 \times Y_2) = 0,$$

$$[J, J](0 \times X_2, Y_1 \times 0) = 0 \quad \text{and} \quad [J, J](X_1 \times 0, 0 \times Y_2) = 0$$

in the expression for $[J, J](X_1 \times X_2, Y_1 \times Y_2)$ where $X \times Y = (X, Y)$. Define a metric on $P_1 \times P_2$ by $g_1 + g_2$, where $g_j = i_j^*G$ is the metric induced on P_j by the almost contact metric G of the cosymplectic manifold $M(\phi, \eta, G)$. The 2-form Ω on $P_1 \times P_2$ given by

$$\Omega = (F_1, 0) + (0, F_2) + (\eta_1', 0) \wedge (0, \eta_2') + (\alpha_1, 0) \wedge (0, \alpha_2)$$

where $F_j = i_j^*\Phi$, $j = 1, 2$ are the fundamental forms of P_1 and P_2, respectively, is the Kaehler form of $P_1 \times P_2$. For, since the fundamental form Φ of M is closed, F_1 and F_2 are closed. Moreover, by (4.1), the η_i and α_i are closed, the latter following since the P_i are totally geodesic submanifolds.

Finally,

$$g(J(X_1 \times X_2),\ Y_1 \times Y_2) = \Omega(X_1 \times X_2,\ Y_1 \times Y_2).$$

COROLLARY 1. *Let P be a totally geodesic hypersurface of a cosymplectic manifold with the induced globally framed f-structure. Then, the direct product of P with itself is Kaehlerian.*

If the linear transformation fields f_1 and f_2 are covariant constant, then by Proposition 7, P_1 and P_2 are totally geodesic hypersurfaces. Thus, we obtain

COROLLARY 2. *Let P_1 and P_2 be hypersurfaces of a cosymplectic manifold with the induced globally framed f-structures $(f_i, \eta_i', \alpha_i, g_i)$. Then, if the $(1, 1)$ tensor fields f_i are covariant constant with respect to their Riemannian connections, $P_1 \times P_2$ is a Kaehler manifold.*

Theorem 17 may be improved by considering f-structures of rank $2n - 2$. Indeed, if M and N are spaces endowed with normal quasi-symplectic framed metric f-structures, then $M \times N$ possesses a Kaehlerian structure if the structure tensors are covariant constant.

7. Integrability of globally framed f-structures

We have seen that a totally geodesic hypersurface of a cosymplectic manifold with the induced globally framed f-structure is covariant constant. Replacing the condition that the hypersurface P be totally geodesic ($HX = 0$ for all X) by the condition that the vector field A is parallel ($fHX = 0$ for all X), we prove

THEOREM 18. *Let $P(f, E, A, g)$ be a globally framed f-manifold and suppose that E' and A are parallel vector fields. Then, the f-structure on P is normal if and only if there exists a symmetric affine connection with respect to which f is parallel and the connection γ of $V(P)$ is flat.*

Proof. Suppose there is a symmetric affine connection D' such that $D'f = 0$. Then, the tensor field $S = [f, f] + d\eta' \otimes E' + d\alpha \otimes A$ vanishes. Conversely, assume that S vanishes. In terms of the Riemannian connection D we define a new connection D' by $D'_X Y = D_X Y + T(X, Y)$ where

$$T(X, Y) = -\tfrac{1}{4}f\{D_X(fY) + D_Y(fX) - f(D_X Y + D_Y X)\}$$
$$-\tfrac{1}{4}\{D_{fY}(fX) - fD_{fY}X + fD_X(fY) + D_XY\}.$$

Then

$$\begin{aligned} 4(D'_X f)Y &= 4[(D_X f)Y + T(X, fY) - fT(X, Y)] \\ &= 2[\eta'(Y)fD_XE' + \alpha(Y)fD_XA] \\ &\quad - 3[\eta'((D_X f)Y)E' + \alpha((D_X f)Y)A] \\ &\quad + \eta'(Y)(D_{E'}f)X + \alpha(Y)(D_Af)X \\ &\quad - \eta'((D_Y f)X)E' - \alpha((D_Y f)X)A. \end{aligned}$$

From equations (4.1) and (2.12), we get

$$\eta'((D_X f)Y) = 0 \quad \text{and} \quad \alpha((D_X f)Y) = 0.$$

Moreover, $D_{E'}f$ and $D_A f$ vanish. For,

$$D_{E'}(fX) = D_{fX}E' + [E', fX] = f[E', X]$$

(see §3). Hence,

$$(D_{E'}f)X = -fD_{E'}X + f[E', X] = f(D_{E'}X - D_X E' - D_{E'}X) = 0$$

since $DE' = 0$. Similarly, $D_A f$ is zero. We conclude then that $D'_X f = 0$.

On the other hand,

$$D'_X Y - D'_Y X - [X, Y] = D_X Y - D_Y X - [X, Y] + T(X, Y) - T(Y, X)$$
$$= \tfrac{1}{4}\{[fX, fY] - f[fX, Y] - f[X, fY] + f^2[X, Y]\} = 0$$

since η' and α are closed.

A similar result for hermitian manifolds was obtained by Walker [9].

8. Topology of globally framed f-spaces

Let $P(f, \eta^a, g)$, $a = 1, \ldots, \dim P - r$, be a covariant constant even dimensional globally framed f-manifold, henceforth called a *K-manifold.* As we have already seen $P(\tilde{f}, g)$ is a Kaehler manifold. If P is compact its topology can therefore be studied from several points of view. In the first instance, as a compact Kaehler manifold and secondly, by introducing a theory on $P(f, \eta^a, g)$ analogous to Weil's generalization of Hodge theory on algebraic varieties. Whereas $\tilde{F}$ is the Kaehler 2-form, F plays that role in the latter theory. A parallel study may be carried out for odd dimensional spaces but this is omitted here. If $r = \dim P$, the manifold is Kaehlerian and the ensuing theory is well-known.

Added in proof. The proper generalization along these lines has been given by one of the authors in a paper to appear in the Journal of Differential Geometry, dedication volume in honour of S. S. Chern and D. C. Spencer.

In the sequel, we assume that P is connected. If in addition P is compact, then since the induced structure is Kaehlerian, we have

THEOREM 19. *The p-th betti number of a compact K-manifold is even if p is odd and the even dimensional betti numbers are different from zero. Moreover, with respect to the induced complex structure the* η^{2i-1}, $i = 1, \ldots, n - r/2$, *are holomorphic differentials, so the first betti number exceeds* $\dim P - r$.

Remark. The operator C on p-forms defined by

$$C_f\alpha(X_1, \cdots, X_p) = \alpha(fX_1, \cdots, fX_p)$$

annihilates the harmonic differentials η^a, $a = 1, \cdots, 2n - r$. Moreover, if P is compact, it can be shown that C maps harmonic forms into harmonic forms.

Define dual operators L and Λ on P by $L = \epsilon(F)$ and $\Lambda = \iota(F)$ where ϵ and ι are respectively the exterior and interior product operators. A p-form ($p \geq 2$) is called *effective* if it is annihilated by Λ. For $p = 0$ or 1 every form is said to be effective. Since $\iota(F) = (-1)^p * \epsilon(F) *$ on p-forms where $*$ is the Hodge star isomorphism, $\Lambda = (-1)^p *L*$.

An orthonormal basis of P_p of the form $\{X_A, X_{A^*}, E_a\}$, $A = 1, \cdots, r/2$, $X_{A^*} = fX_A$; $a = 1, \cdots, 2n - r$, $\dim P = 2n$, will be called an *f-basis*. Such a basis always exists. To see this, let $P_p = \{X \in P_p \mid g(X, E_a) = 0, a = 1, \cdots, 2n - r\}$. Equations (6.1)–(6.3) and $\eta^a = g(E_a, \cdot)$ show that $f|_{P'_p}$ is an almost complex structure on P'_p and $g|_{P'_p}$ is an hermitian metric. If an orthonormal basis of P'_p of the form $\{X_A, f|_{P'_p} X_A\}$, $A = 1, \cdots, r/2$, is then chosen, an f-basis of P_p is obtained.

In terms of an f-basis $\{X_A, X_{A^*}, E_a\}$ with dual basis $\{\omega_A, \omega_{A^*}, \eta^a\}$,

$$L = \sum_{A=1}^{r/2} \epsilon(\omega_A)\epsilon(\omega_{A^*}), \qquad \Lambda = \sum_{A=1}^{r/2} \iota(X_{A^*})\iota(X_A).$$

Since $\iota(X_A)$ is an anti-derivation, $\Lambda F = r/2$.

A p-form α on P is said to have *tridegree* (λ, μ, ν) if it is expressible as a sum of decomposable forms

$$\omega_{A_1} \wedge \cdots \wedge \omega_{A_\lambda} \wedge \omega_{B_1^*} \wedge \cdots \wedge \omega_{B_\mu^*} \wedge \eta^{a_1} \wedge \cdots \wedge \eta^{a_\nu}.$$

If

$$\alpha = \omega_{A_1} \wedge \cdots \wedge \omega_{A_\lambda} \wedge \omega_{B_1^*} \wedge \cdots \wedge \omega_{B_\mu^*} \wedge \eta^{a_1} \wedge \cdots \wedge \eta^{a_\nu},$$

we shall denote by α_h the 'horizontal' part

$$\omega_{A_1} \wedge \cdots \wedge \omega_{A_\lambda} \wedge \omega_{B_1^*} \wedge \cdots \wedge \omega_{B_\mu^*}$$

and by α_v the 'vertical' part

$$\eta^{a_1} \wedge \cdots \wedge \eta^{a_\nu}.$$

Thus, $\alpha = \alpha_h \wedge \alpha_v$. Clearly,

$$\Lambda(\alpha_h \wedge \alpha_v) = \Lambda\alpha_h \wedge \alpha_v . \tag{8.1}$$

LEMMA 1. *On a framed metric f-manifold, L and Λ satisfy*

$$\Lambda L\alpha - L\Lambda\alpha = (r/2 + \nu - p)\alpha$$

for any p-form α of tridegree (λ, μ, ν).

Proof. By linearity it suffices to consider the decomposable forms α_h and $\alpha_h \wedge \alpha_v$. The result then follows from formula (8.1) and the corresponding relation for almost hermitian spaces.

We shall require the following operators:

$$d' = \sum_A \epsilon(\omega_A)D_{X_A}, \qquad d'' = \sum_A \epsilon(\omega_{A^*})D_{X_{A^*}}, \qquad d^0 = \sum_a \epsilon(\eta^a)D_{E_a},$$

$$\delta' = -\sum_A \iota(X_A)D_{X_{A^*}}, \quad \delta'' = -\sum_A \iota(X_{A^*})D_{X_A}, \quad \delta^0 = -\sum_a \iota(E_a)D_{E_a},$$

$A = 1, \cdots, r/2; a = 1, \cdots, 2n - r$. Then,

$$d = d' + d'' + d^0 \quad \text{and} \quad \delta = \delta' + \delta'' + \delta^0.$$

LEMMA 2. *On a K-manifold*

$$\delta' L - L\delta' = -d'', \qquad \delta'' L - L\delta'' = d',$$
$$\delta^0 L = L\delta^0, \qquad \delta L - L\delta = d' - d''.$$

Proof. Since L commutes with D, $\epsilon(\eta^a)$ and $\iota(E_a)$, the proof is a computation similar to the corresponding one for Kaehler manifolds. It is important to note the role played by $DF = 0$.

LEMMA 3. *On a K-manifold*

$$d'd' = 0, \qquad d'd'' + d''d' = 0,$$
$$d''d'' = 0, \qquad d^0 d' + d'd^0 = 0,$$
$$d^0 d^0 = 0, \qquad d^0 d'' + d'' d^0 = 0.$$

Proof. Since $dd = 0$, the result follows by comparing tridegrees.

LEMMA 4. *On a K-manifold*

$$dL = Ld, \qquad \Lambda\delta = \delta\Lambda,$$
$$d'L = Ld', \qquad d''L = Ld'', \qquad d^0 L = Ld^0,$$
$$\Lambda\delta' = \delta'\Lambda, \qquad \Lambda\delta'' = \delta''\Lambda, \qquad \Lambda\delta^0 = \delta^0\Lambda.$$

Several interesting consequences may be derived from Lemmas 2 and 4. To begin with, we have

LEMMA 5. *On a K-manifold*

$$d'\delta'' + \delta''d' = 0, \qquad d''\delta' + \delta'd'' = 0.$$

Proof. Immediate from Lemma 2 and the dual of Lemma 3.

LEMMA 6. *On a K-manifold*

$$d'\delta' + \delta'd' = d''\delta'' + \delta''d''.$$

Proof. From Lemma 2, the expression $\delta''L\delta' - \delta'L\delta'' + L\delta'\delta'' - \delta''\delta'L$ is equal to $d''\delta'' + \delta''d''$ from the first relation and to $d'\delta' + \delta'd'$ from the second.

Let $\wedge_h^p$ denote the linear space of horizontal p-forms.

LEMMA 7. *On a K-manifold*

$$\Delta \mid_{\wedge_h^p} = 2(d'\delta' + \delta'd')\mid_{\wedge_h^p} = 2(d''\delta'' + \delta''d'')\mid_{\wedge_h^p}.$$

LEMMA 8. *On a K-manifold*

$$\Delta L = L\Delta, \qquad \Delta\Lambda = \Lambda\Delta.$$

Hence, L and Λ send harmonic forms into harmonic forms.

Proof. Apply Lemmas 2–4. That $\Delta\Lambda = \Lambda\Delta$ is a consequence of the fact that $*$ commutes with Δ.

LEMMA 9. *On a K-manifold the forms $F^p = F \wedge \cdots \wedge F$ (p times) are harmonic of degree $2p$ for every integer $p \leqq r/2$.*

LEMMA 10. *On a framed metric f-manifold*

$$(\Lambda L^k - L^k\Lambda)\alpha = k(r/2 + \nu - p - k + 1)L^{k-1}\alpha$$

for any p-form α of tridegree (λ, μ, ν), $p \leqq r/2 + \nu - 2k + 2$.

Proof. By recursion on the integer k using Lemma 1.

THEOREM 20. *On a framed metric f-manifold a p-form α of tridegree (λ, μ, ν), $p \leqq r/2 + \nu + 1$, may be uniquely expressed as a sum*

$$\alpha = \textstyle\sum_{k=0}^{s} L^k\psi_{p-2k}, \tag{8.2}$$

where the ψ_{p-2k} are effective forms of degree $p - 2k$ and $s = [p/2]$.

Proof. The theorem is trivial for $p = 0, 1$. Proceeding inductively, assume its validity for $p \leqq r/2 + \nu - 2$. Then, to any p-form β is associated a unique p-form α such that

$$\Lambda L\alpha = \beta, \qquad p \leqq r/2 + \nu - 1. \tag{8.3}$$

For,

$$\beta = \textstyle\sum_{k=0}^{s} L^k\theta_{p-2k}$$

where the θ_{p-2k} are effective. By (8.2) and Lemma 10

$$\begin{aligned}\Lambda L\alpha &= \textstyle\sum_{k=0}^{s} \Lambda L^{k+1}\psi_{p-2k} \\ &= \textstyle\sum_{k=0}^{s} (k+1)(r/2 + \nu - p + k)L^k\psi_{p-2k}.\end{aligned}$$

Since $p \leqq r/2 + \nu - 1$, $r/2 - p + \nu + k \neq 0$, so, in order that (8.3) hold, it is sufficient to take

$$\psi_{p-2k} = \frac{\theta_{p-2k}}{(k+1)\,(r/2 + \nu - p + k)}, \qquad k = 0, 1, \cdots, s.$$

By uniqueness, this is also necessary. The rest of the proof is omitted.

Denote by $\wedge^{\lambda,\mu,\nu}$ the linear space of forms of tridegree (λ, μ, ν).

COROLLARY 1. *On a framed metric f-manifold, ΛL is an automorphism of $\wedge^{\lambda,\mu,\nu}$ for $p \leqq r/2 + \nu - 1$ and of $\wedge^p$ for $p \leqq r/2 - 1$.*

COROLLARY 2. *On a framed metric f-manifold, L is an isomorphism of $\wedge^{\lambda,\mu,\nu}$ into $\wedge^{\lambda+1,\mu+1,\nu}$ for $p \leqq r/2 + \nu - 1$ and of $\wedge^p$ into $\wedge^{p+2}$ for $p \leqq r/2 - 1$.*

Since Δ commutes with L and Λ on a K-manifold, we obtain

COROLLARY 3. *On a K-manifold, a harmonic p-form α of tridegree (λ, μ, ν), $p \leqq r/2 + \nu + 1$, may be unqiuely expressed as a sum*

$$\alpha = \sum_{k=0}^{s} L^k \psi_{p-2k},$$

where the ψ_{p-2k} are effective harmonic forms of degree $p - 2k$ and $s = [p/2]$.

COROLLARY 4. *The betti numbers b_p of a compact K-manifold satisfy the monotonicity condition $b_p \leqq b_{p+2}$, $p \leqq r/2 - 1$. Moreover, $b_q \neq 0$ for all q.*

Proof. The first part follows from Lemma 8 and Corollary 2, whereas the the second half is a consequence of Lemma 9 and the fact that the η^a, $a = 1, \cdots, 2n - r$ are harmonic forms.

The difference $b_p - b_{p-2}$ may be measured in terms of the dimension e_p of the space of effective harmonic forms of degree p, $p \leqq r/2 + 1$. For, by Corollary 3

$$\wedge_H^p = \wedge_{He}^p \oplus L \wedge_{He}^{p-2} \oplus \cdots \oplus L^s \wedge_{He}^{p-2}, \qquad s = [p/2],$$

where $\wedge_H^p$ and $\wedge_{He}^p$ denote the linear spaces of harmonic and effective harmonic p-forms, respectively. Hence, $\wedge_H^{p+2} = \wedge_{He}^{p+2} \oplus L\wedge_H^p$. By Lemma 8 and Corollary 2, $\dim L\wedge_H^p = \dim \wedge_H^p$ from which $b_{p+2} = e_{p+2} + b_p$, $p \leqq r/2 - 1$.

THEOREM 21. *On a compact K-manifold*

$$e_p = b_p - b_{p-2}, \qquad p \leqq r/2 + 1.$$

Since the $\eta^a \wedge \eta^b$, $a \neq b$ are effective harmonic forms of degree 2, we obtain

COROLLARY 1. *On a compact K-manifold*

$$b_2 \geqq 1 + (2n - r)(2n - r - 1)/2.$$

Let $\tilde{e}_p$ denote the dimension of the space of effective harmonic p-forms on the Kaehler manifold $P(\tilde{f}, g)$. If a and b are of opposite parity, the harmonic 2-forms $\eta^a \wedge \eta^b$, $a \neq b$, are effective with respect to the operator Λ, but not in the Kaehler metric. To see this, let $\tilde{L}$ and $\tilde{\Lambda}$ be the operators of Hodge-Weil on $P(\tilde{f}, g)$. Then, by (6.5), since $\tilde{\Lambda} = (-1)^p * \tilde{L} *$ on p-forms,

$$\tilde{\Lambda} = \Lambda - 2 \sum \iota(E^{2i-1})\iota(E^{2i}),$$

from which $\tilde{\Lambda}(\eta^a \wedge \eta^b) = -2\sum (\delta_{2i-1}^b \delta_{2i}^a - \delta_{2i}^b \delta_{2i-1}^a)$. However, we do have the following result.

COROLLARY 2. *On a compact K-manifold*

$$\tilde{e}_p = e_p, \qquad p \leqq r/2 + 1.$$

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