PROJECTIVE DIAGRAMS OF INTERLOCKING SEQUENCES

BY

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1. Introduction

A complex over an abelian category $\mathcal{A}$ with enough projectives is projective as an object in the category of complexes precisely when the kernel of each of its (boundary) maps is projective and the homology of the complex is zero in each dimension. This paper shows that the projective objects in a more general category of diagrams of interlocking sequences over $\mathcal{A}$ are those diagrams with the kernel of each map in the diagram projective and with each of the sequences exact.

Let N be a fixed integer greater than 2. Denote by $\mathcal{R}$ the pointed category whose objects are pairs (s, t) of integers satisfying $s - N < t < s$, whose hom-sets are given by

$$\begin{aligned} \mathcal{R}((s, t), (u, v)) &\cong Z_2 \quad \text{when } u - N < t \leq v < s \leq u \\ &\cong \{0\} \quad \text{otherwise,} \end{aligned}$$

and whose composition

$$\mathcal{R}((u, v), (w, x)) \otimes \mathcal{R}((s, t), (u, v)) \rightarrow \mathcal{R}((s, t), (w, x))$$

is the isomorphism $Z_2 \otimes Z_2 \rightarrow Z_2$ when all three hom-sets are Z_2, and zero otherwise. (Here $\otimes$ denotes the coproduct of pointed sets.)

For $(s, t), (u, v) \in \mathcal{R}$, $t \leq v$, $s \leq u$, the symbol $(s, t; u, v)$ will denote the non-zero element of $\mathcal{R}((s, t), (u, v))$ when it has one, and the zero element otherwise; the rule of composition in $\mathcal{R}$ may be expressed by

$$(u, v; w, x) \cdot (s, t; u, v) = (s, t; w, x).$$

Put $\delta = (s, t; u, v)$. The integer $l_\delta = u - s + v - t$ is called the *length* of δ. If $l_\delta \geq N - 1$ then δ is zero. If $l_\delta = N - 2$ then δ is non-zero precisely when $u = N + t - 1$, $v = s - 1$. If $l_\delta = 0$ then $u = s$, $v = t$. If $l_\delta = 1$ then δ is non-zero and either $u = s + 1$, $v = t$ or $u = s$, $v = t + 1$. If δ is non-zero (i.e. $u - N < t \leq v < s \leq u$) then we also define integers $m_\delta = s - v - 1$ and $n_\delta = N - l_\delta - 2$. Notice $0 \leq m_\delta \leq n_\delta \leq N - 2$.

An $\mathcal{R}$-*sequence* is a diagram in $\mathcal{R}$ of the form

$$(u, v) \xrightarrow{(u, v; t, v)} (t, v) \xrightarrow{(t, v; t, u)} (t, u)$$

or

$$(t, v) \xrightarrow{(t, v; t, u)} (t, u) \xrightarrow{(t, u; v + N, u)} (v + N, u)$$

for $v < u < t < v + N$.

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The following diagram illustrates the maps of $\mathfrak{R}$ of length 1 in the case $N = 5$.

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & {1, 0} \arrow[dr] & & {2, 1} \arrow[dr] & & {3, 2} \arrow[dr] & & {4, 3} \arrow[dr] & & {5, 4} \arrow[dr] & \\
{} \arrow[ur] \arrow[dr] & & {2, 0} \arrow[ur] \arrow[dr] & & {3, 1} \arrow[ur] \arrow[dr] & & {4, 2} \arrow[ur] \arrow[dr] & & {5, 3} \arrow[ur] \arrow[dr] & & {} \\
 & {2, -1} \arrow[ur] \arrow[dr] & & {3, 0} \arrow[ur] \arrow[dr] & & {4, 1} \arrow[ur] \arrow[dr] & & {5, 2} \arrow[ur] \arrow[dr] & & {6, 3} \arrow[ur] \arrow[dr] & \\
{} \arrow[ur] & & {3, -1} \arrow[ur] & & {4, 0} \arrow[ur] & & {5, 1} \arrow[ur] & & {6, 2} \arrow[ur] & & {}
\end{tikzcd}
$$

The category of diagrams of interlocking sequences we wish to consider is the category $\mathfrak{D}$ of pointed functors from $\mathfrak{R}$ to $\mathfrak{A}$ and natural transformations. A pointed functor $D : \mathfrak{R} \to \mathfrak{A}$ will be called *exact* if it takes each $\mathfrak{R}$-sequence into an exact sequence of $\mathfrak{A}$.

The aim of this paper is to prove the following result.

THEOREM 1. *The pointed functor $D : \mathfrak{R} \to \mathfrak{A}$ is a projective object of $\mathfrak{D}$ if and only if it is exact and the kernel of each of the maps*

$$D(s, t; u, v) : D(s, t) \to D(u, v)$$

is a projective object of $\mathfrak{A}$.

Motivation for considering the category $\mathfrak{D}$ may be provided by the following. Let A be a complex over $\mathfrak{A}$, and let

$$0 = A^0 \leq A^1 \leq \cdots \leq A^{N-1} = A$$

be a filtration of A of length $N - 1$. An object D of $\mathfrak{D}$ may now be defined. For any integer r and for integers p, q such that $0 \leq q < p < N$, put

$$D(p - rN, q - rN) = H_{2r}(A^p/A^q)$$

and

$$D(q - (r - 1)N, p - rN) = H_{2r-1}(A^p/A^q).$$

This determines $D(s, t)$ uniquely for all $(s, t) \in \mathfrak{R}$. It is a routine matter to check that the inclusions of the filtration and the boundary maps of A induce maps $D(s, t; u, v) : D(s, t) \to D(u, v)$ in $\mathfrak{A}$ as required to make D an object of $\mathfrak{D}$. Moreover, D takes the $\mathfrak{R}$-sequences to portions of the long exact homology sequences of the short exact sequences

$$0 \to A^y/A^z \to A^x/A^z \to A^x/A^y \to 0$$

of complexes over $\mathfrak{A}$. So D is exact. Such diagrams appear in [W, p. 98].

2. One way is easy

Let $E_{yz} : \mathfrak{D} \to \mathfrak{A}$ be the functor which evaluates at $(y, z) \in \mathfrak{R}$. The left adjoint J_{yz} of E_{yz} exists and takes an object A of $\mathfrak{A}$ to the functor $J_{yz}\, A : \mathfrak{R} \to \mathfrak{A}$ as follows.

(*a*)

$$
\begin{aligned}
(J_{yz}\, A)(s, t) &= A \text{ when } s - N < z \leq t < y \leq s, \\
&= 0 \text{ otherwise;}
\end{aligned}
$$

(b) for $u - N < t \leq v < s \leq u$, the map

$$(J_{yz} A)(s, t; u, v) : (J_{yz} A)(s, t) \to (J_{yz} A)(u, v)$$

is the identity of A when $s - N < z \leq t < y \leq s$, $u - N < z \leq v < y \leq u$, and zero otherwise.

Suppose for each $(y, z) \in \mathcal{R}$ an object $A(y, z)$ of $\mathcal{A}$ is given. The coproduct

$$D = \sum\nolimits_{y-N<z<y} J_{yz} A(y, z)$$

exists in $\mathfrak{D}$; in fact, it is given as follows.

(a) $D(s, t) = \sum_{s-N<z\leq t<y\leq s} A(y, z)$ (a finite direct sum);

(b) for $u - N < t \leq v < s \leq u$, the map

$$D(s, t; u, v) : D(s, t) \to D(u, v)$$

corresponds to the matrix whose typical element $A(y, z) \to A(y', z')$ is the identity map when $y = y'$, $z = z'$, and is the zero map otherwise.

$\mathfrak{D}$ is an abelian category with exact sequences those sequences which go to exact sequences in $\mathcal{A}$ under all the functors E_{yz}, $(y, z) \in \mathcal{R}$. The theory of [EM, Ch. II §3] caters for this situation.

LEMMA 2. *The pointed functor $D : \mathcal{R} \to \mathcal{A}$ is a projective object of $\mathfrak{D}$ if and only if it is a retract of an object of the form*

$$\sum\nolimits_{y-N<z<y} J_{yz} A(y, z),$$

where $A(y, z)$ is a projective object of $\mathcal{A}$ for $y - N < z < y$. ▌

PROPOSITION 3. *If D is a projective object of $\mathfrak{D}$, then it is exact and the kernel of each of the maps $D(s, t; u, v)$ is a projective object of $\mathcal{A}$.*

PROOF. The properties in question are preserved by retracts. The direct sum of a finite number of exact sequences in $\mathcal{A}$ is exact, and so by Lemma 2 it suffices to prove the proposition for $D = J_{yz} A$, where A is projective and $(y, z) \in \mathcal{R}$. The kernel of any map in D is either 0 or A, and so projective. Suppose

$$(s, t) \to (u, v) \to (w, x)$$

is an $\mathcal{R}$-sequence. If $u - N < z \leq v < y \leq u$ then it is readily checked that $s - N < z \leq t < y \leq s$ is true precisely when $w - N < z \leq x < y \leq w$ is false (in fact, this property, for all $(y, z) \in \mathcal{R}$, characterizes the $\mathcal{R}$-sequences —but this is not needed); so D takes the $\mathcal{R}$-sequence to either

$$A \xrightarrow{1} A \to 0 \quad \text{or} \quad 0 \to A \xrightarrow{1} A,$$

each of which is exact. Otherwise $D(u, v) = 0$, and so D necessarily takes the $\mathcal{R}$-sequence into an exact sequence. ▌

3. The necessary machinery

Let $\mathcal{K}$ denote the pointed category with three objects -1, 0, 1 and with only two non-zero non-identity maps $-1 \to 0$, $0 \to 1$. Let $\mathcal{K}_n$ denote the tensor product (as pointed categories) of n (≥ 0) copies of $\mathcal{K}$. The object of $\mathcal{K}_0$ will be denoted by 1; the objects of $\mathcal{K}_n$ for $n > 0$ are the functions

$$a : \Omega_n = \{1, 2, \cdots, n\} \to \{-1, 0, 1\}.$$

In particular, we let $o \in \mathcal{K}_n$ denote the function from Ω_n to $\{-1, 0, 1\}$ given by $o(i) = 0$ for all $i \in \Omega_n$. An *n-corner* is an object a of $\mathcal{K}_n$ such that $a(i) \neq 0$ for all $i \in \Omega_n$. For $i \in \Omega_n$ and $\lambda \in \{-1, 0, 1\}$, let $T_i^\lambda : \mathcal{K}_{n-1} \to \mathcal{K}_n$ be the pointed functor which takes $a \in \mathcal{K}_{n-1}$ to $T_i^\lambda a \in \mathcal{K}_n$ given by

$$\begin{aligned} (T_i^\lambda a)(j) &= a(j) && \text{for } 1 \leq j < i, \\ &= \lambda && \text{for } j = i, \\ &= a(j-1) && \text{for } i < j \leq n. \end{aligned}$$

The maps $-1 \to 0$, $0 \to 1$ in the *i-th* copy of $\mathcal{K}$ in $\mathcal{K}_n$ give a sequence

$$T_i^{-1} \to T_i^0 \to T_i^1$$

of natural transformations for each $i \in \Omega_n$.

The category of pointed functors $F : \mathcal{K}_n \to \mathcal{A}$ and natural transformations will be denoted by $\mathfrak{F}_n$. An object F of $\mathfrak{F}_n$ is called an *n-dimensional three-diagram* in $\mathcal{A}$. We say F is *exact* if, for each $i \in \Omega_n$, the sequence

$$0 \to FT_i^{-1} \to FT_i^0 \to FT_i^1 \to 0$$

is exact in $\mathfrak{F}_{n-1}$. An exact 1-dimensional three-diagram in $\mathcal{A}$ is a *short exact sequence* in $\mathcal{A}$. We say $F \in \mathfrak{F}_n$ *takes projective values* if, for each $a \in \mathcal{K}_n$, Fa is a projective object of $\mathcal{A}$.

PROPOSITION 4. *If F is an exact n-dimensional three-diagram which takes projective values, then F is a projective object of $\mathfrak{F}_n$.*

Proof. If $n = 1$ then F is the direct sum of the two short exact sequences

$$0 \to F(-1) \to F(-1) \to 0 \to 0$$

and

$$0 \to 0 \to F(1) \to F(1) \to 0$$

which are easily checked to be projective in $\mathfrak{F}_1$; so F is. For $n > 1$ an exact n-dimensional three-diagram in $\mathcal{A}$ is a short exact sequence in $\mathfrak{F}_{n-1}$, so the result follows by induction. ▮

For $a, b \in \mathcal{K}_n$, $a \leq b$ means $a(i) \leq b(i)$ for all $i \in \Omega_n$. Let

$$Sa = \{c \mid c \text{ is an } n\text{-corner and } c(i) = a(i) \text{ when } a(i) \neq 0\}.$$

Let $S = So$ be the set of n-corners.

Remark 5. If $a \leq a'$, $c \in Sa$, $c' \in Sa'$ and $c' \leq c$, then $c \in Sa' \Leftrightarrow c' \in Sa$. ▮

Remark 6. $ST_i^0 a$ is the disjoint union of $ST_i^{-1}a$ and $ST_i^1 a$, for $1 \leq i \leq n$ and $a \in \mathcal{K}_{n-1}$.

Suppose F is a function which assigns to each n-corner c an object Fc of $\mathcal{A}$. Define $\sum F \in \mathfrak{F}_n$ by

(i) $(\sum F)a = \sum_{c \in S_a} Fc$ for $a \in \mathcal{K}_n$;
(ii) for $a, a' \in \mathcal{K}_n$ such that $a \leq a'$, $(\sum F)a \to (\sum F)a'$ is the map corresponding to the matrix with typical element $Fc \to Fc'$ the identity map when $c = c'$, and the zero map otherwise.

It follows from Remark *6*, that $\sum F$ is an exact n-dimensional three-diagram. In fact,

$$(\textstyle\sum F)T_i^0 = (\sum F)T_i^{-1} \oplus (\sum F)T_i^1.$$

Remark 7. *If* $F \in \mathfrak{F}_n$, $1 \leq i \leq n$ and $-1 \leq \lambda \leq 1$, then $\sum (FT_i^\lambda) = (\sum F)T_i^\lambda$. ▮

A *splitting* of $F \in \mathfrak{F}_n$ is an isomorphism $f : F \to \sum F$ in $\mathfrak{F}_n$ such that f_c is the identity of Fc for each n-corner c. If such an f exists then we say F *splits*.

Proposition 8. *Exact n-dimensional three-diagrams, which take projective values, split.*

Proof. Suppose

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

is exact, C is projective and $f : A \to A'$, $g : C \to C'$ are maps, all in some abelian category. Let r be a left inverse of i. Then the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[r, "p"] \arrow[d, "\binom{fr}{gp}"'] & C \arrow[d, "g"] \\
A' \arrow[r, "\binom{1}{0}"'] & A' \oplus C' \arrow[r, "{(0,\,1)}"'] & C'.
\end{tikzcd}$$

For $n = 1$ the result follows from the above by taking $A' = A, C' = C, f = 1_A$, $g = 1_B$. We prove the proposition by induction on n. Suppose the result is true for $n - 1$ where $n > 1$. Let F be an exact n-dimensional three-diagram taking projective values. Then FT_1^λ is exact and takes projective values. By Proposition 4, FT_1^λ is projective in $\mathfrak{F}_{n-1}$. The sequence

$$0 \to FT_1^{-1} \to FT_1^0 \to FT_1^1 \to 0$$

is exact in $\mathfrak{F}_{n-1}$. By induction there exist splittings

$$f : FT_1^{-1} \to \textstyle\sum (FT_1^{-1}), \qquad g : FT_1^1 \to \sum (FT_1^1).$$

By the above, and using Remark 7, there exists a commutative diagram

$$\begin{tikzcd} FT_1^{-1} \arrow[r] \arrow[d, "f"'] & FT_1^0 \arrow[r] \arrow[d, "h"'] & FT_1^1 \arrow[d, "g"'] \\ (\sum F)T_1^{-1} \arrow[r] & (\sum F)T_1^0 \arrow[r] & (\sum F)T_1^1; \end{tikzcd}$$

that is, there exists an isomorphism (h is an isomorphism by the short five-lemma) $k : F \to \sum F$ which agrees with f at points in the image of T_1^{-1}, and agrees with g at points in the image of T_1^1. Each n-corner is either in the image of T_1^{-1} or in the image of T_1^1, so k is the identity at each n-corner. So F splits. ▌

PROPOSITION 9. *Suppose F, F' are functions which assign an object of $\mathcal{A}$ to each n-corner. Then $\mathfrak{F}_n(\sum F, \sum F')$ is isomorphic to the additive group of matrices $\sum_{c \in S} Fc \to \sum_{c \in S} F'c$ with elements $Fc \to F'c'$ zero whenever $c' \nleq c$; the isomorphism is given by*

$$f = (f_a) \longmapsto f_o .$$

Proof. Suppose $f : \sum F \to \sum F'$; we show that f_o is such a matrix. If c, c' are n-corners and $c' \nleq c$, then there exists $j \in \Omega_n$ such that $c(j) = -1$, $c'(j) = 1$, so $c \in ST_j^{-1}o$ and $c' \in ST_j^1 o$. Thus $Fc \to F'c'$, induced by f_o, factors as

$$Fc \to (\textstyle\sum F)T_j^{-1}o \to (\sum F)o \xrightarrow{f_o} (\sum F')o \to (\sum F')T_j^1 o \to F'c';$$

but this is zero since the following diagram commutes:

$$\begin{tikzcd} (\sum F)T_j^{-1}o \arrow[r] \arrow[d, "f_{T_j^{-1}o}"'] & (\sum F)o \arrow[r] \arrow[d, "f_0"'] & (\sum F)T_j^1 o \arrow[d, "f_{T_j^1 o}"] \\ (\sum F')T_j^{-1}o \arrow[r] & (\sum F')o \arrow[r] & (\sum F')T_j^1 o. \end{tikzcd}$$

Suppose a map $g(c, c') : Fc \to Fc'$ in $\mathcal{A}$ is given for each pair c, c' of n-corners with $c' \leq c$. For $a \in \mathcal{K}_n$, let

$$f_a : (\textstyle\sum F)a \to (\sum F')a$$

be the matrix $\sum_{c \in Sa} Fc \to \sum_{c \in Sa} F'c$ with typical element $Fc \to F'c'$ equal to $g(c, c')$ when $c' \leq c$, and equal to the zero map otherwise. From Remark 5 it follows that, for $a \leq a'$, the following square commutes:

$$\begin{tikzcd} \sum_{c \in Sa} Fc \arrow[r] \arrow[d, "f_a"'] & \sum_{c \in Sa'} Fc \arrow[d, "f_a'"] \\ \sum_{c \in Sa} F'c \arrow[r] & \sum_{c \in Sa'} F'c, \end{tikzcd}$$

where the rows are the maps of the diagrams $\sum F$, $\sum F'$. Thus

$$f = (f_a) : \textstyle\sum F \to \sum F'$$

is a well-defined map of $\mathfrak{F}_n$ such that $f_0 = (g(c, c'))$. So $f \longmapsto f_0$ is surjective.

Suppose $f = (f_a) : \sum F \to \sum F'$ is a map with $f_o = 0$. We show that

$f \mapsto f_o$ is an isomorphism by proving that $f_a = 0$ for each $a \in \mathcal{K}_n$. Take $a \in \mathcal{K}_n$, and define $b \in \mathcal{K}_n$ by

$$\begin{aligned} b(i) &= a(i) \quad \text{when } a(i) \neq -1, \\ &= 0 \qquad \text{when } a(i) = -1. \end{aligned}$$

Then $Sa \leq Sb \leq So$, and the following squares commute:

$$\begin{array}{ccc} (\sum F)o & \rightarrow & (\sum F)b \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle f_b} \\ (\sum F')o & \rightarrow & (\sum F')b \end{array} \qquad \begin{array}{ccc} (\sum F)a & \rightarrow & (\sum F)b \\ {\scriptstyle f_a}\downarrow & & \downarrow{\scriptstyle f_b} \\ (\sum F')a & \rightarrow & (\sum F')b. \end{array}$$

The horizontal maps of the first diagram are projections, so $f_b = 0$; those of the second are coprojections, so $f_a = 0$ as required. ∎

PROPOSITION 10. *Suppose F, F' are exact n-dimensional three-diagrams which take projective values. Suppose, for each $a \in \mathcal{K}_n$ except $a = o$, a map $f_a : Fa \rightarrow F'a$ is given, such that, whenever $o \neq a \leq a' \neq o$, the following square commutes:*

$$\begin{array}{ccc} Fa & \xrightarrow{f_a} & F'a \\ \downarrow & & \downarrow \\ Fa' & \xrightarrow[f_{a'}]{} & F'a'. \end{array}$$

Then there exists a map $f_o : Fo \rightarrow F'o$ such that $f = (f_a) : F \rightarrow F'$ is a map of $\mathfrak{F}_n$.

Proof. By Proposition 8 and the nature of this proposition, we may suppose $F = \sum F$, $F' = \sum F'$. Let m, $p \in \mathcal{K}_n$ be the objects given by $m(i) = -1$, $p(i) = 1$ for all $i \in \Delta_n$. Suppose c, c' are n-corners with $c' \leq c$ but not both $c' = m$ and $c = p$. Then there exists $j \in \Omega_n$ such that $c(j) = c'(j)$. Define $b_j \in \mathcal{K}_n$ by

$$\begin{aligned} b_j(i) &= c(j) \quad \text{for } i = j, \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Then c, $c' \in Sb_j$ and $c' \leq c$, so

$$f_{b_j} : \textstyle\sum_{e \in Sb_j} Fe \rightarrow \sum_{e \in Sb_j} F'e$$

induces a map $Fc \rightarrow F'c'$. If also $c(k) = c'(k)$ for some $k \neq j$ (say $j < k$), we show that the resulting map $Fc \rightarrow F'c'$ induced by f_{b_k} is the same as the one induced by f_{b_j}. Let $d \in \mathcal{K}_n$ be given by

$$\begin{aligned} d(i) &= c(j) \quad \text{for } i = j, \\ &= c(k) \quad \text{for } i = k, \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

From the hypothesis of the proposition, f_{b_j} and f_{b_k} both induce $f_d : Fd \rightarrow F'd$, and this induces a unique (Proposition 9) map $Fc \rightarrow F'c'$.

Choose any map $Fp \rightarrow F'm$ in $\mathfrak{A}$ whatever-for example, the zero map.

Now we have determined maps $Fc \to F'c'$ in $\mathfrak{A}$ for all n-corners c, c' with $c' \leq c$. Let $f_o : \sum_{c \in S} Fc \to \sum_{c \in S} F'c$ be the matrix with typical element $Fc \to F'c'$ this determined map when $c' \leq c$, and the zero map otherwise. Let $h = (h_a) : F \to F'$ be the unique map (Proposition 9) with $h_o = f_o$. By definition of f_o, the following squares commute:

$$\begin{array}{ccc} FT_i^{-1}o & \rightarrow & Fo \\ {\scriptstyle f_{T_i^{-1}o}}\downarrow & & \downarrow{\scriptstyle f_o} \\ F'T_i^{-1}o & \rightarrow & F'o \end{array} \qquad \begin{array}{ccc} Fo & \rightarrow & FT_i^1o \\ {\scriptstyle f_o}\downarrow & & \downarrow{\scriptstyle f_{T_i^1o}} \\ F'o & \rightarrow & F'T_i^1o \end{array}$$

for $i \in \Omega_n$, where the rows of the first square are coprojections, and those of the second are projections. The corresponding diagrams commute with f replaced by h. But $f_0 = h_0$; so

$$f_{T_i^\lambda o} = h_{T_i^\lambda o} \quad \text{for } \lambda = \pm 1.$$

Each $a \in \mathcal{K}_n$ is either o or of the form $T_i^\lambda a'$ for some $a' \in \mathcal{K}_{n-1}$, $i \in \Omega_n$, $\lambda = \pm 1$. So $h_a = f_a$ for all $a \in \mathcal{K}_n$. So f_o has the required property. ∎

4. Proof of Theorem 1

Let $\mathfrak{L}$ denote the pointed category whose objects are symbols $\delta = (s, t; u, v)$ where $(s, t), (u, v) \in \mathfrak{R}$, $t \leq v$ and $s \leq u$; whose hom-sets are given by

$$\mathfrak{L}(\delta, \delta') \cong Z_2 \text{ when } s' - N < t \leq t' < s \leq s', u' - N < v \leq v' < u \leq u',$$
$$u' - N < t \leq v' < s \leq u';$$
$$\cong \{0\} \quad \text{otherwise;}$$

and whose composition corresponding to $Z_2 \otimes Z_2 \to Z_2$ is the isomorphism.

Let $\delta = (s, t,; u, v)$, where $u - N < t \leq v < s \leq u$, be an object of $\mathfrak{L}$. Let $\mathfrak{L}_\delta$ denote the full subcategory of $\mathfrak{L}$ with objects those symbols $\delta' = (s', t'; u', v')$ such that $s - N < t' \leq t < s' \leq s, u' - N < v \leq v' < u \leq u'$. We shall write l, m, n for l_δ, m_δ, n_δ unless confusion is likely.

Given $a \in \mathcal{K}_n$, set

$$\begin{aligned}
\varepsilon &= \varepsilon(a) = \max \{0\} \cup \{i \mid 0 < i \leq m, a(i) = -1\}, \\
\zeta &= \min \{m + 1\} \cup \{i \mid 0 < i \leq m, a(i) = 1\}, \\
\eta &= \max \{m\} \cup \{i \mid m < i \leq n, a(i) = -1\}, \\
\theta &= \min \{n + 1\} \cup \{i \mid m < i \leq n, a(i) = 1\}, \\
\sigma &= s - \varepsilon \\
\tau &= u - l - \eta - 1, \\
\nu &= u + n - \theta + 1, \\
\phi &= s - \zeta.
\end{aligned}$$

The following inequalities are simply deduced:

$$0 \leq \varepsilon \leq m, \quad 1 \leq \zeta \leq m + 1, \quad m \leq \eta \leq n, \quad m + 1 \leq \theta \leq n + 1,$$
$$s - N < \tau \leq t < \sigma \leq s, \quad \nu - N < v \leq \phi < u \leq \nu.$$

If $\bar{a} \in \mathcal{K}_n$, and $\bar{\varepsilon} = \varepsilon(\bar{a})$ etc, then $\varepsilon \leq \eta$, $\eta \leq \bar{\varepsilon} + n$, $\theta \leq \bar{\zeta} + n$, $\zeta \leq \bar{\theta}$. If further $a \leq \bar{a}$ then $\bar{\varepsilon} \leq \varepsilon$, $\bar{\zeta} \leq \zeta$, $\bar{\eta} \leq \eta$, $\bar{\theta} \leq \theta$. So $\bar{\sigma} - N < \tau \leq \bar{\tau} < \sigma \leq \bar{\sigma}$, $\bar{\nu} - N < \phi \leq \bar{\phi} < \nu \leq \bar{\nu}$.

Define $\Delta = \Delta_\delta : \mathcal{K}_n \to \mathcal{L}_\delta$ by

$$\Delta(a) = (\sigma, \tau; \nu, \phi);$$

$\Delta(a \leq \bar{a})$ is the non-zero element of $\mathcal{L}_\delta(\Delta(a), \Delta(\bar{a}))$ when it has one, and the zero element otherwise. For $a \leq \bar{a}$,

$$\begin{aligned} \mathcal{K}_n(a, \bar{a}) = 0 &\Leftrightarrow \text{there exists } j \in \Omega_n \text{ such that} \\ &\qquad a(j) = -1 \text{ and } \bar{a}(j) = 1 \\ &\Leftrightarrow \text{either } \bar{\zeta} \leq \varepsilon \text{ or } \bar{\theta} \leq \eta \\ &\Leftrightarrow \text{either } \sigma \leq \bar{\phi} \text{ or } \tau \leq \bar{\nu} - N \\ &\Leftrightarrow \mathcal{L}_\delta(\Delta(a), \Delta(\bar{a})) = 0. \end{aligned}$$

So Δ is a pointed functor.

For $\delta' = (s', t'; u', v') \in \mathcal{L}_\delta$, let $\Gamma_\delta \delta'$ denote the object of $\mathcal{K}_n$ given by

$$\begin{aligned} (\Gamma_\delta \delta')i &= -1 \quad \text{for } 0 < i \leq s - s' \text{ or } m < i \leq m + t - t', \\ &= 0 \quad \text{for } s - s' < i \leq v - v' + m \\ &\qquad\quad \text{or } m + t - t' < i \leq n + u - u', \\ &= 1 \quad \text{for } v - v' + m < i \leq m \text{ or } n + u - u' < i \leq n. \end{aligned}$$

Then $\Delta(\Gamma_\delta \delta') = \delta'$. It is not necessarily the case that $\Gamma\Delta(a) = a$, for $a \in \mathcal{K}_n$; but there does exist $b \in \mathcal{K}_n$ such that $\Delta(b) = \Delta(a)$, $b \leq a$ and $b \leq \Gamma\Delta(a)$; namely,

$$\begin{aligned} b(i) &= -1 \quad \text{for } 0 < i \leq \varepsilon \text{ or } m < i \leq \eta, \\ &= a(i) \quad \text{otherwise.} \end{aligned}$$

These results are expressed by the following.

Remark 11. For each $\delta' \in \mathcal{L}_\delta$, the fibre category of the functor Δ_δ over δ' is non-empty and pathwise connected. ▌

Given an object D of $\mathfrak{D}$, we define a pointed functor $\hat{D} : \mathcal{L} \to \mathcal{A}$ by:

(i) for $\delta = (s, t; u, v) \in \mathcal{L}$, $\hat{D}\delta$ is the image in $\mathcal{A}$ of the map

$$D\delta : D(s, t) \to D(u, v);$$

(ii) for $\mathcal{L}(\delta, \delta') \cong Z_2$, $\hat{D}(\delta \to \delta')$ is the map $\hat{D}\delta \to \hat{D}\delta'$ induced vertically on images by the square

$$\begin{array}{ccc} D(s, t) & \to & D(u, v) \\ \downarrow & & \downarrow \\ D(s', t') & \to & D(u', v'). \end{array}$$

For non-zero $\delta \in \mathcal{L}$, put $F_\delta = \hat{D}\Delta_\delta \in \mathfrak{F}_{n_\delta}$.

LEMMA 12. *If D is an exact object of $\mathfrak{D}$ and $\delta = (s,\ t;\ u,\ v)$, where $u - N < t \leq v < s \leq u$, then $F = F_\delta$ is an exact object of $\mathfrak{F}_{n_\delta}$.*

Proof. Suppose $i \in \Omega_n$ $(n = n_\delta)$. Given $a \in \mathcal{K}_n$ such that $a(i) = 0$, define $a', a'' \in \mathcal{K}_n$ by

$$a'(j) = a''(j) = a(j) \quad \text{for } j \neq i,\ a'(i) = -1,\ a''(i) = 1.$$

We must show that the following sequence is exact:

$$0 \to Fa' \to Fa \to Fa'' \to 0.$$

Let $\varepsilon = \varepsilon(a)$, $\sigma = \sigma(a)$ etc; so $Fa = \hat{D}(\sigma,\ \tau;\ \nu,\ \phi)$. Eight exhaustive cases must be distinguished.

(i) $\zeta \leq i \leq \varepsilon$. Here $Fa' = Fa = Fa''$ and $\sigma \leq \phi$, so Δa is zero. The sequence is thus trivial.

(ii) $0 < i \leq \varepsilon, i \leq \zeta$. Here $Fa' = Fa$ and $\sigma = s - \varepsilon \leq v + m - i + 1$, so $\Delta a'' = (\sigma,\ \tau;\ \nu,\ \phi + m - i + 1) = 0$. The sequence is thus

$$0 \to Fa \xrightarrow{1} Fa \to 0 \to 0.$$

(iii) $\varepsilon < i \leq m, \zeta \leq i$. Here $Fa = Fa''$ and $s - i \leq v + m - \zeta + 1 = \phi$, so $\Delta a' = (s - i,\ \tau;\ \nu,\ \phi) = 0$. The sequence is thus

$$0 \to 0 \to Fa \xrightarrow{1} Fa \to 0.$$

(iv) $\varepsilon < i < \zeta$. In this case the sequence becomes

$$0 \to \hat{D}(\omega,\ \tau;\ \nu,\ \phi) \to \hat{D}(\sigma,\ \tau;\ \nu,\ \phi) \to \hat{D}(\sigma,\ \tau;\ \nu,\ \omega) \to 0$$

where $\omega = s - i = v + m - i + 1$. A chase of the following commutative diagram, using the exactness of the sequences

$$D(\omega,\ \phi) \to D(\nu,\ \phi) \to D(\nu,\ \omega)$$

$$D(\omega,\ \tau) \to D(\omega,\ \phi) \to D(N + \tau,\ \phi)$$

$$D(\nu,\ \tau) \to D(\nu,\ \phi) \to D(N + \tau,\ \phi),$$

yields the exactness required.

$$\begin{tikzcd}
D(\omega,\tau) \arrow[rr] \arrow[dr] \arrow[dd] & & D(\sigma,\tau) \arrow[rr, equal] \arrow[dr] \arrow[dd] & & D(\sigma,\tau) \arrow[dr] & \\
& Fa' \arrow[rr] \arrow[dd] & & Fa \arrow[rr] \arrow[dd] & & Fa'' \arrow[dd] \\
D(\omega,\phi) \arrow[dr] \arrow[dd] & & D(\nu,\tau) \arrow[dr] & & & \\
& D(\nu,\phi) \arrow[rr, equal] & & D(\nu,\phi) \arrow[rr] \arrow[dlll] & & D(\nu,\omega) \\
D(N+\tau,\phi) & & & & &
\end{tikzcd}$$

The remaining four cases

(v) $\theta \leq i \leq \eta$;
(vi) $m < i \leq \eta, i \leq \theta$;
(vii) $\eta < i \leq n, \theta \leq i$; and
(viii) $\eta < i < \theta$;

may be dealt with correspondingly. ▌

LEMMA 13. *If $D \in \mathfrak{D}$ is exact and the kernel of each of the maps*

$$D\delta : D(s, t) \to D(u, v)$$

is a projective object of $\mathfrak{A}$, then each F_δ is exact and takes projective values.

Proof. It suffices to show that the image $\hat{D}_\delta$ of each

$$D\delta = D(s, t; u, v) : D(s, t) \to D(u, v)$$

is projective, where $u - N < t \leq v < s \leq u$. If $t = v$ then, from the exact sequence

$$D(s, t) \to D(u, v) \to D(u, s),$$

we have that $\hat{D}\delta$ is the kernel of $D(u, v; u, s)$, and hence projective. If $t < v$ then, from (viii) of Lemma 13, we have a short exact sequence

$$0 \to \hat{D}(s, t; u, v) \to \hat{D}(s, v; u, v) \to \hat{D}(s, v, N + t, v) \to 0.$$

By the $t = v$ case we have that the second and third terms in this sequence are projective; and hence, so is the first. ▌

THEOREM 14. *If $D \in \mathfrak{D}$ is exact and the kernel of each of the maps*

$$D(s, t; u, v) : D(s, t) \to D(u, v)$$

is a projective object of $\mathfrak{A}$, then

$$D \cong \textstyle\sum_{y-N<z<y} J_{yz} A(y, z)$$

in $\mathfrak{D}$, where $A(y, z)$ is the projective object of $\mathfrak{A}$ given by

$$A(y, z) = \hat{D}(y, z; z + N - 1, y - 1) \quad \textit{for } y - N < z < y.$$

Proof. Let $\mathfrak{L}^n$ denote the full subcategory of $\mathfrak{L}$ whose objects are those objects δ of $\mathfrak{L}$ such that $n_\delta \leq n$. In particular, $\mathfrak{L}^{n-2} = \mathfrak{L}$ and

$$\mathfrak{L}^0 = \{(y, z; z + N - 1, y - 1) \mid y - N < z < y\}$$

as a discrete pointed category (only zero maps). With the objects $A(y, z)$ as in the theorem, put $D' = \sum_{y-N<z<y} J_{yz} A(y, z)$. Let $\hat{D}_n$, $\hat{D}'_n$ denote the restrictions of $\hat{D}$, $\hat{D}'$ to $\mathfrak{L}^n$.

By induction on n we shall prove

(*) *for $0 \leq n \leq N - 2$ there exists a natural isomorphism*

$$f : \hat{D}_n \to \hat{D}'_n .$$

Then, by taking $n = N - 2$, we shall have $\hat{D} \cong \hat{D}'$, which certainly implies $D \cong D'$ as asserted by the theorem.

For $n = 0$ we may take $f = 1$ in (*) since $\hat{D}_0 = \hat{D}_0'$.

Suppose $0 < n \leq N - 2$ and that there exists a natural isomorphism $f : \hat{D}_{n-1} \to \hat{D}'_{n-1}$. For $\delta \in \mathfrak{L}$ such that $n_\delta = n$ there exists $f_\delta : F_\delta\, o = \hat{D}\delta \to D'\delta = F'_\delta\, o$ such that $g^\delta = (g_a^\delta) : F_\delta \to F'_\delta$, given by

$$g_a^\delta = f_{\Delta_\delta a} \quad \text{for } a \neq o,\ g_o^\delta = f_\delta$$

is a map of $\mathfrak{F}_n$. This is by Proposition 10 applied to the family

$$\{f_{\Delta_\delta a} : F_\delta a \to F'_\delta a \mid o \neq a \in \mathcal{K}_n\}$$

which satisfies the hypothesis of the proposition since f is natural and since $n_{\Delta_\delta a} \leq n_\delta$ for all $a \in \mathcal{K}_n$ with equality precisely when $a = o$. By the short five-lemma, f_δ is an isomorphism. So, for each $\delta \in \mathfrak{L}^n$, we have an isomorphism $f_\delta : \hat{D}\delta \to \hat{D}'\delta$. It remains to show that $f = (f_\delta) : \hat{D}_n \to \hat{D}'_n$ is natural.

Given a non-zero map $\delta \to \bar{\delta}$ of $\mathfrak{L}^n$, where $\delta = (s, t; u, v)$, $\bar{\delta} = (\bar{s}, \bar{t}; \bar{u}, \bar{v})$, we must show that the following square commutes:

$$(1) \qquad \begin{array}{ccc} \hat{D}\delta & \xrightarrow{f_\delta} & \hat{D}'\delta \\ \downarrow & & \downarrow \\ \hat{D}\bar{\delta} & \xrightarrow{f_{\bar{\delta}}} & \hat{D}'\bar{\delta}. \end{array}$$

If $\delta, \bar{\delta} \in \mathfrak{L}^{n-1}$ then (1) commutes since f restricted to $\mathfrak{L}^{n-1}$ is natural. Suppose then $\delta \notin \mathfrak{L}^{n-1}$; a similar argument will apply in the case $\bar{\delta} \notin \mathfrak{L}^{n-1}$. Put $\delta' = (s, t; \bar{u}, \bar{v})$. Then $\delta \to \bar{\delta}$ factors as $\delta \to \delta' \to \bar{\delta}$. Commutativity of (1) will thus follow from commutativity of the two squares:

$$(2) \qquad \begin{array}{ccc} \hat{D}\delta & \xrightarrow{f_\delta} & \hat{D}'\delta \\ \downarrow & & \downarrow \\ \hat{D}\delta' & \xrightarrow{f_{\delta'}} & \hat{D}'\delta' \end{array}$$

$$(3) \qquad \begin{array}{ccc} \hat{D}\delta' & \xrightarrow{f_{\delta'}} & \hat{D}'\delta' \\ \downarrow & & \downarrow \\ \hat{D}\bar{\delta} & \xrightarrow{f_{\bar{\delta}}} & \hat{D}'\bar{\delta}. \end{array}$$

If $\delta' \in \mathfrak{L}^n$ then $u = \bar{u}$, $v = \bar{v}$, so $\delta' = \delta$, and so (2) commutes. If $\delta' \notin \mathfrak{L}^n$ set $a = \Gamma_\delta\, \delta'$, and note that $a(i) \neq -1$ for all $i \in \Omega_n$, so $o \leq a$. So the square

$$\begin{array}{ccc} F_\delta o & \xrightarrow{g_o^\delta} & F'_\delta o \\ \downarrow & & \downarrow \\ F_\delta a & \xrightarrow[g_a^\delta]{} & F'_\delta a \end{array}$$

commutes; but this is just (2). So (2) commutes. If $\bar{\delta} \notin \mathfrak{L}^n$ then $\delta' \notin \mathfrak{L}^n$ and so (3) commutes since f restricted to $\mathfrak{L}^{n-1}$ is natural. So consider the case $\bar{\delta} \in \mathfrak{L}^n$. If $\delta' \in \mathfrak{L}^n$ then $\delta' = \bar{\delta}$ so (3) commutes. If $\delta' \notin \mathfrak{L}^n$, set $\bar{a} = \Gamma_{\bar{\delta}}\, \delta'$ and note $\bar{a} \leq o$. So the square

$$\begin{tikzcd} F_{\bar{\delta}}\bar{a} \arrow[r, "g_{\bar{a}}^{\bar{\delta}}"] \arrow[d] & F'_{\bar{\delta}}\bar{a} \arrow[d] \\ F_{\bar{\delta}}o \arrow[r, "g_{o}^{\bar{\delta}}"'] & F'_{\bar{\delta}}o \end{tikzcd}$$

commutes; but this is just (3). So (3) commutes. ∎

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