ON FINITE SOLVABLE LINEAR GROUPS

BY

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Introduction

The following theorem of N. Itô [3] is well known.

Let G be a finite solvable group and let p be a prime. Let G have a faithful representation of degree n over the complex number field. If $n < p$, then G has a normal abelian p-Sylow subgroup unless p is odd, $n = p - 1$ and p is a Fermat prime.

We shall prove here the following generalization.

THEOREM. *Let G be a finite solvable group, p a prime. Let G have a faithful irreducible representation of degree n over the complex number field. Then G has an abelian normal p-Sylow subgroup unless, for some positive integer m, $n = mp$ or $n = mq^s$ where q^s is a positive power of a prime with $q^s \equiv \pm 1 \pmod{p}$.*

The theorem is "best possible" in the sense that for each of the exceptional values of n mentioned there is a finite solvable group which has a faithful irreducible representation of degree n over the field of complex numbers and which does not have a normal p-Sylow subgroup (§2). The proof of the theorem is not applicable to p-solvable groups as is the proof given by Itô for his theorem and leaves open the question as to whether or not there is an analogue for p-solvable groups. The reader is referred to [1] for another kind of generalization of Itô's theorem. We shall make frequent use of the following statement which is a well known consequence of Clifford's theorem and the Frobenius reciprocity theorem.

LEMMA. *Let H be a normal subgroup of prime index p of the finite group G and let θ be an irreducible complex character of H with inertia group $I(\theta)$ in G. If $I(\theta) = H$, then the induced character θ^* is an irreducible character of G and is the only irreducible character of G whose restriction to H contains θ; $\theta^* \mid H$ is a sum of p distinct conjugate characters of H. If $I(\theta) = G$, then there are p distinct characters of G which are extensions of θ and these are the only irreducible characters of G whose restriction to H contains θ.*

1. Proof of the theorem

Let G be a counterexample to the theorem of minimal order. A contradiction is obtained after a series of steps. Let χ denote the given faithful irreducible character of G.

Received January 20, 1969.

(I) *Let H be a proper subgroup of G such that the degree of each irreducible constituent of $\chi \mid H$ is neither a multiple of p nor a multiple of any prime power $q^s > 1$ with $q^s \equiv \pm 1 \pmod{p}$. Then H has a normal abelian p-Sylow subgroup.*

Proof. Let $\chi_1, \cdots, \chi_t$ be the irreducible constituents of $\chi \mid H$ and let their kernels be $K_1, \cdots, K_t$. By minimality of the order $\mid G \mid$ of G, H/K_i has a normal abelian p-Sylow subgroup for all i. Therefore

$$H \cong H/(K_1 \cap \cdots \cap K_t)$$

has a normal abelian p-Sylow subgroup because the latter group is isomorphic to a subgroup of the direct product of the groups H/K_i, $i = 1, \cdots, t$.

(II) *If H is a proper normal subgroup of G, then H has a normal abelian p-Sylow subgroup.*

Proof. It follows from Clifford's theorem that all irreducible constituents of $\chi \mid H$ have the same degree z. Since $\chi(1)$ is not a multiple of p and not a multiple of q^s for any prime power $q^s \equiv \pm 1 \pmod{p}$, $q^s > 1$, the same is true of z and (II) follows from (I).

As an easy consequence we have

(III) *A proper normal subgroup of G cannot contain a Sylow p-subgroup of G.*

Now let P_0 be the maximal normal p-subgroup of G. All irreducible constituents of $\chi \mid P_0$ have the same degree which is a power of p but not a multiple of p; hence $\chi \mid P_0$ is a sum of linear characters and P_0 must be abelian. Let N/P_0 be the maximal normal subgroup of G/P_0 which has order prime to p and P_1/N be the maximal normal p-subgroup of G/N. It then follows from (II) that $P_1 = G$ for otherwise the definition of P_0 would be contradicted. For the same reason we have $\mid P_1{:}N \mid = p$. We write $G = PN$ where P is a Sylow p-subgroup of G and $\mid P{:}P_0 \mid = p$. Notice that $\chi \mid N$ is irreducible for otherwise $p \mid \chi(1)$ by the lemma.

(IV) $P_0 \leq Z(G)$ *where $Z(G)$ is the center of G.*

Proof. Suppose (IV) is false. Let M be a normal subgroup of G such that $P_0 \leq M$ and N/M is a chief factor of G. By Clifford's theorem, $\chi \mid M = e\sum_{i=1}^{r} \theta_i$ where e is a positive integer and $\theta_1, \cdots, \theta_r$ are distinct irreducible characters of M conjugate in G. They are also conjugate in N since $\chi \mid N$ is irreducible.

Suppose $r = 1$ and $\chi \mid M = e\theta_1$. Then by the lemma all irreducible constituents of $\chi \mid PM$ are extensions of θ_1. (I) now implies that P is abelian. Hence $P \leq C(P_0) \lhd_{\neq} G$ where $C(P_0)$ is the centralizer of P_0 in G. This contradicts (III) and therefore $r > 1$.

Since $p \nmid r$, $P \leq I(\theta_1)$, replacing θ_1 by a suitable conjugate if necessary. Therefore P normalizes $I(\theta_1) \cap N$ and since N/M is abelian, N normalizes it also. Thus $I(\theta_1) \cap N \lhd G$. Since N/M is a chief factor of G, $I(\theta_1) \cap N$ is

M or N. The latter is impossible since $|N:I(\theta_1)\cap N| = r > 1$. Therefore $I(\theta_1)\cap N = M$ and $\theta_1^N = \chi \mid N$ where θ_1^N is the character of N induced by θ_1. Hence $\chi(1) = |N:M|\,\theta_1(1) = q^s\theta_1(1)$ for some prime $q \neq p$.

Let F denote the subgroup, $M \leq F \leq N$ such that F/M is the set of fixed points of N/M under the action of PM/M. Then $F \lhd G$ and so $F = M$ or $F = N$. If $F = N$, then $G/M = (PM/M) \times (N/M)$ and PM is a proper normal subgroup of G contradicting (III). Therefore PM/M acts fixed-point-free on N/M. Hence $q^s \equiv 1 \pmod p$. (IV) is now proved.

(V) $P_0 = \langle 1\rangle, G = PN, |P| = p, p \nmid |N|$.

Proof. Since $P_0 \leq Z(G)$, Burnside's transfer theorem yields that N has a normal p-complement N_1 which is also a normal p-complement of G. $\chi \mid P_0 = \chi(1)\lambda$ where λ is a linear character of P_0. Also $\chi \mid P$ contains a linear character μ since $p \nmid \chi(1)$. Then $\mu \mid P_0 = \lambda$. Further we may consider μ as a character of G/N_1. Then $\bar{\mu}\chi$ is a faithful, irreducible character of G/P_0 having the same degree as χ. If $P_0 \neq \langle 1\rangle$, minimality of $|G|$ yields $P \lhd G$ and then P is abelian since $\chi \mid P$ is a sum of linear characters. This proves (V).

Now let $\langle 1\rangle \leq M < N < PN = G$ be a series of subgroups of G such that N/M is a chief factor of G. Then $|N:M| = q^t$ for some prime q and $q^t \equiv 1 \pmod p$ since P acts fixed-point-free on N/M. Let $\chi \mid M = e\sum_{i=1}^r \theta_i$ be the Clifford decomposition, $r \geqq 1$. Since $p \nmid r$, we again obtain $I(\theta_1)\cap N \lhd G$ and $I(\theta_1)\cap N = M$ or N.

If $I(\theta_1)\cap N = M$, then $\theta_1^N = \chi \mid N$ and $q^t \mid \chi(1)$. So we may assume $I(\theta_1)\cap N = N$. Then $\chi \mid M = e\theta_1$ and by the lemma all irreducible constituents of $\chi \mid PM$ are extensions of θ_1. By (I), $P \lhd PM$. Since $P \cap M = \langle 1\rangle$, $P \leqq C(M) \lhd G$. By (III), $C(M) = G$ and it follows that $M = Z(G)$. Let Q be a Sylow q-subgroup of N and let Z_1 be the normal q-complement of $Z(G)$. Then $G = PQ \times Z_1$ and by (III), $G = PQ$, $Z_1 = \langle 1\rangle$, $N = Q$ and $M = Z(G) \leq Z(Q)$. Since $\chi \mid Q$ is irreducible, Q is non-abelian. Since $Z(Q)$ is invariant under P, we must have $Z(G) = Z(Q) = M$. In particular, $Z(Q)$ is cyclic. If $Q' <_{\neq} Z(Q)$, a transfer theorem ([4], page 173) yields that G/Q' has a normal subgroup of index q which is not the case. Therefore $Q' = Z(Q)$ and Q/Q' is elementary abelian. It follows [2] that Q is extra-special, $t = 2s$ for some s and each non-linear character of Q has degree q^s. Therefore $\chi(1) = q^s$. Since $p \mid q^t - 1$ and $q^t - 1 = (q^s - 1)(q^s + 1)$, this is a contradiction. The proof of the theorem is complete.

2. Exceptional cases

We first show that if H is a solvable group which does not have a normal p-Sylow subgroup and which has a faithful irreducible complex character of degree z, then for each m, there is a finite solvable group G which does not have a normal p-Sylow subgroup and which has a faithful irreducible complex character of degree mz. For, let

$$G = \langle y \rangle (H_1 \times \cdots \times H_m)$$

where each H_i is an isomorphic copy of H and where y is an element of order m which permutes the H_i cyclically. Let λ be the given character of H_1 considered as a character of

$$H_1 \times \cdots \times H_m / H_2 \times \cdots \times H_m .$$

Then the induced character λ^* is a faithful, irreducible character of G of degree mz as desired.

First let r be a prime different from p and let R_i be a group of order r, $i = 1, \cdots, p$. Adjoin to $R_1 \times \cdots \times R_p$ an element w of order p which permutes the R_i cyclically and let $H = \langle w \rangle (R_1 \times \cdots \times R_p)$. Let λ be a faithful linear character of

$$R_1 \times \cdots \times R_p / R_2 \times \cdots \times R_p .$$

Then λ^* is a faithful irreducible character of H of degree p and H is solvable but does not have a normal p-Sylow subgroup.

Next let q be a prime different from p and s a positive integer such that $q^s \equiv \pm 1 \pmod p$. Then it may be verified that there is an extraspecial q-group Q of order q^{2s+1} which has an automorphism α of order p which acts trivially on $Z(Q)$. Let θ be a non-linear irreducible complex character of Q. Then [2] $\theta(1) = q^s$ and it follows from $\sum_Q |\theta|^2 = |Q|$ that θ vanishes outside of $Z(Q)$ and is faithful. Hence α fixes θ and by the lemma, θ has an extension to an irreducible character of the semidirect product $H = \langle \alpha \rangle Q$. Therefore H has a faithful irreducible complex character of degree q^s and does not have a normal p-Sylow subgroup.

References

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