FIRST COHOMOLOGY GROUPS OF SOME LINEAR GROUPS OVER FIELDS OF CHARACTERISTIC TWO

BY

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Introduction

A *derivation* (or crossed homomorphism) from a group G to a left (resp. right) G-module V is a map $\delta : G \to V$ such that $\delta(ST) = S(\delta T) + \delta S$ (resp. $\delta(ST) = (\delta S)T + \delta T$) for all $S, T \in G$. An *inner derivation* (or principal crossed homomorphism) from G to a left (resp. right) G-module V is a derivation $\delta : G \to V$ for which there exists an element $v_0 \in V$ with $\delta T = Tv_0 - v_0$ (resp. $\delta T = v_0 T - v_0$) for all $T \in G$. The derivations from G to a (right or left) G-module V form an abelian group Der (G, V) under point-wise addition, and the inner derivations form a subgroup, Inn (G, V). If V is a K-space, then Der (G, V) can be regarded as a K-space in the natural way. Inn (G, V) is then also a K-space, and so

$$\text{Der }(G, V)/\text{Inn }(G, V) \cong H^1(G, V),$$

the first cohomology group of G with coefficients in V [14, p. 130–131].

In this paper we use the representation of $H^1(G, V)$ as Der (G, V)/Inn (G, V) to compute the K-dimension of $H^1(G, V)$ for certain linear groups G over K and their standard modules V. In particular, we compute $H^1(G, V)$ for $G = Sp_{2n}(K)$ with $n \geq 2$ and K either of odd characteristic or perfect of characteristic two, and for $G = O_{2n}(K)$ with $n \geq 2$ and K perfect of characteristic two. In addition, viewing S_n as a linear group on the $(n - 1)$- (or $(n - 2)$-) dimensional $\mathbf{F}_2$-space V for n odd (or even), we compute $H^1(S_n, V)$ for $n \geq 5$. Each of these cohomology groups is found to have K-dimension at most one.

1. Preliminaries

In this section we collect some of the basic definitions and results on symplectic and orthogonal groups over perfect fields of characteristic two that will be used throughout this paper.

First a few remarks on notation and language. We will use G, $G(V)$, $G_n(K)$ and $G(F)$ interchangeably to name the group G of transformations on the n-dimensional K-space V that preserve the form F on V, or to name the corresponding matrix group. We will also denote linear transformations and their matrix representations (and vectors and their representations as n-tuples) by the same symbol. A *transvection* T is a linear transformation such

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that the image Im $(T - 1)$ of $T - 1$ is a line, and the kernel Ker $(T - 1)$ of $T - 1$ is a hyperplane. If T is a transvection, we will say that Im $(T - 1)$ is the *center* of T and Ker $(T - 1)$ is its *axis*. The symplectic transvections T on V are of the form $Tv = v + B(x, v)x$ for all $v \in V$, where $x \in V$ is arbitrary. The orthogonal transvections T are all of the form $Tv = v + B(x, v)x$ for all v, where $Q(x) = 1$.

THEOREM 1.1. *The symplectic group $Sp(V)$ is generated by the subgroups $\mathfrak{S}_{\langle x\rangle}$ and $\mathfrak{S}_{\langle y\rangle}$, where $x, y \in V$ is a hyperbolic pair and $\mathfrak{S}_{\langle x\rangle} = \{T \in Sp(V) : Tx = x$ and $T\bar{z} = \bar{z}$ for all $\bar{z} \in \langle x\rangle^{\perp}/\langle x\rangle\}$.*

Proof. Arguing as in [16, Lemma 4, p. 194], we have that $H = \langle\mathfrak{S}_{\langle x\rangle}, \mathfrak{S}_{\langle y\rangle}\rangle$ is transitive on the lines of V, and hence that $\mathfrak{S}_{\langle v\rangle} \leq H$ for all $v \in V$. Therefore H contains every symplectic transvection and so [2, Theorem 3.25, p. 139] equals $Sp(V)$.

THEOREM 1.2. *Suppose $V = U \oplus W$, where $U, W \leq V$ are totally isotropic. Then $Sp(V) = \langle Sp(V)_U, Sp(V)_W\rangle$.*

Proof. If the dimension of V is 2, $Sp(V) = SL(V)$, and the result is clear. Suppose the dimension $2n$ of V is at least 4, and assume 1.2 is true for spaces of dimension less than $2n$. Let $H = \langle Sp(V)_U, Sp(V)_W\rangle$. Choose a hyperbolic pair x, y with $x \in U$ and $y \in W$. For $T \in Sp(V)_x$, $T = RS$ for suitable $R \in \mathfrak{S}_{\langle x\rangle} \leq Sp(V)_U$ and S fixing $\langle x, y\rangle$ point-wise. By the induction hypothesis, $S \in H$. Hence $Sp(V)_x \leq H$. Similarly $Sp(V)_y \leq H$. Therefore, by 1.1, $H = Sp(V)$.

The map $Q : V \rightarrow K$ is a *quadratic form* on V if

$$Q(\alpha x + \beta y) = \alpha^2 Q(x) + \beta^2 Q(y) + \alpha\beta B(x, y)$$

for all $x, y \in V$, $\alpha, \beta \in K$, where B is a bilinear form on V. We will say B is *associated with* Q, Q is *associated with* B. If the characteristic of K is 2, the form B determined by Q is alternate. From this point, we will consider only fields of characteristic 2, unless we explicitly say otherwise. We will further restrict our attention to those quadratic forms Q for which B is non-degenerate, unless we specify otherwise, so that $O(Q) \leq Sp(B)$.

We denote the *index* of the form Q by $\nu(Q)$ or $\nu(V)$. Since the index of a quadratic form on a $2n$-dimensional space over a perfect field is n or $n - 1$ [4, Theorem 1.3.2, p. 13], for K perfect we write $O(+1, K)$ or $O(+1, V)$ for the group of a form of maximal index, and $O(-1, K)$ or $O(-1, V)$ for the group of a form of non-maximal index.

THEOREM 1.3. *$O(Q)$ is primitive in its action on the singular lines of V, its standard module.*

Proof. The theorem is trivial if the dimension $2n$ of V is 2. For $n \geq 2$ we will show that O is of rank 3 in its action on the singular lines of V. Choose a hyperbolic pair of singular vectors $x, y \in V$. Consider $O_{\langle x\rangle}$ acting on $\Gamma(x)$, the set of singular lines off $\langle x\rangle^{\perp}$. Choose $\langle z\rangle \in \Gamma(x)$ and assume $B(x, z) = 1$.

The map $T : \langle x, y\rangle \to \langle x, z\rangle$ defined by $Tx = x$ and $Ty = z$ is a Q-isomorphism and, by Witt's theorem [4, Theorem 1.4.1, p. 16], may be extended to an element of $O_{\langle x\rangle}$. Hence $O_{\langle x\rangle}$ is transitive on $\Gamma(x)$.

Let $\Delta(x)$ be the set of singular lines $\neq \langle x\rangle$ on $\langle x\rangle^{\perp}$. By Witt's theorem, for $\langle v\rangle, \langle u\rangle \in \Delta(x)$ such that $\langle v,\rangle\ \langle u\rangle \leq \langle x, y\rangle^{\perp}$, there exists $T \in O(\langle x, y\rangle^{\perp})$ taking v to u. T may be extended to an element of $O_{\langle x\rangle}$ by defining $Tx = x$, $Ty = y$. If $\langle \alpha x + u\rangle, \langle v\rangle \in \Delta(x)$, with $u, v \in \langle x, y\rangle^{\perp}$ and $\alpha \neq 0$, then since u is singular, Witt's theorem implies that there is a $T \in O(\langle x, y\rangle^{\perp})$ taking v to u. Define $A \in O_{\langle x\rangle}$ by $Ax = \alpha x$, $As = \alpha B(Ts, w)x + Ts$ for $s \in \langle x, y\rangle^{\perp}$, and $Ay = \alpha Q(w)x + w + (1/\alpha)y$, where $w \in \langle x, y\rangle^{\perp}$ is such that $B(u, w) = 1$. Then $A\langle v\rangle = \langle \alpha x + u\rangle$. Hence $O_{\langle x\rangle}$ is transitive on $\Delta(x)$.

Now suppose a set I of singular lines is a set of imprimitivity for O. Since O is of rank 3 on the singular lines, for every $\langle v\rangle \in I$, $I \cap \Delta(v) = \emptyset$ or $\Delta(v)$ and $I \cap \Gamma(v) = \emptyset$ or $\Gamma(v)$. Choose $\langle x\rangle \in I$. Suppose $I \cap \Delta(x) = \Delta(x)$ (so $\nu(Q) \geq 2$). Then for $\langle z\rangle \in \Delta(x)$, $I \cap \Delta(z) \neq \emptyset$, and so $\Delta(z) \subseteq I$. Since $\langle x\rangle \neq \langle z\rangle$, $\langle x\rangle^{\perp} \neq \langle z\rangle^{\perp}$. Choose $\langle y'\rangle \leq \langle z\rangle^{\perp}$, $\nleq \langle x\rangle^{\perp}$. Assume $B(x, y') = 1$, and let $y = y' + Q(y')x$. Then $\langle y\rangle \in I \cap \Gamma(x)$, so $\Gamma(x) \subseteq I$, and I contains all the singular lines of V. Suppose, on the other hand, that $I \cap \Gamma(x) = \Gamma(x)$. Let $\langle z\rangle \in \Gamma(x)$. Then $\Gamma(z) \subseteq I$. If $n = 2$ and $\nu(Q) = 1$, then $\Delta(x) = \emptyset$, and we are done. Assume $\nu(Q) \geq 2$, and choose a singular $\langle u\rangle \in \langle x, y\rangle^{\perp}$. Then $\langle u + z\rangle \in \Gamma(x) \subseteq I$, and so $\langle u + z\rangle \in I \cap \Delta(z)$. Hence again I contains all singular lines.

We remark that for $x \in V$ singular, $O^{+}(V)_{\langle x\rangle}$ is a *maximal parabolic subgroup*. Thus $O^{+}(V)$, and so $O(V)$, is primitive on the singular lines of V (see the proof of 1.12).

THEOREM 1.4. *Let K be perfect. Let x in the K-space V be singular, and let T be the transvection taking $v \in V$ to $v + B(v, x)x$. Then if $O = O(V)$, $O \cap O^{T} = O_x$.*

Proof. If $S \in O_x$, then for $v \in V$, $TSTv = Sv$, and $S \in O \cap O^{T}$; thus $O_x \leq O \cap O^{T}$. If $K = \mathbf{F}_2$ (the field of two elements), O_x is maximal in O by 1.3, and we are done. So assume $K \neq \mathbf{F}_2$. Let $S \in O \cap O^{T}$; then

$$Q(TSTv) = Q(v) + B(Sv, x)^2 + B(v, x)^2 + B(v, x)^2 B(Sx, x)^2 = Q(v),$$

and hence

$$B(Sv, x)^2 = B(v, x)^2(1 + B(Sx, x)^2)$$

for all $v \in V$. If $v \in \langle x\rangle^{\perp}$, we see $Sv \in \langle x\rangle^{\perp}$. Suppose y is chosen singular with $B(x, y) = 1$, and suppose $Sx = \alpha x + u$, with $u \in \langle x, y\rangle^{\perp}$. If $w \in \langle x, y\rangle^{\perp}$, $Sw = \beta(w)x + Tw$ with $\beta(w) \in K$ and $T \in 0(\langle x, y\rangle^{\perp})$. Thus $0 = B(x, w) = B(Sx, Sw) = B(Tw, u)$ for all $w \in \langle x, y\rangle^{\perp}$; so $u = 0$, $S \in O_{\langle x\rangle}$ and

$$Q(TSTv) = Q(v) + B(Sv, x)^2 + B(v, x)^2.$$

Hence $B(v, S^{-1}x + x) = 0$ for all $v \in V$, and $S \in O_x$. Therefore $O \cap O^T \leq O_x$.

THEOREM 1.5. *If $\nu(Q) \geq 1$, B is associated with Q, and K is perfect, then $O(Q)$ is maximal in $Sp(B)$.*

Proof. First we show that, for $T \in Sp(B)$, if $T(x)$ is singular whenever x is singular, then $T \in O(Q)$. Since K is perfect, it suffices to show that $Q(Tx) = 1$ whenever $Q(x) = 1$. Choose $x \in V$ with $Q(x) = 1$. The theorem is trivial if $O(Q) = O_2(+1, \mathbf{F}_2)$, for then $O(Q)$ is of order 2 and $Sp(B)$ is of order 6. Excluding this case, $O(Q)$ is irreducible [4, Theorem 1.6.7, p. 33], so there are singular vectors off $\langle x\rangle^{\perp}$. Choose y singular such that $B(x, y) = 1$. Then $x + y$ is singular, so $0 = Q(Tx + Ty) = Q(Tx) + 1$, and $Q(Tx) = 1$.

Now we show that if $T \in Sp(B)$, $T \notin O(Q)$, then $\langle O(Q), T\rangle = Sp(B)$. Let $G = \langle O(Q), T\rangle$. Since $T \notin O(Q)$, there is a singular x such that Tx is non-singular. $O(Q)$ contains a transvection with center $\langle Tx\rangle$, so G contains a transvection with center $\langle x\rangle$. $O(Q)$ is transitive on the singular vetors, so G contains all symplectic transvections with singular centers. Since G thus contains all the symplectic transvections with center $\langle x\rangle$, G contains all those with center $\langle Tx\rangle$. K is perfect, so Q takes all values in K^* $(= K - \{0\})$ on $\langle Tx\rangle$. For each $\alpha \in K$, $O(Q)$ is transitive on the set of $v \in V$ with $Q(v) = \alpha$. Hence G also contains every symplectic transvection having a non-singular center. Since $Sp(B)$ is generated by the symplectic transvections, $G = Sp(B)$.

THEOREM 1.6. *Let K be a perfect field, and let B and Q be associated forms on the K-space V. Define a map u: $Sp(B) \to V$ by*

$$B(u(T), Tv) = \sqrt{Q(Tv) + Q(v)} \qquad \textit{for all } v \in V, \textit{for } T \in Sp(B).$$

Then (i) *u is a derivation*; (ii) *$u(T) = 0$ if and only if $T \in O(Q)$*; (iii) *for T the transvection taking $v \in V$ to $v + B(x, v)x$, $u(T) = \sqrt{1 + Q(x)}\, x$; and* (iv) *$X^2 + X = Q(u(T))$ has a solution in K for every $T \in Sp(B)$.*

Proof. (i), (ii) and (iii) are easily verified. By (iii), for a transvection T taking $v \in V$ to $v + B(x, v)x$, $Q(u(T)) = Q(x)^2 + Q(x)$. Suppose for T, $S \in Sp(B)$, $Q(u(T)) = \alpha^2 + \alpha$ and $Q(u(S)) = \beta^2 + \beta$, with $\alpha, \beta \in K$. Then

$$Q(u(TS)) = \gamma^2 + \gamma \quad \text{for } \gamma = \sqrt{Q(Tu(S)) + \beta + \alpha^2} \in K.$$

Since $Sp(B)$ is generated by transvections, we see that $X^2 + X = Q(u(T))$ has a solution in K for every $T \in Sp(B)$.

THEOREM 1.7 *Let K be a perfect field, and let x, y be a hyperbolic pair of non-singular vectors with $Q(x) = 1$. Then a symplectic transformation A is in $O(Q)_{\langle x\rangle}$ if and only if $Ax = x$, $Av = B(Tv, u(T))x + Tv$ for all $v \in \langle x, y\rangle^{\perp}$, and $Ay = \alpha x + u(T) + y$, where $T \in Sp(\langle x, y\rangle^{\perp})$, u is defined as in* 1.6, *and α is a solution in K of $X^2 + X = Q(u(T))$.*

Proof. Clearly $A \in Sp(B)_{\langle x\rangle}$ if and only if $Ax = \beta x$, $Av = \beta B(Tv, u)x + Tv$ for all $v \in \langle x, y\rangle^{\perp}$, and $Ay = \alpha x + u + (1/\beta)y$, with $T \in Sp(\langle x, y\rangle^{\perp})$, $\alpha \in K$,

$\beta \in K^*$. $A \in Sp(B)$ is a Q-isometry if and only if $\beta = 1$, $\alpha^2 + \alpha = Q(u)$ and $u = u(T)$. For such an $A \in O(Q)_{\langle x \rangle}$, write $A = A(T, \alpha)$.

We remark further that for every $T \in Sp(\langle x, y\rangle^{\perp})$, there are two elements, $A(T, \alpha)$ and $A(T, \alpha + 1)$ in $O_{\langle x \rangle}$.

THEOREM 1.8. *Let $n \geq 3$, let K be a perfect field, and let $O = O_{2n}(K)$. Then for x, y a hyperbolic pair of non-singular vectors $O = \langle O_{\langle x \rangle}, O_{\langle y \rangle}\rangle$.*

Proof. Clearly $O_{\langle x \rangle}$ contains all the orthogonal transvections centered in $\langle x\rangle^{\perp}$. Suppose $\langle u \rangle$ is a non-singular line off both $\langle x\rangle^{\perp}$ and $\langle y\rangle^{\perp}$. Then we may assume $u = x + v + \beta y$, with $v \in \langle x, y\rangle^{\perp}$ and $\beta \in K^*$. If $v \neq 0$, then since $O(\langle x, y\rangle^{\perp})$ is irreducible, there is a singular vector $w \in \langle x, y\rangle^{\perp}$, $\notin \langle v\rangle^{\perp}$; say $B(v, w) = 1$. Let T be the transvection in $Sp(\langle x, y\rangle^{\perp})$ taking $s \in \langle x, y\rangle^{\perp}$ to $s + B(s, w)w$. Then $u(T) = w$ by 1.6, and $Tv = w + v$. Let $A = A(T, 0)$. Then

$$Au = v + (\beta + 1)w + y \in \langle y\rangle^{\perp}.$$

If $u = x + \beta y$, choose $T \in Sp(\langle x, y\rangle^{\perp})$, $T \notin O(\langle x, y\rangle^{\perp})$, and let $A = A(T, \alpha)$. Then

$$Au = (\alpha\beta + 1)x + \beta u(T) + \beta y.$$

If $Au \notin \langle y\rangle^{\perp}$, proceed as above to obtain an $A' \in O_{\langle x \rangle}$ such that $A'Au \in \langle y\rangle^{\perp}$.

Thus for u non-singular off $\langle x\rangle^{\perp}$ and off $\langle y\rangle^{\perp}$, there is an $A \in O_{\langle x \rangle}$ such that $Au \in \langle y\rangle^{\perp}$. $O_{\langle y \rangle}$ contains the orthogonal transvection with center $\langle Au\rangle$, so $\langle O_{\langle x \rangle}, O_{\langle y \rangle}\rangle$ contains the orthogonal transvection with center $\langle u\rangle$. Thus $\langle O_{\langle x \rangle}, O_{\langle y \rangle}\rangle$ contains every orthogonal transvection. Since O is generated by the orthogonal transvections [8, Proposition 14, p. 42], $O = \langle O_{\langle x \rangle}, O_{\langle y \rangle}\rangle$.

THEOREM 1.9. *Suppose $n \geq 2$. Then $O = O_{2n}(Q) = \langle O_{\langle x \rangle}, O_{\langle y \rangle}\rangle$ for x, y a hyperbolic pair of singular vectors.*

Proof. By 1.3, $O_{\langle x \rangle}$ and $O_{\langle y \rangle}$ are maximal subgroups of O. Since $n \geq 2$, $O_{\langle x \rangle} \neq O_{\langle y \rangle}$, so $O = \langle O_{\langle x \rangle}, O_{\langle y \rangle}\rangle$.

LEMMA. *Let K be an arbitrary field, and let V be a K-space with a bilinear form B. Let $x_1, \cdots, x_k$ be linearly independent vectors in V, and let T_i be defined by*

$$T_i(v) = v + B(x_i, v)x_i \qquad \text{for all } v \in V,\ i = 1, \cdots, k.$$

Then $T_1 \cdots T_k\, v = v$ if and only if $v \in \bigcap_{i=1}^{k} \langle x_i\rangle^{\perp}$.

Proof. Let $T = T_1 \cdots T_k$. Obviously if $v \in \bigcap_{i=1}^{k} \langle x_i\rangle^{\perp}$, then $Tv = v$. We will prove the converse by induction on k. If $k = 1$, it is clear. Suppose $k > 1$ and assume the lemma is true for fewer than k vectors x_i. For $v \in V$,

$$T_2 \cdots T_k\, v = v + \textstyle\sum_{i=2}^{k} \alpha_i x_i \quad \text{for some } \alpha_i \in k.$$

Thus

$$Tv = v + B(x_1, v)x_1 + \textstyle\sum_{i=2}^{k} \alpha_i(x_i + B(x_i, x_1)x_1).$$

If $Tv = v$, the linear independence of the x_i implies $\alpha_i = 0$ for $i = 2, \cdots, k$.

But then $T_2 \cdots T_k v = v$, and the induction hypothesis implies $v \in \bigcap_{i=2}^{k} \langle x_i \rangle^{\perp}$; so $T_1 v = v$ and $v \in \langle x_1 \rangle^{\perp}$ as well.

THEOREM 1.10. *Let K be a perfect field (of characteristic two). If $O = O_{2n}(K) \neq O_2(+1, \mathbf{F}_2)$, then there exists $T \in O$ such that $T + 1$ is non-singular.*

Proof. O is irreducible, so $\langle x \in V : Q(x) = 1 \rangle$ must be V. Hence V has a basis $x_1, \cdots, x_{2n}$ with $Q(x_i) = 1$ for $i = 1, \cdots, 2n$. Define $T_i, i = 1, \cdots, 2n$, as in the lemma, and let $T = T_1 \cdots T_{2n}$. Then since $\bigcap_{i=1}^{2n} \langle x_i \rangle^{\perp} = 0$, T has no non-zero fixed points, and so $T + 1$ is non-singular.

COROLLARY. *For K a perfect field, if $\delta : Sp(V) \to V$ is a non-zero derivation such that $\delta \mid O(V) = 0$, $O \neq O_2(+1, \mathbf{F}_2)$, then δ is non-inner. In particular, the derivation u defined in 1.6 is non-inner.*

Let $C(Q)$ denote the *Clifford algebra* of the quadratic form Q, and $C^+(Q)$ its even subalgebra. The elements of $O(Q)$ induce automorphisms of $C^+(Q)$ and so of its center, Z. For $T \in O(Q)$, write $D(T)$ for the automorphism of Z induced by T. Z has a K-basis consisting of 1 and $z_{\mathfrak{B}}$; here $\mathfrak{B} = \{x_1, \cdots, x_{2n}\}$ is a symplectic basis of V with $B(x_i, x_j) = \delta_{j,2n-i+1}$ ($\delta_{rs} = 1$ if $r = s$, and $\delta_{rs} = 0$ otherwise) and $z_{\mathfrak{B}} = x_1 x_{2n} + \cdots + x_n x_{n+1}$ [4, Theorem 11.2.3, p. 44]. $z_{\mathfrak{B}}$ satisfies $z_{\mathfrak{B}}^2 + z_{\mathfrak{B}} = \Delta_{\mathfrak{B}}(Q)$, where the *pseudo-discriminant*

$$\Delta_{\mathfrak{B}}(Q) = \sum_{i=1}^{n} Q(x_i) Q(x_{2n-i+1}).$$

Let $x \in V$ be non-singular, and let T be the orthogonal transvection defined by $Tv = v + (1/Q(x))B(x, v)x$ for all $v \in V$. Complete x to a symplectic basis $\mathfrak{B}$ as above, with $x = x_1$. Then $D(T)z_{\mathfrak{B}} = z_{\mathfrak{B}} + 1$, and so D is a homomorphism of $O(Q)$ onto the group of automorphisms of Z over K. Therefore, if z is any generator for Z over K of the form $z_{\mathfrak{B}}$ for some symplectic basis $\mathfrak{B}$, and if T is any element of $O(Q)$, $D(T)(z) = z + d(T)$, where $d(T) = 0$ or 1 according as $T \in \operatorname{Ker} D$ or not. The *rotation subgroup* $O^+(Q) \leq O(Q)$ is defined to be $\operatorname{Ker} D$ (or $\operatorname{Ker} d$). The map d is the *Dickson Invariant*; it is a homomorphism from $O(Q)$ into the additive group of K.

For B associated with Q, the elements of $Sp(B)$ not in $O(Q)$ do not induce automorphisms of $C(Q)$. However, for $T \in Sp(B)$ and for any symplectic basis $\mathfrak{B}$, $z_{T\mathfrak{B}}$ is an element of Z, so $z_{T\mathfrak{B}} = \alpha z_{\mathfrak{B}} + \beta$ for some $\alpha, \beta \in K$. From the relation $z_{\mathfrak{B}}^2 + z_{\mathfrak{B}} = \Delta_{\mathfrak{B}}(Q)$ we obtain

$$\alpha^2 z_{\mathfrak{B}}^2 + \beta^2 + \alpha z_{\mathfrak{B}} + \beta = \Delta_{T\mathfrak{B}}(Q) \in K;$$

hence $\alpha^2 = \alpha$ and $\alpha = 1$. Then $z_{T\mathfrak{B}} = z_{\mathfrak{B}} + \beta$. For $T \notin O(Q)$, Dieudonné [9] extends d to $Sp(B)$, defining $d(T) = z_{T\mathfrak{B}} + z_{\mathfrak{B}}$. However, this definition depends on the choice of the basis $\mathfrak{B}$ as well as on the choice of Q. Writing $d_{\mathfrak{B}}$ to denote this dependence, $d_{\mathfrak{B}}(ST) = d_{T\mathfrak{B}}(S) + d_{\mathfrak{B}}(T)$ for $T, S \in Sp(B)$. When $Sp(B)$ is simple, d cannot be a homomorphism, hence $d_{T\mathfrak{B}}(S)$ must be different from $d_{\mathfrak{B}}(S)$ for some $T \in Sp(B)$. If $Sp(B) = Sp_2(\mathbf{F}_2)$ or $Sp_4(\mathbf{F}_2)$

and Q has non-maximal index, a direct computation shows that there exists $T \in Sp(B)$ such that $d_{\mathfrak{B}}(T)$ depends on the choice of $\mathfrak{B}$.

Let K be perfect, and let B_1 and Q_1 be associated forms on the K-space U. Form the K-space $V = U \oplus W$, where W is a 2-dimensional K-space with associated forms B_2 and Q_2. Then a qudaratic form Q and its associated bilinear form B can be defined on V by $Q \mid U = Q_1$, $Q \mid W = Q_2$, and $B(u, w) = 0$ for $u \in U$, $w \in W$. Choose a hyperbolic pair of non-singular vectors $x, y \in W$ and assume $Q(x) = 1$. By 1.7, $A \in O_{\langle x \rangle}$ has the form $Ax = x$, $Av = B(u(T), Tv)x + Tv$ for all $v \in U$ and $Ay = \alpha x + u(T) + y$, where $T \in Sp(B_1)$, $u : Sp(B_1) \to U$ is the derivation defined in 1.6, and $\alpha \in K$ is a solution of $X^2 + X = Q_1(u(T))$. Write $A = A(T, \alpha)$.

Let T_0 be the transvection taking $v \in V$ to $v + B(x, v)x$. If d is the Dickson Invariant on $O(Q)$, $d(T_0) = 1$. Now $T_0 A(T, \alpha) = A(T, \alpha + 1)$, so, since d is a homomorphism, $A(T, \alpha) \in O^+(Q)$ if and only if $A(T, \alpha + 1) \notin O^+(Q)$. That is, the subgroup $O^+(Q)_{\langle x \rangle}$ contains exactly one element $A(T, \alpha)$ for each $T \in Sp(B_1)$. Thus we have defined a function $a : Sp(B_1) \to K$ by $a(T) = \alpha$ if $A(T, \alpha) \in O^+(Q)_{\langle x \rangle}$.

Suppose $T \in O(Q_1)$. Choose a symplectic basis $\mathfrak{B}_1$ for $U$and complete $\mathfrak{B}_1$ with x, y to a symplectic basis $\mathfrak{B}$ for V. Then

$$D(A(T, \alpha))z_{\mathfrak{B}} = x(\alpha x + y) + D(T)z_{\mathfrak{B}_1} = \alpha + xy + z_{\mathfrak{B}_1} + d_1(T)$$
$$= \alpha + d_1(T) + z_{\mathfrak{B}},$$

where d_1 is the Dickson Invariant on $O(Q_1)$. If $A(T, \alpha) \in O^+(Q)$, then $A(T, \alpha) \in \operatorname{Ker} D$, and $\alpha = d_1(T)$. But for $A(T, \alpha) \in O^+(Q)$, $\alpha = a(T)$. Thus for $T \in O(Q_1)$, $a(T) = d_1(T)$, and a extends the Dickson Invariant on $O(Q_1)$ to $Sp(B_1)$. Although this extension depends on the choice of the quadratic form Q_1, it is independent of the choice of the basis $\mathfrak{B}_1$.

THEOREM 1.11. *Let Q be a quadratic form on the K-space V for K a perfect field. Let $x, y \in V$ be a hyperbolic pair of non-singular vectors with $Q(x) = 1$. Then if V is of dimension at least 4, the linear transformation $A \in O^+(Q)_{\langle x \rangle}$ if and only if $Ax = x$, $Av = B(u(T), Tv)x + Tv$ for all $v \in \langle x, y \rangle^{\perp}$, and $Ay = a(T)x + u(T) + y$, where $T \in Sp(\langle x, y \rangle^{\perp})$, u is defined as in 1.6, and a is defined as above. If $T \in O(\langle x, y \rangle^{\perp})$, $a(T) = d(T)$, where d is the Dickson Invariant on $O(\langle x, y \rangle^{\perp})$.*

THEOREM 1.12. *Let Q be of maximal index on the $\mathbf{F}_2$-space V of dimension $2n \geq 4$. Let $V = U \oplus W$ with U, W totally singular. Then*

$$O^+(V) = \langle O(V)_U, O(V)_W \rangle.$$

Proof. First we will show that $O_U, O_W \leq O^+$. Choose bases $u_1, \cdots, u_n$ and $w_1, \cdots, w_n$ of U and W respectively such that $B(u_i, w_j) = \delta_{ij}$, $i, j = 1, \cdots, n$. For $X \in \operatorname{Hom}(U, U)$ and $X' \in \operatorname{Hom}(W, W)$, define $T(X, X')$ on V by $T(X, X')u = Xu$ and $T(X, X')w = X'w$ for all $u \in U$, $w \in W$. Then

$T(X, X') \in Sp(B)$ if and only if $X' = X^{-1t}$ for $X \in GL_n(\mathbf{F}_2)$. Write $T(X) = T(X, X^{-1t})$. For $Y \in \text{Hom}(W, U)$, define $S(Y)$ on V by $S(Y)u = u$ and $S(Y)w = Yw + w$ for all $u \in U$, $w \in W$. Then $S(Y) \in Sp(B)$ if and only if $Y = Y^t$. Clearly $T(X) \in O(V)$ for every $X \in GL_n(\mathbf{F}_2)$, and $S(Y) \in O(V)$ if and only if Y is alternate. Thus we see that O_U is generated by the elements $T(X)$ and $S(Y)$ for $X \in GL_n(\mathbf{F}_2)$ and Y alternate, $n \times n$.

Let $\mathfrak{B} = \{u_1, \cdots, u_n, w_1, \cdots, w_n\}$, and define $z_{\mathfrak{B}}$ as above. Let

$$S_{ij} = S(E_{ij} + E_{ji}), \qquad i \neq j, i, j = 1, \cdots, n,$$

where E_{rs} is the $n \times n$ matrix having a 1 in the intersection of the *r-th* row and the *s-th* column, and all other entries zero. Then we see that $S_{ij} z_{\mathfrak{B}} = z_{\mathfrak{B}}$, so $S_{ij} \in O^+$, $i \neq j$, $i, j = 1, \cdots, n$. Since $S(Y)S(Y') = S(Y + Y')$, and since the $E_{ij} + E_{ji}$, $i \neq j$, $i, j = 1, \cdots, n$ generate the alternate $n \times n$ matrices additively, $S(Y) \in O^+$ for every alternate $n \times n$ Y.

Let X be a transvection in $GL(U)$ with center $\langle x \rangle$, and choose $\langle y \rangle$ so that $U = \langle y \rangle \oplus \text{Ker}\ (X - 1)$. Complete a basis $x_1 = x, x_2, \cdots, x_{n-1}$ for $\text{Ker}\ (X - 1)$ with $x_n = y$ to a basis for U, and choose a basis $y_1, \cdots, y_n$ for W so that $B(x_i, y_j) = \delta_{i,j}$, $i, j = 1, \cdots, n$. Then if

$$\mathfrak{B} = \{x_1, \cdots, x_n, y_1, \cdots, y_n\},$$

$T(X)z_{\mathfrak{B}} = z_{\mathfrak{B}}$, and $T(X) \in O^+$. Since $GL(U) = SL(U)$ is generated by transvections, $T(X) \in O^+$ for all $X \in GL_n(\mathbf{F}_2)$. Hence $O_U \leq O^+$. Similarly $O_W \leq O^+$.

To show that $\langle O_U, O_W \rangle = O^+$, we will draw on the Lie Theory. We refer the reader to [15] and [5] or [3] for a discussion of the relevant material. O^+ is the Chevalley group coming from the Lie algebra of type D_n. With respect to the original basis $\mathfrak{B}$, the $2n \times 2n$ diagonal elements

$$h = \textstyle\sum_{i=1}^n \lambda_i E_{ii} - \sum_{i=1}^n \lambda_i E_{n+i,n+i}, \qquad \lambda_i \in \mathbf{F}_2, i = 1, \cdots n,$$

of D_n yield the positive roots $r = \lambda_p - \lambda_q$, $1 \leq p < q \leq n$, corresponding to the root vectors

$$X_r = \begin{vmatrix} E_{pq} & 0 \\ 0 & E_{qp} \end{vmatrix},$$

and $r = \lambda_p + \lambda_q$, $1 \leq p < q \leq n$, corresponding to the root vectors

$$X_r = \begin{vmatrix} 0 & E_{pq} - E_{qp} \\ 0 & 0 \end{vmatrix}.$$

For $r = -(\lambda_p + \lambda_q)$, the root vector is

$$X_r = \begin{vmatrix} 0 & 0 \\ E_{pq} - E_{qp} & 0 \end{vmatrix}.$$

We have a fundamental set $F = \{a_1, \cdots, a_n\}$ of roots with

$$a_i = \lambda_i - \lambda_{i+1}, i = 1, \cdots, n-1 \quad \text{and} \quad a_n = \lambda_{n-1} + \lambda_n .$$

$x_r(\tau) = 1 + \tau X_r$ for r a root and $\tau \in \mathbf{F}_2$, and

$$O^+ = \langle x_r(\tau) : \tau \in \mathbf{F}_2 \quad \text{and} \quad r \text{ a root}\rangle.$$

O^+ has $\mathbf{B} - \mathbf{N}$ structure for

$$\mathbf{B} = \langle x_r(\tau) : \tau \in \mathbf{F}_2 \text{ and } r \text{ a positive root}\rangle$$

and

$$\mathbf{N} = \langle \omega_{a_i} : i = 1, \cdots, n\rangle,$$

where

$$\omega_a = \phi_a \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since ϕ_a maps $SL_2(\mathbf{F}_2)$ onto $\langle x_a(\tau), x_{-a}(\tau)\rangle$, with

$$\phi_a \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} = x_a(\tau) \quad \text{and} \quad \phi_a \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix} = x_{-a}(\tau),$$

we see that $\omega_a = x_a(1)x_{-a}(1)x_a(1)$.

For $S \subseteq F$, we have the *parabolic subgroups* $G_S = \mathbf{BN}_S\mathbf{B}$, with $\mathbf{N}_S = \langle \omega_a : a \in S\rangle$. For S a maximal subset of F, G_S is maximal in O^+. We will show that O_U is a maximal parabolic subgroup. We see immediately that $\mathbf{B} \leq O_U$. Since

$$(1 + E_{pq})(1 + E_{qp})(1 + E_{pq}) = R_{pq} = 1 + E_{pp} + E_{qq} + E_{pq} + E_{qp},$$

$$\omega_r = \begin{vmatrix} R_{pq} & 0 \\ 0 & R_{qp} \end{vmatrix} \quad \text{for } r = \lambda_p - \lambda_q, 1 \leq p < q \leq n.$$

For $r = \lambda_p + \lambda_q$,

$$\omega_r = \begin{vmatrix} 1 + E_{pp} + E_{qq} & E_{pq} + E_{qp} \\ E_{pq} + E_{qp} & 1 + E_{pp} + E_{qq} \end{vmatrix}.$$

Now it is easily seen that $\omega_{a_i} \in O_U$ for $i = 1, \cdots, n-1$. So $G_S \leq O_U$ for $S = \{a_1, \cdots, a_{n-1}\}$. Hence $O_U = G_S$, O_U is maximal in O^+, and $O^+ = \langle O_U, O_W\rangle$.

However, we can say more. Let

$$U_i = \langle u_1, \cdots, u_i\rangle \quad \text{for } i \leq n-2,$$

and let

$$U_{n-1} = \langle u_1, \cdots, u_{n-2}, w_n\rangle.$$

Then we see that $\mathbf{B} \leq O^+_{U_i}$. Furthermore, for $j \neq i$, $\omega_{a_j} \in O^+_{U_i}$, for $i = 1, \cdots, n-1$. So $G_{S_i} = O^+_{U_i}$ for $S_i = \{a_j : j \neq i\}$. In particular, $O^+_{\langle u_1\rangle}$ is a maximal subgroup, so we see again that O^+, and hence O, is primitive on the singular lines of V.

2. $H^1(G, V)$ for $G = SL(V), GL(V), Sp(V)$

Now we consider some of the groups $H^1(G, V)$ with V a finite-dimensional K-space and $G \leq GL(V)$. Note that $H^1(G, V)$ and $H^1(G, V^*)$ (V^* being the dual space of V) are isomorphic as K-spaces, so throughout this paper, derivations will be from G to V or to V^*, whichever is more convenient. Note also the elementary fact that for δ a derivation from a group G to a unitary module for G, $\delta(1_G) = 0$.

D. G. Higman implicitly computed the dimension of $H^1(G, V)$ when $G = SL(V)$. His results include:

THEOREM 2.1. *Let V be of dimension n over an arbitrary field K. If $n \geq 4$, the K-dimension of $H^1(SL(V), V)$ is zero. If $n = 2$ and $K = \mathbf{F}_2$, the dimension is again zero. If $n = 3$ and $K = \mathbf{F}_2$, the dimension is at most one* [12, Lemma 4, p. 441].

As a corollary of this we have:

THEOREM 2.2. *If the dimension n of the K-space V (K arbitrary) is at least 4, then the K-dimension of $H^1(GL(V), V)$ is zero. If $n = 2$ and $K = \mathbf{F}_2$, the dimension of $H^1(GL(V), V)$ is again zero.*

Proof. In the cases under consideration, the derivations from $SL(V)$ to V are all inner, so for $\delta \in \operatorname{Der}(GL(V), V)$, $\delta \mid SL(V)$ is an inner derivation. If necessary, change δ by subtracting off an inner derivation and assume $\delta \mid SL(V) = 0$.

Let $T \in GL(V)$ and $S \in SL(V)$. Since $SL(V) \trianglelefteq GL(V)$, there exists $U \in SL(V)$ with $ST = TU$. Hence $S(\delta T) = \delta T$, and this equality holds for all $S \in SL(V)$. However, $SL(V)$ has no non-zero fixed points, so $\delta T = 0$ and $\delta = 0$. Thus the original δ was inner.

We also have:

THEOREM 2.3. *If the characteristic of K is not two, then the K-dimension of $H^1(Sp(V), V)$ is zero.*

Proof. Let $\delta \in \operatorname{Der}(Sp(V), V)$ and let $T \in Sp(V)$. Then $T(-1_V) = (-1_V)T$ implies

$$T\delta(-1) + \delta T = (-1)(\delta T) + \delta(-1),$$

so $\delta T = (T - 1)(-1/2)\delta(-1)$, and δ is an inner derivation.

For other similar results, see [11, Section 14].

3. $H^1(G, V)$ for $G = Sp(K)$ or $O(K)$, K a perfect field of characteristic two, $\neq \mathbf{F}_2$

Throughout this section, K will be a perfect field of characteristic two having more than two elements, unless specified otherwise.

THEOREM 3.1. *Let V be of dimension $2n$ over K. Then the K-dimension of $H^1(O(V), V)$ is zero.*

Proof. The proof will be by induction on the dimension $2n$ of V.

LEMMA 3.2. *For V of dimension 2 over K (possibly equal to $\mathbf{F}_2$), the dimension of $H^1(O(V), V)$ is zero.*

Proof. Choose a hyperbolic pair $x, y \in V$ with $Q(x) = 1$ and $Q(y) = \sigma$. If $K \neq \mathbf{F}_2$, we can assume $\sigma \neq 0$. Then

$$Q(\alpha x + \beta y) = \alpha^2 + \alpha\beta + \beta^2\sigma \quad \text{for } \alpha, \beta \in K.$$

Let A be the quotient of the ring of polynomials in the indeterminate θ by the ideal generated by $\theta^2 + \theta + \sigma$. Then 1 and θ form a K-basis for A. The map sending θ to $\theta + 1$ induces an automorphism of A; write $\bar{t}$ for the image of $t \in A$ under this automorphism. If $t = \alpha + \beta\theta$, $\alpha, \beta \in K$, then

$$t\bar{t} = \alpha^2 + \alpha\beta + \beta^2\sigma.$$

Hence we have a model for V and Q with $V = A$ and $Q(t) = t\bar{t}$ for $t \in A$. Working within this model, let $O = O(Q)$. Suppose $T_0 \in O_1$, $T_0 \neq 1$. Then $T_0(1) = 1$ and $T_0(1 + \theta) = 1 + T_0\,\theta$. $Q(1 + \theta) = Q(\theta) = \sigma$, so

$$\sigma = Q(T_0(1 + \theta)) = 1 + T_0\,\theta + \overline{T_0\,\theta} + Q(T_0\,\theta).$$

Hence $1 + T_0\,\theta + \overline{T_0\,\theta} = 0$. If $T_0\,\theta = \alpha + \beta\theta$, we see $\beta = 1$ and $\alpha^2 = \alpha$, $\alpha = 1$. Therefore $O_1 = \langle T_0\rangle$ and $T_0\,t = \bar{t}$ for every $t \in A$.

Denote by S_t the left multiplication by $t \in A$, so $S_t\,v = tv$ for $v \in A$. Let $U = \{S_t : t \in A, t\bar{t} = 1\}$. Then $U \leq O$, and U is isomorphic to a subgroup of the group of units in A. Identify $S_t \in U$ with $t \in A$. Then for $u \in U$ and T_0 as above, $T_0\,uT_0^{-1} = \bar{u}$, and T_0 normalizes U. If $S \in O$, $S(1) = u\cdot 1$ for some $u \in U$, and $u^{-1}S$ fixes 1. Therefore, $S \in U\langle T_0\rangle$; that is, $O = U\langle T_0\rangle$.

Now let $\delta \in$ Der (O, A). Then, since U is commutative, $(u + 1)\delta v = (v + 1)\delta u$ for all $u, v \in U$. If U is trivial, $\delta \mid U = 0$. Otherwise, choose $u_0 \neq 1$ in U. Then for every $v \in U$, $\delta v = (v + 1)\delta u_0(u_0 + 1)$, and $\delta \mid U$ is an inner derivation.

Suppose $U \neq 1$. If necessary, change δ by subtracting off an inner derivation, and assume $\delta \mid U = 0$. $T_0\,u = \bar{u}T_0$, so $\delta T_0 = \bar{u}\delta T_0$ for all $u \in U$. Since there is a $u \neq 1$ in U, δT_0 must be zero. Therefore $\delta = 0$, and the original δ was inner. If $U = 1$, $O = \langle T_0\rangle$. Because T_0 is an involution, $T_0(\delta T_0) = \delta T_0$. But $T_0(\delta T_0) = \overline{\delta T_0}$, so $\delta T_0 \in K$. Suppose $\delta T_0 = \alpha$. With respect to the basis $1, \theta$ of V, the matrix of $T_0 + 1$ is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We thus see that δ is the inner derivation based on $\alpha\theta$.

Return now to the proof of 3.1. Let $\delta \in \operatorname{Der}(O(V), V)$. Suppose the dimension $2n$ of V is greater than two, and assume the theorem is true for spaces of dimension less than $2n$. Since $2n \geq 4$, we may choose a hyperbolic pair x, y of singular vectors.

First we will show that δ is homologous to zero on $O_{\langle x,y\rangle}$. Let $U = \langle x, y\rangle^{\perp}$, so $V = \langle x, y\rangle \oplus U$. $S \in O_{\langle x,y\rangle}$ has the form $Sw = Aw$, $Su = Tu$ for $w \in \langle x, y\rangle$, $u \in U$ with $A \in O(\langle x, y\rangle)$ and $T \in O(U)$. Write $S = S(A, T)$. Suppose

$$\delta S(A, 1) = a(A) + f(A) \quad \text{with } a(A) \in \langle x, y\rangle \text{ and } f(A) \in U.$$

Then $S(A, 1)S(C, 1) = S(AC, 1)$ implies that

$$a(AC) = Aa(C) + a(A) \quad \text{and} \quad f(AC) = f(A) + f(C).$$

That is, a is a derivation on $O(\langle x, y\rangle)$, and so, by 3.2, a is inner. If necessary, change δ by an inner derivation based on a vector in $\langle x, y\rangle$ and assume $a = 0$. Similarly, suppose

$$\delta S(1, T) = b(T) + g(T) \quad \text{with } b(T) \in \langle x, y\rangle \text{ and } g(T) \in U.$$

Arguing as above, change δ, if necessary, by an inner derivation based on a vector in U, and assume $g = 0$. Now

$$S(A, 1)S(1, T) = S(1, T)S(A, 1)$$

implies that $(A + 1)b(T) = 0$ and $(T + 1)f(A) = 0$ for every $A \in O(\langle x, y\rangle)$ and every $T \in O(U)$. Since $K \neq \mathbf{F}_2$, by 1.10, A and T can be chosen in $O(\langle x, y\rangle)$ and $O(U)$, respectively, with $A + 1$ and $T + 1$ non-singular. Therefore, b and f are both identically zero, and $\delta \mid O_{\langle x,y\rangle} = 0$.

Now we will show that $\delta \mid O_{\langle x\rangle} = 0$. Write $V = \langle x\rangle \oplus U \oplus \langle y\rangle$. An element $A \in Sp(V)_{\langle x\rangle}$ has the form $Ax = \alpha x$, $Au = \alpha B(w, Tu)x + Tu$ for all $u \in U$, and $Ay = \beta x + w + (1/\alpha)y$, with $\alpha \in K^*$, $\beta \in K$, $w \in U$, and $T \in Sp(U)$. If A is a Q-isomorphism, then $T \in O(U)$ and $\beta = \alpha Q(w)$. So $A \in O_{\langle x\rangle}$ determines and is determined by $T \in O(U)$, $w \in U$, and $\alpha \in K^*$. Write $A = A(T, w, \alpha)$. Since $\delta \mid O_{\langle x,y\rangle} = 0$, $\delta A(T, 0, \alpha) = 0$. Consider the subgroup of $O_{\langle x\rangle}$ consisting of all $A(1, w, 1) = S(w)$. Suppose

$$\delta S(w) = p(w) + h(w) + q(w),$$

with $p(w) \in \langle x\rangle$, $h(w) \in U$, and $q(w) \in \langle y\rangle$. Since $S(w)S(v) = S(w + v)$, it follows that

$$p(w + v) = p(w) + p(v) + B(w, h(v)) \tag{1}$$

$$h(w + v) = h(w) + h(v) + wq(v) \tag{2}$$

$$q(w + v) = q(w) + q(v) \tag{3}$$

for all $w, v \in U$.

Since $S(0) = 1$ and $\delta(1) = 0$, $p(0) = 0$, $h(0) = 0$, and $q(0) = 0$. Thus relation (2) implies $vq(v) = 0$ for all $v \in U$, and so $q = 0$.

A routine computation gives $A(T, 0, \alpha)S(w) = S(\alpha Tw)A(T^{-1}, 0, \alpha)$. Hence

$$(4) \qquad \alpha p(w) = p(\alpha Tw)$$

$$(5) \qquad Th(w) = h(\alpha Tw).$$

By relations (2) and (5), for $\alpha \neq 1$ in K^*, $h((\alpha + 1)w) = 0$. Therefore, again by (5), $h = 0$. Relation (1) now becomes $p(w + v) = p(w) + p(v)$. Relation (4) implies then, that $p((T + 1)w) = 0$ for every $T \in O(U)$ and every $w \in U$. By 1.10, $p = 0$. Thus $\delta S(w) = 0$ for every $w \in U$.

Let $A(T, w, \alpha)$ be an arbitrary element of $O_{\langle x \rangle}$. Clearly

$$A(T, w, \alpha) = A(1, 0, \alpha)S(w)A(T, 0, 1),$$

so $\delta A(T, w, \alpha) = 0$ and $\delta \mid O_{\langle x \rangle} = 0$.

Define S_0 by $S_0 x = y$, $S_0 y = x$ and $S_0 \mid U = 1$, so $S_0 \in O_{\langle x,y \rangle}$ and $\delta S_0 = 0$. $O_{\langle y \rangle} = S_0 O_{\langle x \rangle} S_0$, so we also have $\delta \mid O_{\langle y \rangle} = 0$. By 1.9, $O = \langle O_{\langle x \rangle}, O_{\langle y \rangle} \rangle$, and therefore $\delta = 0$, and the original δ was inner. Hence the K-dimension of $H^1(O(V), V)$ is zero.

THEOREM 3.3. *Let V be a K-space of dimension $2n \geq 2$. Then the dimension of $H^1(Sp(V), V)$ over K is one.*

Proof. Let Q and B be associated forms on V, with $\nu(Q) \geq 1$, and let $O = O(Q)$. Choose $x \in V$, singular, and let T_0 be the transvection taking $v \in V$ to $v + B(v, x)x$. By 1.5, O is maximal in $Sp(V)$, so $Sp(V) = \langle O, T_0 O T_0 \rangle$.

Suppose $\delta \in \operatorname{Der}(Sp(V), V)$. O and $T_0 O T_0$ are orthogonal groups in $Sp(V)$, $T_0 O T_0$ being the group of the form QT_0. By 3.1, the restrictions of δ to O and $T_0 O T_0$ respectively are inner derivations. We may assume $\delta \mid O = 0$. Suppose $v_0 \in V$ is such that for all $A \in T_0 O T_0$, $\delta A = (A + 1)v_0$. By 1.4, $O \cap (T_0 O T_0) = O_x$. For $A \in O_x$, $\delta A = (A + 1)v_0$; that is $Av_0 = v_0$ for every $A \in O_x$. However, the fixed points of O_x are all on $\langle x \rangle$, so $v_0 = \alpha x$ for some $\alpha \in K$.

Hence the action of δ on O and $T_0 O T_0$ is determined up to a scalar multiple. If $T \in Sp(V)$, δT is determined by the action of δ on O and on $T_0 O T_0$. Therefore, δ is a scalar multiple of the derivation δ_0 which is zero on O and is an inner derivation based on x on $T_0 O T_0$. So we see that the dimension of $H^1(Sp(V), V)$ is at most one.

Recall the derivation u of 1.6. By the corollary to 1.10, u is not an inner derivation, so the dimension of $H^1(Sp(V), V)$ is exactly one. Moreover, by 1.6, $u(T_0) = x$ and $u \mid O = 0$, so for $S \in O$, $u(T_0 S T_0) = (T_0 S T_0 + 1)x$. Thus we see that in fact $u = \delta_0$.

4. $H^1(G, V)$ for $G = Sp(\mathbf{F}_2), O(\mathbf{F}_2)$

The proofs of 4.1 and 4.2 below require only that the underlying field K be perfect. By 3.1 and 3.3, only the case where $K = \mathbf{F}_2$ is actually needed, but since it does not significantly alter the arguments, the more general results are stated and proved. The proofs in Section 3 were also given, however, since they are much simpler.

THEOREM 4.1. *Let K be a perfect field, and let V be a K-space of dimension $2n \geqq 8$. Then the K-dimension of $H^1(Sp(V), V)$ is one.*

Proof. Let $\delta \in \operatorname{Der}(Sp(V), V)$. V can be written as the sum of two totally isotropic subspaces, $V = U \oplus W$, and bases $u_1, \cdots, u_n$ and $v_1, \cdots, v_n$ for U and W respectively can be chosen such that $B(u_i, v_j) = \delta_{ij}$ [5, Theorem 1.3.2, p. 13]. Then $Sp(V)_U$ is generated by the elements $T(X)$ and $S(Y)$ (notation as in the proof of 1.12) with $X \in GL_n(K)$ and Y $n \times n$ symmetric over K. Write $X^{-1t} = X^*$, and suppose

$$\delta T(X) = k(X) + l(X^*), \quad \text{with } k(X) \in U \text{ and } l(X^*) \in W.$$

Then $T(X)T(Z) = T(XZ)$ implies $Xk(Z) + k(X) = k(XZ)$, and

$$X^*l(Z^*) + l(X^*) = l(X^*Z^*).$$

So we see that $k \in \operatorname{Der}(GL(U), U)$ and $l \in \operatorname{Der}(GL(W), W)$. By 2.2, k and l are inner. If necessary, change δ by an inner derivation based on a vector in U and again by an inner derivation based on a vector in W, and assume k and l are zero.

Now let $\delta S(Y) = r(Y) + s(Y)$, with $r(Y) \in U$ and $s(Y) \in W$. Then $S(Y)S(Y') = S(Y + Y')$ implies

$$r(Y) + r(Y') = Ys(Y') + r(Y + Y') \tag{1}$$

$$s(Y) + s(Y') = s(Y + Y'). \tag{2}$$

Since $r(0) = 0$, relation (1) implies $Ys(Y) = 0$ for all symmetric Y. Hence, if Y is non-singular, $s(Y) = 0$. Relation (1) is symmetric in Y and Y', so $Ys(Y') = Y's(Y)$ for all symmetric Y, Y'. Taking $Y' = 1$, we obtain $s(Y) = 0$ for all symmetric Y.

Now relation (1) becomes $r(Y + Y') = r(Y) + r(Y')$. From

$$T(X)S(Y) = S(XYX^t)T(X)$$

it follows that $Xr(Y) = r(XYX^t)$ for all non-singular X and all symmetric Y. If $XYX^t = Y$, then $(X + 1)r(Y) = 0$. Let

$$Y = Y_{ij} = \alpha E_{ij} + \alpha E_{ji}, \quad \text{with } \alpha \in K^* \text{ and } i \neq j,\ i, j = 1, \cdots, n.$$

If we choose $i = 1$ and $j = 2$, then for

$$X = \begin{vmatrix} 1 & 1 & \\ 1 & 0 & \\ & & X' \end{vmatrix},$$

with X' a non-singular $(n - 2) \times (n - 2)$ matrix, we see that $XY_{12}X^t = Y_{12}$. If $n - 2$ is even, we may apply 1.10 and assume that X' may be chosen with

$X' + 1$ non-singular. If $n - 2$ is odd, we have two cases. If $n - 2 = 3$, let

$$X' = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}.$$

If $n - 2 \geq 5$, let

$$X' = \begin{vmatrix} X'' & \\ & X''' \end{vmatrix}, \quad \text{with} \quad X'' = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

and X''' chosen such that $X''' + 1$ is non-singular. Thus, in all cases, we may choose X' so that $X' + 1$, and so $X + 1$, is non-singular. Hence we have $r(Y_{12}) = 0$. Similarly $r(Y_{ij}) = 0$ for all $i \neq j$. The Y_{ij} generate the alternate matrices additively, so by (1), $r(Y) = 0$ for all alternate Y.

Now let $Y_{ii} = \alpha E_{ii}$, $i = 1, \cdots, n$, with $\alpha \in K^*$. Consider Y_{11} and write

$$r(Y_{11}) = \alpha_1 u_1 + \cdots + \alpha_n u_n .$$

If

$$X = \begin{vmatrix} 1 & 0 & \\ 1 & 1 & \\ & & X' \end{vmatrix}$$

with X' a non-singular $(n - 2) \times (n - 2)$ matrix such that $X' + 1$ is non-singular, then $XY_{11} X^t = Y_{11}$, and we see from $(X + 1)r(Y_{11}) = 0$ that $\alpha_2 = \cdots = \alpha_n = 0$. So $r(Y_{11}) = \alpha_1 u_1$, with $\alpha_1 \in K$. Similarly, $r(Y_{ii}) = \alpha_i u_i$, with $\alpha_i \in K$, $i = 2, \cdots, n$.

Return now to the relation $Xr(Y) = r(XYX^t)$. If $Y = E_{ii}$ and X, with entries χ_{lm}, is chosen such that χ_{ii} is the only non-zero entry in the i-th column of X, a direct calculation shows $XYX^t = \chi_{ii}^2 E_{ii}$, and hence $\chi_{ii}\, r(E_{ii}) = r(\chi_{ii}{}^2 E_{ii})$. That is,

$$r(\alpha E_{ii}) = \sqrt{\alpha}\, r(E_{ii}) \quad \text{for } i = 1, \cdots, n \text{ and } \alpha \in K^*$$

(and trivially for $\alpha = 0$). If X is chosen with entries χ_{lm} such that $\chi_{ki} = 1$ for some $k \neq i$ and $\chi_{ji} = 0$ for all $j \neq k$, then

$$XE_{ii} X^t = (\chi_{ki})^2 E_{kk} = E_{kk},$$

and

$$Xr(E_{ii}) = \chi_{ki}\, \alpha_i u_k = \alpha_i u_k .$$

Therefore $\alpha_i u_k = \alpha_k u_k$, and $\alpha_i = \alpha_k$ for all $i, k = 1, \cdots, n$.

Let $\alpha = \alpha_1 = \cdots = \alpha_n$. Then, since r is additive, if $Y = (\eta_{lm})$ is a symmetric $n \times n$ matrix, $r(Y) = \alpha \sum_{i=1}^n \sqrt{\eta_{ii}}\, u_i$. So, up to a scalar multiple, r is completely determined; that is, for $Y = (\eta_{lm})$, $r(Y) = \alpha r_0(Y)$, where $r_0(Y) = \sum_{i=1}^n \sqrt{\eta_{ii}}\, u_i$.

An arbitrary element A of $Sp(V)_U$ has the form $Au = Xu$, $Aw = Yw + X^* w$ for $u \in U$, $w \in W$, with $X \in GL_n(K)$ and Y $n \times n$ symmetric over K such that

$YX^t = XY^t$. Thus $A = S(YX^t)T(X)$, and $\delta A = \alpha r_0(YX^t)$. Let A_0 be defined by $Au_i = w_i$ and $Aw_i = u_i$, $i = 1, \cdots, n$. If $\delta A_0 = u_0 + w_0$ with $u_0 \in U$, $w_0 \in W$, then, since $A_0 T(X) A_0 = T(X^*)$, $Xu_0 = w_0$ for all $X \in GL_n(K)$. Therefore $u_0 = w_0 = 0$, and $\delta A_0 = 0$. Since $A_0 \, Sp(V)_U \, A_0 = Sp(V)_W$, δ is determined up to the same scalar multiple α on $Sp(V)_W$. By 1.2,

$$Sp(V) = \langle Sp(V)_U, Sp(V)_W \rangle,$$

so the dimension of $H^1(Sp(V), V)$ is at most one. However, by the corollary to 1.10, the derivation $u : Sp(V) \to V$ of 1.6 is non-inner, so the dimension of $H^1(Sp(V), V)$ is exactly one.

THEOREM 4.2. *Under the hypothesis of* 4.1, *the dimension of* $H^1(O(+1, V), V)$ *over* K *is zero.*

Proof. Let $O = O(+1, V)$ and let $\delta \in \mathrm{Der}\,(O, V)$. Proceeding as in the proof of 4.1, we write $V = U \oplus W$, where U and W are totally singular subspaces of V, and choose bases $\{u_i\}$ and $\{w_i\}$ of U and W respectively such that $B(u_i, w_j) = \delta_{ij}$. As before, we first consider δ on O_U. Since U and W are totally singular, $T(X) \in O_U$ for every $X \in GL_n(K)$, and $S(Y) \in O_U$ if and only if Y is alternate. Arguing as in 4.1, we may assume $T(X) = 0$ for all $X \in GL_n(K)$.

As before, let $\delta S(Y) = r(Y) + s(Y)$. Then using relation (1) in the proof of 4.1, $Ys(Y) = 0$ for all alternate Y. If n is even, there exists non-singular, alternate Y, and we again obtain $s = 0$. If n is odd, choose

$$Y_1 = \begin{vmatrix} 0 & & \\ & Y' & \\ & & 1 \end{vmatrix} \quad \text{and} \quad Y_2 = \begin{vmatrix} Y' & & \\ & 2 & \\ & & 0 \end{vmatrix}$$

with the $Y'_i (n - 1) \times (n - 1)$ non-singular alternate matrices. Then $s(Y_1) = \sigma_1 w_1$ and $s(Y_2) = \sigma_2 w_n$, $\sigma_1, \sigma_2 \in K$, by (1). For any alternate Y, we have $Ys(Y_i) = Y_i s(Y)$, and we see that again $s = 0$. Now, exactly as in the preceding argument, we have $r(Y) = 0$ for all alternate Y, and $\delta \mid O_U = 0$.

The element A_0 is also in O, and $\delta A_0 = 0$, so $A_0 O_U A_0 = O_W$ implies $\delta \mid O_W = 0$. By 1.12, $O^+ = \langle O_U, O_W \rangle$, hence $\delta \mid O^+ = 0$. For $T \in O$ and $S \in O^+$, $ST = TS'$ for some $S' \in O^+$. Then $S\delta T = \delta T$, and δT is a fixed point for O^+. But O^+ is irreducible [4, Theorem 1.6.7, p. 33], so $\delta T = 0$ and $\delta = 0$. Therefore the dimension of $H^1(O, V)$ is zero.

COROLLARY 4.3. *Under the hypotheses of* 4.1, *the* K*-dimension of* $H^1(O^+(+1, V)V)$ *is zero.*

Proof. Arguing as for $S(Y)$, we can show directly that $(A_0 \, S(Y) A_0) = 0$ for every alternate Y, and so $\delta \mid O_W = 0$.

THEOREM 4.4. *If* V *is an* $\mathbf{F}_2$*-space of dimension* 4, *then the* $\mathbf{F}_2$*-dimension of* $H^1(Sp(V), V)$ *is one.*

Proof. Since by 2.2, the derivations from $GL_2(\mathbf{F}_2)$ to its standard module are all inner, we can use without change the proof of 4.1.

THEOREM 4.5. *If V is a 4-dimensional $\mathbf{F}_2$-space, the $\mathbf{F}_2$-dimension of*

$$H^1(O(+1, V), V)$$

is zero.

Proof. First we construct a model for $O(+1, V)$. Let V be the 2×2 matrices over $\mathbf{F}_2$, and for $X \in V$ define $Q(X) = \det X$. Then Q is a quadratic form whose associated bilinear form is non-degenerate. The subspace of all matrices of the form

$$\begin{vmatrix} 0 & \alpha \\ 0 & \beta \end{vmatrix}$$

is totally singular, so $\nu(Q) = 2$. For $A, C \in SL_2(\mathbf{F}_2)$, define a transformation $S(A, C)$ on V by $S(A, C)X = A\ X\ C$, $X \in V$. Clearly $S(A, C) \in O(Q)$. Let T be the transformation on V given by $TX = X^t$; then T is also in $O(Q)$. The group generated by the $S(A, C)$ is isomorphic to $SL_2(\mathbf{F}_2) \times SL_2(\mathbf{F}_2)$ and so has order 36. T is not in this group, so the order of $\langle SL_2(\mathbf{F}_2) \times SL_2(\mathbf{F}_2), T\rangle$ is at least 72. The order of $O(Q)$ is 72, so

$$O(Q) \simeq \langle SL_2(\mathbf{F}_2) \times SL_2(\mathbf{F}_2), T\rangle.$$

Let $\delta \in \operatorname{Der}(O(Q), V)$. Since $S(A, 1)S(1, C) = S(1, C)S(A, 1)$, we have

$$(*) \qquad (A + 1)\delta S(1, C) = \delta S(A, 1)(C + 1).$$

Choose $C_0 \in SL_2(\mathbf{F}_2)$ with $C_0 + 1$ non-singular. Then

$$\delta S(A, 1) = (S(A, 1) + 1)(\delta S(1, C_0)/(C_0 + 1)),$$

and δ is inner on the $S(A, 1)$. Assume $\delta S(A, 1) = 0$ for all $A \in SL_2(\mathbf{F}_2)$. Then $(*)$ implies $\delta S(1, C)$ is a fixed point for $SL_2(\mathbf{F}_2)$, and so $\delta S(1, C) = 0$ for all $C \in SL_2(\mathbf{F}_2)$.

Now $TS(A, 1)X = S(1, A^t)TX$ for all $X \in V$, so $\delta T = S(1, A^t)\delta T$. Thus $(\delta T)(A^t + 1) = 0$ for all $A \in SL_2(\mathbf{F}_2)$, so $\delta T = 0$ and $\delta = 0$.

THEOREM 4.6. *If K is a perfect field and V is a K-space of dimension at least* 10, *then the K-dimension of $H^1(O(-1, V), V)$ is zero.*

Proof. Choose $x, y \in V$ with $Q(x) = 1$, $Q(y) = \sigma \neq 0$, and $B(x, y) = 1$, such that $\langle x, y\rangle$ contains no singular vectors, and let $U = \langle x, y\rangle^{\perp}$. Let $O = O(-1, V)$, and let $\delta \in \operatorname{Der}(O, V)$. Since the dimension of U is at least 8 and $Q \mid U$ is a form of maximal index, by 4.2 we may use the arguments of 3.1 and assume that $\delta \mid O_{\langle x,y\rangle} = 0$.

Now we will show that $\delta \mid O_{\langle x\rangle} = 0$. By 1.7, the elements of $O_{\langle x\rangle}$ are the $A(T, \alpha)$ for $T \in Sp(U)$ and $\alpha \in K$ a solution of $X^2 + X = Q(u(T))$. Since

$$A(T, \alpha)A(1, 1) = A(T, \alpha + 1) \quad \text{and} \quad \delta A(1, 1) = 0,$$

$\delta A(T, \alpha)$ is independent of α. Let $\delta A(T, \alpha) = p(T) + h(T) + q(T)$ with $p(T) \in \langle x \rangle$, $h(T) \in U$, and $q(T) \in \langle y \rangle$. $A(T, \alpha)A(S, \beta) = A(TS, *)$ implies

$$(1) \qquad p(TS) = p(T) + p(S) + B(u(T), Th(S))$$

$$(2) \qquad h(TS) = Th(S) + h(T) + u(T)q(S)$$

$$(3) \qquad q(TS) = q(T) + q(S).$$

Since the dimension of U is at least 8, $Sp(U)$ is simple, and so (3) implies $q = 0$.

Now relation (2) implies that h is a derivation on $Sp(U)$, and 4.1 tells us that h is homologous to a scalar multiple of the non-inner derivation u. Since $T \in O(U)$ implies $A(T, \alpha) \in O_{\langle x,y \rangle}$, $h \mid O(U) = 0$. Hence, by 1.10, we may suppose $h(T) = \lambda u(T)$ for all $T \in Sp(U)$, $\lambda \in K$.

Relation (1) thus becomes $p(TS) = p(T) + p(S) + \lambda B(u(T), Tu(S))$. Recall the extension a of the Dickson invariant d given in 1.11: $A(T, \alpha) \in O^+_{\langle x \rangle}$ if and only if $\alpha = a(T)$, for all $T \in Sp(U)$. The invariant a satisfies

$$a(TS) = a(T) + a(S) + B(u(T), Tu(S)),$$

so if $\lambda \neq 0$, $k = p + \lambda a$ is a homomorphism on $Sp(U)$. Since $Sp(U)$ is simple and $k \mid O^+(U) = 0$, $k = 0$ and $p = \lambda a$. However, $p \mid O(U) = 0$ and $a \mid O(U) \neq 0$. Hence $\lambda = 0$, and so p and h are zero. Therefore $\delta \mid O_{\langle x \rangle} = 0$.

If R is the transformation taking x to $(1/\sqrt{\sigma})y$, y to $\sqrt{\sigma}\, x$ and fixing U point-wise, $R \in O_{\langle x,y \rangle}$ and $RO_{\langle x \rangle} R = O_{\langle y \rangle}$, so $\delta \mid O_{\langle y \rangle} = 0$. By 1.8,

$$O = \langle O_{\langle x \rangle}, O_{\langle y \rangle} \rangle,$$

so $\delta = 0$. Therefore the dimension of $H^1(O, V)$ is zero.

THEOREM 4.7. *If V is an $\mathbf{F}_2$-space of dimension* 6, *then the $\mathbf{F}_2$-dimension of $H^1(O(-1, V), V)$ is zero.*

Proof. Let $O = O(-1, V)$, and let $\delta \in \mathrm{Der}\,(O, V)$. Choose a hyperbolic pair of non-singular vectors $x, y \in V$ and let $U = \langle x, y \rangle^{\perp}$. By 4.5 we may use the arguments in the proof of 3.1 in order to assume $\delta \mid O_{\langle x,y \rangle} = 0$.

Again, for $A(T, \alpha) \in O_{\langle x \rangle}$, let $\delta A(T, \alpha) = p(T) + h(T) + q(T)$. Relations (1), (2), (3) of 4.6 hold, so q is a homomorphism from $Sp(U)$ to $\langle y \rangle$. If $T \in O(U)$, $A(T, \alpha) \in O_{\langle x,y \rangle}$, so $O(U) \leq \mathrm{Ker}\, q$. Since, by 1.3, $O(U)$ is maximal in $Sp(U)$, and since $O(U)$ is not normal in $Sp(U)$, $q = 0$. From this point the argument of 4.6 may be used without change.

THEOREM 4.8. *Let V be an $\mathbf{F}_2$-space of dimension* 6. *Then the $\mathbf{F}_2$-dimension of $H^1(Sp(V), V)$ is one.*

For the moment assume 4.8 is true. Its proof appears after the proof of 4.9.

THEOREM 4.9. *If V is an $\mathbf{F}_2$-space of dimension* 8, *then the $\mathbf{F}_2$-dimension of $H^1(O(-1, V), V)$ is zero.*

Proof. Let $O = O(-1, V)$ and let $\delta \in \operatorname{Der}(O, V)$. Choose a hyperbolic pair of non-singular vectors $x, y \in V$ and let $U = \langle x, y\rangle^{\perp}$. Arguing as in 3.1, for $S(A, T) \in O_{\langle x,y\rangle}$, we may assume $\delta S(A, T) = g(T)$, $g \in \operatorname{Der}(O(U), U)$.

As usual, for $A(T, \alpha) \in O_{\langle x\rangle}$, let $\delta A(T, \alpha) = p(T) + h(T) + q(T)$. Then the relations (1), (2), (3) of 4.6 hold. In particular, $q = 0$. Now (2) implies h is a derivation on $Sp(U)$. By 4.8, h is homologous to a scalar multiple of the non-inner derivation u; assume $h(T) = \lambda u(T)$ for $T \in Sp(U)$. Hence relation (1) becomes

$$p(TS) = p(T) + p(S) + \lambda B(u(T), Tu(S)).$$

Since $\delta S(A, T) \in U$, $p \mid O = 0$. Now complete the proof as in 4.6.

Now we prove 4.8:

Let $\delta \in \operatorname{Der}(Sp(V), V)$. Choose a hyperbolic pair $x, y \in V$, and let $U = \langle x, y\rangle^{\perp}$. $Sp(V)_U$ consists of the elements $S(A, T)$ with $A \in Sp(\langle x, y\rangle)$ and $T \in Sp(U)$. By 2.1 and 4.4 we may again use the arguments of 3.1 to assume $\delta S(A, T) = \lambda u(T)$, $\lambda \in \mathbf{F}_2$.

With respect to the decomposition $V = \langle x\rangle \oplus U \oplus \langle y\rangle$, the elements of $Sp(V)_x$ are the $A(T, v, \alpha)$ with $T \in Sp(U)$, $v \in U$, and $\alpha \in \mathbf{F}_2$, where

$$A(T, v, \alpha)u = B(v, Tu)x + Tu \quad \text{for } u \in U,$$

and

$$A(T, v, \alpha)y = \alpha x + v + y.$$

Since

$$A(T, v, \alpha)A(1, 0, \beta) = A(T, v, \alpha + \beta) \quad \text{and} \quad \delta A(1, 0, \beta) = 0,$$

$\delta A(T, v, \alpha)$ is independent of α. Let

$$\delta A(1, v, \alpha) = p(v) + h(v) + q(v),$$

with $p(v) \in \langle x\rangle$, $h(v) \in U$, and $q(v) \in \langle y\rangle$. Since

$$A(1, w, \alpha)A(1, v, \beta) = A(1, w + v, *),$$

we have the relations (1), (2), (3) of 3.1, and we may conclude that $q = 0$.

Now (2) implies that $h : U \to U$ is a homomorphism. Say $h(u) = Mu$, for M a 4×4 matrix over $\mathbf{F}_2$. Since

$$A(T, 0, 0)A(1, u, \alpha) = A(1, Tu, \alpha)A(T, 0, 0) \quad \text{and} \quad \delta A(T, 0, 0) = \lambda u(T),$$

we have the (new) relations

$$p(u) = p(Tu) + \lambda B(Tu, u(T)) \tag{4}$$

$$TMu = MTu, \tag{5}$$

for all $T \in Sp(U)$, all $u \in U$. Relation (5) implies $MT = TM$ for all $T \in Sp(U)$, so $M = 0$ or $M = 1_U$.

Case 1. Assume $M = 0$. Then relations (1) and (4) imply $p(T + 1)(u) = 0$ for all $T \in O(U)$, and by 1.10, $p = 0$. Therefore,

$\lambda B(u(T), Tu) = 0$ for all $T \in Sp(U)$ and all $u \in U$, so $\lambda = 0$, and $\delta \mid Sp(V)_x = 0$. Let R be the element of $Sp(V)$ interchanging x and y and fixing U point-wise. Then $RSp(V)_x R = Sp(V)_y$ and $\delta R = 0$ imply $\delta \mid Sp(V)_y = 0$. By 1.1, $Sp(V) = \langle Sp(V)_x, Sp(V)_y \rangle$, so $\delta = 0$ and the original δ was inner.

Case 2. Assume $M = 1_U$. Then relation (1) implies $p + Q = L$ is linear on U, and (4) implies $L(T + 1)(u) = 0$ for all $T \in O(U)$ and all $u \in U$. By 1.10, $L = 0$ and $p = Q$.

$$A(1, u, \alpha)A(T, 0, 0) = A(T, u, \alpha)$$

implies

$$\delta A(T, u, \alpha) = [\lambda B(u(T), u) + Q(u)]x + \lambda u(T) + u.$$

If $\lambda = 0$, then $\delta A(T, u, \alpha) = Q(u)x + u$, so

$$A(T, u, \alpha)\delta A(S, v, \beta) + \delta A(T, u, \alpha) = \delta A(TS, Tv + u, *)$$

implies $B(u, v) = B(u, Tv)$ for all $T \in Sp(U)$ and all $u, v \in U$, which is impossible. Hence $\lambda = 1$. For R as in case 1, $\delta R = 0$ and $RSp(V)_x R = Sp(V)_y$, so δ is also completely determined on $Sp(V)_y$. Therefore, δ is completely determined on $Sp(V) = \langle Sp(V)_x, Sp(V)_y \rangle$.

So we see that the dimension of $H^1(Sp(V), V)$ over $\mathbf{F}_2$ is at most one. The derivation $u : Sp(V) \to V$ is non-inner, so the dimension of $H^1(Sp(V), V)$ is exactly one.

5. S_n as a subgroup of $Sp_m(\mathbf{F}_2)$

Let V be a K-space of dimension n over K, and let $x_1, \cdots, x_n$ be a basis for V. S_n, the symmetric group on the letters $\{1, \cdots, n\}$ can be viewed as a subgroup of $GL(V)$ by identifying $\pi \in S_n$ with $T(\pi) \in GL(V)$, where $T(\pi)(x_i) = x_{\pi(i)}$. Define $\eta \in V^*$ by

$$\eta\left(\textstyle\sum_{i=1}^n \alpha_i x_i\right) = \textstyle\sum_{i=1}^n \alpha_i,$$

and let $H = \operatorname{Ker} \eta$. Let $x_0 = \sum_{i=1}^n x_i$. Then if n is odd, $V = H \oplus \langle x_0 \rangle$, and if n is even, $\langle x_0 \rangle \leq H$. Define a bilinear form B on V by

$$B\left(\textstyle\sum \alpha_i x_i, \sum \beta_i x_i\right) = \textstyle\sum_{i \neq j} \alpha_i \beta_j.$$

Then B is alternate, its matrix being $1 + E$, where E is the matrix each of whose entries is one. If n is even, $E^2 = 0$ and B is non-degenerate. If n is odd, E has rank 1, so B has rank $n - 1$. With respect to B, $\langle x_0 \rangle^\perp$ is V or H, according as n is odd or even. Hence B is nondegenerate on H or on $H/\langle x_0 \rangle$ according as n is odd or even.

S_n is contained in the group of B on V. Furthermore, $\langle x_0 \rangle$ and H are stable under S_n, so S_n may be viewed as a subgroup of $Sp(H)$ or $Sp(H/\langle x_0 \rangle)$, according as n is odd or even. The transposition (ij), $i \neq j$, in S_n corresponds to the transvection with center $\langle x_i + x_j \rangle$ in $SL(V)$, so S_n is a subgroup of $Sp_m(K)$ generated by transvections, where m is $n - 1$ or $n - 2$, according as n is odd or even.

If Q is defined on V by $Q(\sum \alpha_i x_i) = \sum_{i<j} \alpha_i \alpha_j$, then Q is quadratic form on V, Q is associated with B, and $S_n \leqq O(Q)$. If n is odd, $S_n \leqq O(\varepsilon, H)$, the group of $Q \mid H$, where ε is $+1$ or -1 according as $Q \mid H$ is of maximal index or not.

To determine ε, suppose $n = 2k + 1$ and choose a basis $u_1, v_1, \cdots, u_m, v_m$ for H with $u_i = \sum_{j=1}^{2i} x_j$, $v_i = x_{2i} + x_{2i+1}$. Let $P_i = \langle u_i, v_i \rangle$. Then the P_i are mutually perpendicular with respect to B. $Q(v_i) = 1$, $Q(u_i) = i(2i-1)$ and $B(u_i, v_i) = 1$. Therefore

$$Q(\alpha u_i + \beta v_i) = \alpha^2 i + \alpha\beta + \beta^2.$$

Thus $Q \mid P_i$ is of index 1 if i is even and of index 0 or 1, according as $X^2 + X + 1$ is irreducible over K or not, if i is odd. That is, $\varepsilon = 1$ if $X^2 + X + 1$ is reducible over K. Suppose $X^2 + X + 1$ is irreducible over K. Let

$$Q_i = Q \mid (P_1 \oplus \cdots \oplus P_i).$$

Then $\nu(Q_1) = 0$, $\nu(Q_2) = 1$, $\nu(Q_3) = 3$, $\nu(Q_4) = 4$. The pattern persists, so that if $k \equiv 0$ or 3 modulo 4, $\varepsilon = 1$, and if $k \equiv 1$ or 2 modulo 4, $\varepsilon = -1$. Equivalently, if $n \equiv 1$ or 7 modulo 8, $\varepsilon = 1$, and if $n \equiv 3$ or 5 modulo 8, $\varepsilon = -1$. In particular, since the order of $O_4(-1, \mathbf{F}_2)$ is $120 = 5\,!$, we have $S_5 \simeq O_4(-1, \mathbf{F}_2)$.

Now suppose n is even. Since the order of $Sp_4(\mathbf{F}_2)$ is $720 = 6\,!$, $S_6 \simeq Sp_4(\mathbf{F}_2)$. Suppose $n \equiv 0$ modulo 4. Then $Q(x_0) = n(n-1)/2 = 0$ and $\langle x_0 \rangle^\perp = H$, so we can define Q on $H/\langle x_0 \rangle$ by $Q(\bar{v}) = Q(v)$ for $v \in H$ and $\bar{v}$ the coset of v in $H/\langle x_0 \rangle$. Thus for $n \equiv 0$ modulo 4, $S_n \leq O(\varepsilon, H/\langle x_0 \rangle)$, where ε is $+1$ or -1 according as the index of $Q \mid (H/\langle x_0 \rangle)$ is maximal or not. Again we find that if $X^2 + X + 1$ is irreducible over K, $\varepsilon = 1$ if $(n-1) \equiv 1$ or 7 modulo 8, and $\varepsilon = -1$ if $(n-1) \equiv 3$ or 5 modulo 8. In particular, since the order of $O_6(+1, \mathbf{F}_2)$ is $8\,!$, $S_8 \simeq O_6(+1, \mathbf{F}_2)$.

The preceding discussion is taken from [7].

Suppose now that n is odd and $K = \mathbf{F}_2$. Then $S_n \leq O_{n-1}(\mathbf{F}_2)$, the group of $Q \mid H$.

THEOREM 5.1. *If $n \geq 5$ is odd, then the dimension of $H^1(S_n, H)$ over $\mathbf{F}_2$ is zero. In particular, if V is the 4-dimensional $\mathbf{F}_2$-space, the dimension of $H^1(O_4(-1, \mathbf{F}_2), V)$ over $\mathbf{F}_2$ is zero.*

Proof. Let $\delta \in \mathrm{Der}\,(S_n, H)$. Write $x_{ij} = x_i + x_j$, and if σ is the transposition $(ij) \in S_n$, write $x_{ij} = x_\sigma$. Let σ, τ be distinct commuting transpositions in S_n. Then $(\sigma + 1)\delta\tau = (\tau + 1)\delta\sigma$. But

$$\mathrm{Im}\,(\sigma + 1) \cap \mathrm{Im}\,(\tau + 1) = \langle x_\sigma \rangle \cap \langle x_\tau \rangle = 0.$$

Therefore

$$(\sigma + 1)\delta\tau = 0 = (\tau + 1)\delta\sigma,$$

and $\delta\tau \in \mathrm{Ker}\,(\sigma + 1) = \langle x_\sigma \rangle^\perp$, $\quad \delta\sigma \in \langle x_\tau \rangle^\perp$. Thus if $\tau = (ij)$ and $\delta\tau = \sum \alpha_k x_k$, we have

$$B(\sum \alpha_k x_k, x_r + x_s) = \alpha_r + \alpha_s = 0$$

for all $r, s \neq i, j$, and

$$B(\sum \alpha_k x_k, x_i + x_j) = \alpha_i + \alpha_j = 0.$$

Since $\delta\tau \in H$, we have $\delta\tau = \alpha(\tau)x_\tau$, with $\alpha(\tau) \in \mathbf{F}_2$, for every transposition $\tau \in S_n$.

Now let $y = x_{12}$, $z = x_{23}$ and $U = \langle y, z\rangle^\perp$. Then $H = \langle y, z\rangle \oplus U$. The elements of $(S_n)_{\langle y,z\rangle}$ have the form $S(A, T)$, with $A \in O(\langle y, z\rangle)$, $T \in O(U)$. By our first remarks,

$$\delta S(A, 1) = a(A) \in \langle y, z\rangle \quad \text{and} \quad \delta S(1, T) = g(T) \in U.$$

$S(A, 1) \in S_n$ for each $A \in S_3$, the symmetric group on $\{1, 2, 3\}$, so a is a derivation on $S_3 \simeq SL_2(\mathbf{F}_2)$. By 2.1, a is inner, so changing δ, if necessary, by an inner derivation based on a vector in $\langle y, z\rangle$, we may assume $a = 0$.

If $\pi \in S_{n-2}$, the symmetric group on $\{3, \cdots, n\}$, then π has the form

$$\pi(y) = y, \pi(u) = v(\pi)(u)y + T(\pi)(u)$$

for $u \in U$, $\pi(z) = \alpha(\pi)y + u(\pi) + z$ with $v(\pi) \in U^*$, $T(\pi) \in GL(U)$ defined as above, $\alpha(\pi) \in \mathbf{F}_2$ and $u(\pi) \in U$. Suppose

$$\delta(\pi) = p(\pi) + h(\pi) + q(\pi) \quad \text{with } p(\pi) \in \langle y\rangle, h(\pi) \in U, q(\pi) \in \langle z\rangle.$$

Then q is easily seen to be a homomorphism on S_{n-2}. If $\pi \in S_{n-3}$, the symmetric group on $\{4, \cdots, n\}$, then π fixes $\langle y, z\rangle$, so $u(\pi) = 0$ and $v(\pi) = 0$. Thus, since $a = 0$, $q \mid S_{n-3} = 0$. Since $n \geq 5$, $S_{n-3} \not< A_{n-2}$, so $q = 0$. Thus for $\pi \in S_{n-2}$, $\delta\pi \in \langle y\rangle^\perp = \langle y\rangle \oplus U$.

U is not stable for S_{n-2}. For $\pi \in S_{n-2}$ and $u \in U$, write

$$\pi(u) = f_\pi(u)y + \pi^*u, \quad \text{with } \pi^*u \in U.$$

It is easily verified that $\pi^* \in GL(U)$ and $f_\pi \in U^*$. If $\rho, \pi \in S_{n-2}$,

$$(\rho\pi)(u) = f\pi(u)y + f_\rho(\pi^*u)y + \rho^*\pi^*u.$$

Hence $(\rho\pi)^* = \rho^*\pi^*$ and $f_{\rho\pi} = f_\rho\, \pi^* + f_\pi$. We have

$$\delta(\pi) = p(\pi) + h(\pi) \quad \text{with } p(\pi) \in \langle y\rangle \text{ and } h(\pi) \in U,$$

so by computing $\delta(\rho\pi) = p(\rho\pi) + h(\rho\pi)$ we obtain

$$h(\rho\pi) = \rho^*h(\pi) + h(\rho) \tag{1}$$

$$p(\rho\pi) = f_\rho(h(\pi))y + p(\pi) + p(\rho). \tag{2}$$

We know that U is a module for $(S_{n-2})^*$ and so is a module for S_{n-2}. If V' is an $(n - 2)$-dimensional $\mathbf{F}_2$-space and $H' = \operatorname{Ker}(\eta \mid V')$, then H' is also a module for S_{n-2}. In order to use an induction argument to complete the proof of 5.1, we must show that $U \simeq H'$ as S_{n-2}-modules. That is, we must show that there is an isomorphism Φ between the $(n - 3)$-dimensional $\mathbf{F}_2$-spaces U and H' such that $\pi^* u = v$ if and only if $\pi(\Phi u) = \Phi v$ for $\pi \in S_{n-2}$ and $u, v \in U$.

Set $z_{34} = y + x_{34}$, $z_{k,k+1} = x_{k,k+1}$ for $k > 3$. The $z_{k,k+1}$, $k = 3, \cdots, n - 1$, form a basis for U. We can view $x_{34}, x_{45}, \cdots, x_{n-1,n}$ as a basis for H'. Define Φ by $\Phi z_{k,k+1} = x_{k,k+1}$. Now we see for each of the generating transpositions $(i, i + 1)$ of S_{n-2},

$$(i, i + 1)^* z_{k,k+1} = \sum_{j=3}^{n-1} \alpha_{j,k} z_{j,j+1}$$

if and only if

$$(i, i + 1) x_{k,k+1} = \sum_{j=3}^{n-1} \alpha_{j,k} x_{j,j+1},$$

where the $\alpha_{j,k} \in \mathbf{F}_2$.

Now, by (1), $h \in \operatorname{Der}(S_{n-2}, U)$. Since $S_3 \simeq SL_2(\mathbf{F}_2)$, the derivations on S_3 are all inner. Therefore, using an appropriate induction hypothesis, assume h is inner. If necessary, change δ by an inner derivation based on a vector in U, and assume $h = 0$. Then, by (2), p is a homomorphism. Like q, p vanishes on S_{n-3}, so $p = 0$. Since $\delta(12) = 0$, δ is zero on

$$(S_n)_y = (S_n)_{1,2} = S_{n-2} \times \langle (12) \rangle.$$

If $\tau = (13)$, then $\delta\tau = 0$ and $\tau (S_n)_y \tau = (S_n)_z$. Therefore, $\delta \mid (S_n)_z = 0$. Clearly $S_n = \langle (S_n)_y, (S_n)_z \rangle$, so $\delta = 0$, and the original δ was inner.

Now suppose n is even. Again let V be an $\mathbf{F}_2$-space with basis $x_1, \cdots, x_n$, let $H = \operatorname{Ker} \eta$, where $\eta(\sum \alpha_i x_i) = \sum \alpha_i$, let $x_0 = \sum x_i$, and view S_n as a subgroup of $Sp(H/\langle x_0 \rangle)$.

THEOREM 5.2. *If $n \geq 6$ is even, the dimension of $H^1(S_n, H/\langle x_0 \rangle)$ over $\mathbf{F}_2$ is one. In particular, the dimension of $H^1(O_6(+1, \mathbf{F}_2), V)$ is one, for V the 6-dimensional $\mathbf{F}_2$-space.*

Proof. $S_6 \simeq Sp_4(\mathbf{F}_2)$. By 4.4, the $\mathbf{F}_2$-dimension of $H^1(S_6, V)$ is one, for V the 4-dimensional $\mathbf{F}_2$-space. Assume $n \geq 8$ and let $\delta \in \operatorname{Der}(S_n, H/\langle x_0 \rangle)$. As in the proof of 5.1, if σ and τ are two distinct commuting transpositions, $\delta\sigma \in \langle x_\tau \rangle^\perp$ and $\delta\tau \in \langle x_\sigma \rangle^\perp$. So for every transposition $\tau \in S_n$, $\delta\tau \equiv \alpha(\tau) x_\tau$ modulo $\langle x_0 \rangle$, with $\alpha(\tau) \in \mathbf{F}_2$.

H has a basis

$$x_{12}, x_{23}, \cdots, x_{n-2,n-1}, x_0,$$

so $H/\langle x_0 \rangle$ has a basis

$$\bar{x}_{12}, \bar{x}_{23}, \cdots, \bar{x}_{n-2,n-1}.$$

Let $\bar{y} = \bar{x}_{12}$, $\bar{z} = \bar{x}_{23}$, and $U = \langle \bar{y}, \bar{z} \rangle^\perp$, so $H/\langle x_0 \rangle = \langle \bar{y}, \bar{z} \rangle \oplus U$. The elements of $(S_n)_{\langle \bar{y}, \bar{z} \rangle}$ have the form $S(A, T)$ with $A \in Sp(\langle \bar{y}, \bar{z} \rangle)$ and $T \in Sp(U)$. By the remarks above,

$$\delta S(A, 1) = a(A) \in \langle \bar{y}, \bar{z} \rangle \quad \text{and} \quad \delta S(1, T) = g(T) \in U.$$

$S(A, 1) \in S_n$ for each $A \in S_3$, so a is a derivation on $S_3 \simeq SL_2(\mathbf{F}_2)$. By 2.1, we may assume $a = 0$.

If $\pi \in S_{n-2}$, the symmetric group on $\{3, \cdots, n\}$, then on $H/\langle x_0\rangle$, π has the form $\pi\bar{y} = \bar{y}$, $\pi\bar{u} = v(\pi)(\bar{u})\bar{y} + T(\pi)\bar{u}$ for $\bar{u} \in U$, and $\pi\bar{z} = \alpha(\pi)\bar{y} + \bar{u}(\pi) + \bar{z}$, where $v(\pi) \in U^*$, $T(\pi) \in Sp(U)$, $\alpha(\pi) \in \mathbf{F}_2$ and $\bar{u}(\pi) \in U$. Suppose $\delta(\pi) = p(\pi) + h(\pi) + q(\pi)$, with $p(\pi) \in \langle\bar{y}\rangle$, $h(\pi) \in U$, $q(\pi) \in \langle\bar{z}\rangle$. Then, as in the proof of 5.1, we see that $q = 0$. Thus for $\pi \in S_{n-2}$, $\delta\pi \in \langle\bar{y}\rangle^{\perp} = \langle\bar{y}\rangle \oplus U$.

As in the argument for 5.1, write

$$\pi(\bar{u}) = f_\pi(\bar{u})\bar{y} + \pi^*\bar{u} \quad \text{for } \pi \in S_{n-2} \text{ and } \bar{u} \in U.$$

Then $\pi^* \in GL(U)$, $f_\pi \in U^*$, $(\rho\pi)^* = \rho^*\pi^*$ and $f_{\rho\pi} = f_\rho\,\pi^* + f_\pi$. We also have the relations (1) and (2) of 5.1. In order to use induction, we must verify that $U \simeq H'/\langle x_0\rangle$ as S_{n-2}-modules, where $H' = \operatorname{Ker} \eta \mid V'$. Choosing bases

$$\bar{z}_{34} = \bar{y} + \bar{x}_{34}, \; \bar{z}_{k,k+1} = \bar{y}_{k,k+1}, \qquad k = 3, \cdots, n-2,$$

for U and

$$\bar{x}_{34}, \bar{x}_{45}, \cdots, \bar{x}_{n-2,n-1}$$

for $H'/\langle x_0\rangle$, and defining $\Phi : U \to H'/\langle x_0\rangle$ by $\Phi\bar{z}_{k,k+1} = \bar{x}_{k,k+1}$, we obtain the module-isomorphism as for 5.1.

Hence, using a suitable induction hypothesis, we may suppose that the $\mathbf{F}_2$-dimension of $H^1(S_{n-2}, U)$ is one. Define a derivation

$$\delta_0 : S_{n-2} \to \langle\bar{y}\rangle^{\perp}$$

by $\delta_0(\pi) = (\pi + 1)\bar{x}_3$, and then set $\delta_0(\pi) = p_0(\pi) + h_0(\pi)$ with $p_0(\pi) \in \langle y\rangle$ and $h_0(\pi) \in U$. Then we have

$$h_0(\rho\pi) = \rho^*h_0(\pi) + h_0(\rho) \tag{3}$$

$$p_0(\rho\pi) = f_\rho(h_0(\pi))\bar{y} + p_0(\pi) + p_0(\rho). \tag{4}$$

Thus h_0 may be viewed as an element of Der (S_{n-2}, U). Since $\delta_0 \mid S_{n-3} = 0$, we have $h_0 \mid S_{n-3} = 0$ and $p_0 \mid S_{n-3} = 0$. Suppose h_0 is inner; that is, suppose there exists $\bar{u}_0 \in U$ such that

$$(\pi + 1)\bar{x}_3 = (\pi + 1)u_0 + p_0(\pi) \quad \text{for all } \pi \in S_{n-2}.$$

Then $(\pi + 1)(\bar{x}_3 + \bar{u}_0) = p_0(\pi)$, and $\pi(\bar{x}_3 + \bar{u}_0) = \bar{x}_3 + \bar{u}_0$ for all $\pi \in S_{n-3}$. Let u_0 be a preimage for $\bar{u}_0$, with $u_0 = \sum \alpha_k x_k$, $x_3 + u_0 = \sum \alpha_k' x_k$. Then since $\bar{x}_3 + \bar{u}_0$ is a fixed point for S_{n-3}, $B(\sum \alpha_k' x_k, x_i + x_j) = 0$ for $i, j \geq 4$; and since $\bar{u}_0 \in U$,

$$B(\sum \alpha_k x_k, x_1 + x_2) = \alpha_2 + \alpha_3 = 0.$$

Thus we see that $\bar{u}_0 = 0$. But $h_0 \neq 0$, so h_0 must be non-inner.

By (1), $h \in$ Der (S_{n-2}, U). We may assume $h = \lambda h_0$, $\lambda \in \mathbf{F}_2$. Then (2) becomes

$$p(\rho\pi) = \lambda f_\rho(h_0(\pi))\bar{y} + p(\pi) + p(\rho).$$

Since h_0, f and p vanish on S_{n-3},

$$p(\rho\pi) = p(\pi) \text{ and } p(\rho\pi\rho^{-1}) = p(\pi) \quad \text{for } \rho \in S_{n-3}.$$

Clearly $S_{n-2} = S_{n-3} + \sum_{i=4}^{n} (3i)S_{n-3}$, and for $i > 4$, $(3i) = (4i)(34)(4i)$. Thus we see that p is constant on the elements of S_{n-2} not in S_{n-3}. Let $\pi = (34)$, $\rho = (345)$, so $\pi\rho = (35)$. We check easily that $f_\pi(h_0(\rho)) = 0$ and $p(\pi\rho) = 0$, so $p = 0$.

Thus δ is determined up to a scalar multiple λ on $(S_n)_{\bar{y}}$. If $\tau = (13)$, then $\tau(S_n)_{\bar{y}}\,\tau = (S_n)_{\bar{z}}$. Therefore, δ is determined up to the same scalar λ on $(S_n)_{\bar{z}}$. Since $S_n = \langle (S_n)_{\bar{y}}, (S_n)_{\bar{z}} \rangle$, the dimension of $H^1(S_n, H/\langle x_0 \rangle)$ is at most one.

Define $\delta : S_n \to H/\langle x_0 \rangle$ by $\delta\pi = (\pi + 1)\bar{x}_1$. Then δ is a derivation vanishing on S_{n-1}, the symmetric group on $\{2, \cdots, n\}$. Arguing as for h_0, we see that δ must be non-inner. Hence the dimension of $H^1(S_n, H/\langle x_0 \rangle)$ is exactly one, for $n \geq 8$, even.

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