ON THE UNION OF CERTAIN CELLULAR DECOMPOSITIONS OF 3-MANIFOLDS

BY

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Introduction. Suppose that $G_1, G_2, \cdots, G_n$ are cellular upper semicontinuous decompositions of a 3-manifold with boundary M such that for $i = 1, 2, \cdots, n$, M/G_i is homeomorphic to M. Let G be the decomposition of M obtained from the G_i in the following manner. A set g belongs to G if and only if g is a nondegenerate element of some G_i or g is a point in

$$M - (H^*_{G_1} \cup \cdots \cup H^*_{G_n}).$$

The principal result of this paper shows that if the decompositions, $G_1, G_2, \cdots, G_n$, fit together in a "continuous" manner, then G is also a decomposition of M, and M/G is homeomorphic to M. Lamoreaux [5] has established a similar theorem for 0-dimensional cellular decompositions of a 3-manifold. In [7], Lamoreaux's result was extended to a countable number of 0-dimensional decompositions. Perhaps the main utility of such theorems lies in the fact that given a decomposition G of a 3-manifold M, it may be possible to break the decomposition into appropriate pieces which fit toether properly and for which it is known that the decomposition space associated with each piece is homeomorphic to the original space. It follows then that M/G will also be homeomorphic to M. An example of this procedure is given in Corollary 1.

In general, some condition must be imposed on the manner in which the various decompositions are pieced together. For example, Bing has described a cellular decomposition of E^3 with only a countable number of nondegenerate elements such that E^3/G is not homeomorphic to E^3 [2]. In this example, there exists two perpendicular planes, Q_1 and Q_2, such that if g is a nondegenerate element of the decomposition, then $g \subset Q_1$ or $g \subset Q_2$. Hence, if for $i = 1, 2$ we let G_i be the decomposition of E^3 whose nondegenerate elements are precisely those nondegenerate members of G which lie in Q_i, then E^3/G_1 and E^3/G_2 are homeomorphic to E^3 (see, for example, Hamstrom and Dyer [4]), but E^3/G is distinct from E^3.

Notation and terminology. Let G be an upper semicontinuous decomposition of a topological space, X. Then X/G will denote the associated decomposition space, P will denote the natural projection map from X onto X/G, and H_G will denote the collection of nondegenerate elements of G. If U is an open subset of X, then U is said to be *saturated* (with respect to G) in case $U = P^{-1}[P[U]]$.

The statement that M is a *3-manifold with boundary* means that M is a separable metric space such that each point of M has a neighborhood which is a

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3-cell. If A is a subset of M, then A is *cellular* in M if and only if there exists a sequence $C_1, C_2, \cdots$ of 3-cells in M such that (1) for each positive integer i, $C_{i+1} \subset$ Interior C_i and (2) $\bigcap_{i=1}^{\infty} C_i = A$. If M is a 3-manifold with boundary, the statement that G is a *cellular decomposition* of M means that G is an upper semicontinuous decomposition of M and each nondegenerate element of G is a cellular subset of M.

If M is a metric space, A a subset of M, then $S_\varepsilon(A)$ denotes the ε-neighborhood of A. If K is a collection of sets, then $K^* = \bigcup \{k : k \in K\}$.

Let M be a metric space and suppose K is a collection of mutually disjoint subsets of M. If $g \in K$, then K is said to be *continuous* at g in case for each positive number ϵ, there exists an open subset V of M containing g such that if $g' \in K$ and $g' \cap V \neq \emptyset$, then $g \subset S_\epsilon(g')$ and $g' \subset S_\epsilon(g)$.

THEOREM 1. *Suppose that $G_1, G_2, \cdots, G_n$ is a finite collection of cellular decompositions of a 3-manifold with boundary M such that:*

(1) *If $g \in H_{G_i}$ and $g \cap H_{G_j}^* \neq \emptyset$, then $g \in H_{G_j}$.*

(2) *For each $k = 1, 2, \cdots, n$, if $g \in H_{G_k}$, then $\{H_{G_i} : i \neq k\} \cup \{g\}$ is continuous at g.*

(3) *For $i = 1, 2, \cdots, n$, M/G_i is homeomorphic to M. Let*

$$G = \{h : h \in \textstyle\bigcup_{i=1}^{n} H_{G_i} \text{ or } h \text{ is a point of } M - (\bigcup_{i=1}^{n} H_{G_i}^*)\}.$$

Then G is a cellular decomposition of M and M/G is homeomorphic to M.

Proof. We first consider a number of lemmas. Lemma 1 provides a frequently useful characterization of decompositions of metric spaces, and its proof is straightforward.

LEMMA 1. *Let X be a metric space and suppose G is a collection of mutually disjoint compact subsets of X such that $G^* = X$. Then G is an upper semicontinuous decomposition of X if and only if for each sequence $\{x_n\}$, $x_n \in g_n \in G$, which converges to a point $x \in g' \in G$, and for any sequence $\{y_n\}$, $y_n \in g_n$, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ which converges to a point $y \in g'$.*

It is now easy to see that under the hypotheses of Theorem 1, G is in fact an upper semicontinuous decomposition of M. Let $\{x_m\}$ be a sequence in M which converges to a point $x \in M$, and suppose $x \in g' \in G$. For $m = 1, 2, \cdots$ there exists an integer i_m, $1 \leq i_m \leq n$, and a set g_{i_m} such that $x_m \in g_{i_m} \in G_{i_m}$. Let $\{y_n\}$ be any sequence in M where for each m, $y_m \in g_{i_m}$. We must find a subsequence of $\{y_m\}$ which converges to some point in g'.

There exists an integer k, $1 \leq k \leq n$, such that $i_m = k$, for infinitely many m. Hence, there exists a subsequence $\{x_{m_p}\}$ of $\{x_m\}$ such that for $p = 1, 2, \cdots$, $x_{m_p} = x_k \in g_k \in G_k$. $\{x_{m_p}\}$ converges to $x \in g'$, and it follows from the continuity of G at g' that $g' \in G_k$. Since G_k is an upper semicontinuous decomposition of M, the corresponding subsequence $\{y_{m_p}\}$ of $\{y_m\}$ must itself contain a subsequence which converges to a point of g', and, thus by Lemma 1, G is cellular decomposition of M.

A proof of the following lemma may be found in [3].

LEMMA 2. *Let X be a paracompact space. Then every open covering of X has an open star refinement* (i.e., *if $\mathcal{U}$ is an open covering of X, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that for each $V \in \mathcal{U}$, there exists $U \in \mathcal{U}$ such that*

$$\text{Star } V = \bigcup \{W \in \mathcal{V} : W \cap V \neq \emptyset\} \subset U.$$

The next lemma is a well known result in decomposition theory [1].

LEMMA 3. *If G is a cellular decomposition of a separable metric space M, then M/G is a separable metric space.*

If G is an upper semicontinuous decomposition of a metric space M, then G is said to be *shrinkable* in case for each covering $\mathcal{U}$ of H_G^* by saturated open sets of M, for each positive number ε, and for an arbitrary homeomorphism f from M onto M, there exists a homeomorphism h from M onto itself such that
(1) if $x \in M - \mathcal{U}^*$, then $f(x) = h(x)$,
(2) for each $g \in G$, $g \subset \mathcal{U}^*$
(a) diam $h[g] < \varepsilon$ *and*
(b) there exists $D \in \mathcal{U}$ such that $f[g] \cup h[g] \subset f[D]$.

Lemma 4 is established in [8].

LEMMA 4. *Suppose G is a cellular decomposition of a 3-manifold with boundary M. Then M/G is homeomorphic to G if and only if G is shrinkable.*

In fact, in order that M/G be homeomorphic to M, part (2) of the definition of shrinkability may be modified to read "for each $g \in H_G$, etc." (see [6]).

LEMMA 5. *Suppose G is a shrinkable cellular decomposition of a 3-manifold with boundary M. Let $\mathcal{U}$ be a covering of H_G^* by saturated open sets. Then*
(1) *for each $U \in \mathcal{U}$, $h[U] \subset f[\text{Star } U]$ (notation as in the definition of shrinkability), and*
(2) *there exists a star refinement $\mathcal{V}$ of $\mathcal{U}$ by saturated open sets.*

Proof. (1) Suppose $U \in \mathcal{U}$. For $x \in U$, there exists $g \in G$ such that $x \in g \subset U$. Furthermore, there exists $U' \in \mathcal{U}$ such that $f[g] \cup h[g] \subset f[U']$. Since $f[U] \cap f[U'] \neq \emptyset$, $U' \subset \text{Star } U$, and, hence, $h(x) \in f[\text{Star } U]$.
(2) $P[\mathcal{U}] = \{P[U] : U \in \mathcal{U}\}$ is an open covering of $P[H_G^*]$. By Lemma 3, M/G is a metric space, and, hence, by Lemma 2, there exists an open star refinement $\mathcal{Z}$ of $P[\mathcal{U}]$. Then $\mathcal{V} = \{P^{-1}[Z] : Z \in \mathcal{Z}\}$ is the desired refinement.

We now complete the proof of Theorem 1 for the case $n = 2$. We shall show that G is a shrinkable decomposition of M. Let $\mathcal{U}$ be a saturated open cover of H_G^*, ε a positive number, and f an arbitrary homeomorphism from M onto M. By Lemma 5, there exists a saturated (with respect to G) open star refinement $\mathcal{V}$ of $\mathcal{U}$. $\mathcal{V}$ is then also a covering of $H_{G_1}^*$, and since M/G_1 is homeomorphic to M, there exists by Lemma 4 a homeomorphism h_1 from M onto M such that

(1) if $x \in M - \mathcal{V}^*$, $h_1(x) = f(x)$,

(2) if $g \in G_1$ then diam $h_1[g] < \varepsilon$,
(3) if $g \in G_1$ and $g \subset \mathcal{V}^*$ then there exists $V \in \mathcal{V}$ such that

$$f[f] \cup h_1[g] \subset f[V].$$

We note that if $g \in G_2$ and $g \subset \mathcal{V}^*$, then there exists $U \in \mathcal{U}$ such that $f[g] \cup h_1[g] \subset f[U]$. This can be seen as follows. For $g \in G_2$, there exists $V \in \mathcal{V}$ and $U \in \mathcal{U}$ such that $g \subset V$ and Star $[V] \subset U$. Applying Lemma 5 to $\mathcal{V}$, we have that $h_1[V] \subset f[\text{Star } V] \subset f[U]$, and, hence, $f[g] \cup h_1[g] \subset f[U]$.

We denote by $h_1[G_2]$ the cellular decomposition of M whose elements are of the form $h_1[g]$, for $g \in G_2$. It is easily verified that $M/h_1[G_2]$ is homeomorphic to M. We cover $H^*_{h_1[G_2]}$ as follows.

(A) Suppose $g \in H_{G_1} \cap H_{G_2}$. Then diam $h_1[g] < \epsilon$ and there exists $V_g \in \mathcal{V}$ such that $f[g] \cup h_1[g] \subset f[V_g]$. Let W_g be a saturated (with respect to G) open subset of M such that

(1) $g \subset W_g \subset V_g$,
(2) diam $h_1[W_g] < \varepsilon$,
(3) $f[W_g] \cup h_1[W_g] \subset f[V_g]$.

(B) Suppose $g \in H_{G_2}$, $g \notin H_{G_1}$. It was noted above that there exists $U_g \in \mathcal{U}$ such that $f[g] \cup h_1[g] \subset f[U_g]$. Let W_g be a saturated (with respect to G) open subset of M such that

(1) $g \subset W_g \subset U_g$
(2) $f[W_g] \cup h_1[Wg] \subset f[U_g]$
(3) $W_g \cap H^*_{G_1} = \emptyset$.

That condition (3) is possible follows from the continuity of $H_{G_1} \cup \{g\}$ at g. For convenience, we also cover the nondegenerate elements of G which do not belong to H_{G_2}.

(C) Suppose $g \in H_{G_1}$, $g \notin H_{G_2}$. There exists $V_g \in \mathcal{V}$ such that

$$f[g] \cup h_1[g] \subset f[V_g].$$

Let W_g be a saturated (with respect to G) open set such that

(1) $g \subset W_g \subset V_g$,
(2) diam $h_1[W_g] < \varepsilon$,
(3) $f[W_g] \cup h_1[W_g] \subset f[V_g]$.

Let $\mathcal{W} = \{h_1[W_g] : g \in H_G\}$. Then $\mathcal{W}$ is a saturated (with respect to $h_1[G]$) open cover of $h_1[H^*_G]$ and in particular of $h_1[H^*_{G_2}]$. Let $\mathcal{Y}$ be a saturated (with respect to $h_1[G]$) open star refinement of $\mathcal{W}$. Again applying Lemma 4, we have a homeomorphism h_2 from M onto itself with the following properties:

(1) if $x \in M - \mathcal{Y}^*$, $h_2(x) = x$,
(2) if $g \in H_{G_2}$, diam $h_2h_1[g] < \varepsilon$,

(3) if $g \in H_{G_2}$, there exists $Y_g \in \mathcal{Y}$ such that

$$h_1[g] \cup h_2 h_1[g] \subset Y_g .$$

Let $h = h_2 h_1$. We must show that

(i) if $x \in M - \mathcal{U}^*$, $h(x) = f(x)$
(ii) if $g \in H_G$, diam $h[g] < \varepsilon$
(iii) if $g \in H_G$, there exists $U \in \mathcal{U}$ such that

$$f[g] \cup h[g] \subset f[U].$$

(i) Suppose $x \in M - \mathcal{U}^*$. Since $\bigcup \{W_g : g \in G\} \subset \mathcal{U}^*$, $h_1(x) \notin \mathcal{W}^* = \mathcal{Y}^*$. Therefore, $h_2 h_1(x) = h_1(x)$. But $h_1(x) = f(x)$ for $x \in M - \mathcal{U}^*$, and, thus, $h(x) = f(x)$.

(ii) If $g \in H_{G_2}$, then clearly diam $h[g] < \varepsilon$. Suppose then that $g \in H_{G_1}$ and $g \notin H_{G_2}$. There exists $Y \in \mathcal{Y}$ and $g' \in H_G$ such that

$$h_1[g] \subset Y \subset \text{Star } Y \subset h_1[W_{g'}],$$

where $h_1[W_{g'}]$ is a member of $\mathcal{W}$. g' must be an element of H_{G_1} since if not it would follow from our construction (part B) that $W_{g'} \cap H^*_{G_1} = \emptyset$. Therefore, diam $h_1[W_{g'}] < \varepsilon$. We apply Lemma 5 to the decomposition $h_1[G_2]$, the covering $\mathcal{Y}$, the shrinking homeomorphism h_2, and $f =$ identity. We have then that

$$h_2 h_1[g] \subset h_2[Y] \subset f[\text{Star } Y] = \text{Star } Y \subset h_1[W_{g'}],$$

and, hence, diam $h[g] < \varepsilon$.

(iii) Suppose $g \in H_G$. Then as above, there exists $Y \in \mathcal{Y}$ and $g' \in H_G$ such that

$$h_1[g] \subset Y \subset \text{Star } Y \subset h_1[W_{g'}] \quad \text{and} \quad h_2[Y] \subset f[\text{Star } Y] = \text{Star } Y.$$

Thus $h_2 h_1[g] \subset h_2[Y] \subset h_1[W_{g'}]$, and, of course, $h_2 h_1[g] \subset h_2 h_1[W_{g'}]$. But corresponding to $W_{g'}$, there exists a set $V_{g'}$ (or $U_{g'}$) such that

$$f[W_{g'}] \cup h_1[W_{g'}] \subset f[V_{g'}] \text{ (or } f[U_{g'}]).$$

Thus $g \in W_{g'}$ implies that $f[g] \cup h[g] \subset f[V_{g'}]$ (or $f[U_{g'}]$), and since $\mathcal{V}$ is a refinement of $\mathcal{U}$, there exists $U \in \mathcal{U}$, such that $f[g] \cup h[g] \subset f[U]$.

The proof of the theorem for $n > 2$ follows easily by induction, since if $G_1, G_2, \cdots, G_n$ are decompositions of M satisfying the three conditions of the hypothesis of Theorem 1, then $G^j = \{G_1, G_2, \cdots, G_j\}$, $2 \leqq j \leqq n$, will also satisfy these conditions.

A decomposition of a metric space is said to be *nondegenerately continuous* if for each $g \in G$, $H_G \cup \{g\}$ is continuous at g.

COROLLARY 1. *Suppose G is a cellular nondegenerately continuous decomposition of E^3. Suppose there exists a finite collection of planes in E^3, Q_1, Q_2,*

$\cdots$, Q_n *such that for each* $g \in H_G$, g *is contained in at least one of these planes. Then* E^3/G *ishomeomorphic to* E^3.

Proof. For $i = 1, 2, \cdots, n$ let G_i be the decomposition of E^3 such that $H_{G_i} = \{g \in H_G : g \subset Q_i\}$. Then E^3/G_i is homeomorphic to E^3 [4] and since it is readily verified that $G_1, G_2, \cdots, G_n$ satisfy the conditions of Theorem 1, E^3/G is homeomorphic to E^3.

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