RESIDUAL PROPERTIES FOR CLOSED OPERATORS ON FRÉCHET SPACES

BY

F.-H. VASILESCU

1. Introduction

The single-valued extension property is a remarkable one for a very large class of linear operators on a locally convex space. Its first definition is due to Dunford and is related to spectral operators, which possess this property [5], [6]. But there exist simple examples of operators which do not have this property or have it only on a part of their spectrum. So it is very natural to consider a residual part of the spectrum of an operator and define this property "outside" this part [11]. Many properties of a spectral operator (and even for an operator in certain larger classes [7], [4]) can be obtained using only a natural assumption of decomposability of the space with respect to this operator [8]. To study simultaneously the class of unbounded operators with a suitable spectral behaviour and other classes (obtained for example, from direct sums between "good" operators and operators which do not even have the single valued extension property) it is again necessary to consider a residual part of their spectrum [11]. The purpose of our paper is to give some new results related to the single-valued extension property and supplementary assertion for the residually decomposable operators [11].

Our main result is a theorem of existence and uniqueness, for a large class of operators, of a minimal closed set outside which such an operator has a suitable spectral behaviour.

First we need some definition and additional properties. In the sequel $\mathfrak{X}$ will be a Fréchet space [3] (although many considerations are true in more general spaces), $B(\mathfrak{X})$ the set of all continuous linear operators on $\mathfrak{X}$, and $C(\mathfrak{X})$ the set of all closed linear operators on $\mathfrak{X}$. For the spectrum $\sigma(T)$ of an operator $T \in C(\mathfrak{X})$ we shall use the definition of Waelbroeck [12]. Thus a point $\lambda \in \mathbf{C}_\infty (= \mathbf{C} \cup \{\infty\})$ is in $\rho(T)$ if there exists a neighbourhood $V_\lambda \subseteq \mathbf{C}_\infty$ of λ such that $(\mu I - T)^{-1} \in B(\mathfrak{X})$ for any $\mu \in V_\lambda \cap \mathbf{C}$ and the set

$$\{(\mu I - T)^{-1}x;\ \mu \in V_\lambda \cup \mathbf{C}\}$$

is bounded in $\mathfrak{X}$ for any $x \in \mathfrak{X}$. We shall also use the well-known notations: $R(\lambda, T) = (\lambda I - T)^{-1}$, $\mathfrak{D}_T$ for the domain of the operator T and $\sigma(T) = \complement\rho(T)$ (all operations with sets are considered in $\mathbf{C}_\infty$).

Let $T \in C(\mathfrak{X})$ and $x \in \mathfrak{X}$ be fixed. We shall say that $\lambda \in \delta_T(x)$ if in a neighbourhood V_λ of λ there exists an analytic function $f_x : V_\lambda \to \mathfrak{D}_T$ (not necessarily unique) such that $(\mu I - T)f_x(\mu) = x$ for $\mu \in V_\lambda \cap \mathbf{C}$. Such an analytic func-

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tion $f_x(\mu)$ is called *T-associated* with x [11]. We put $\gamma_T(x) = \complement\, \delta_T(x)$. We know that there exists a unique maximal open set $\Omega_T \subseteq \mathbf{C}_\infty$ with this property: if $\omega \subseteq \Omega_T$ is an open set and $f_0 : \omega \to \mathfrak{D}_T$ is an analytic function such that $(\mu I - T)f_0(\mu) \equiv 0$ for $\mu \in \omega \cap \mathbf{C}$ then $f_0(\mu) \equiv 0$ for $\mu \in \omega$, and we denote $S_T = \complement\, \Omega_T$ [11]. We shall also put

$$\sigma_T(x) = \gamma_T(x) \cup S_T \quad \text{and} \quad \rho_T(x) = \delta_T(x) \cap \Omega_T$$

for any $x \in \mathfrak{X}$ [11]. It is easy to see that in $\rho_T(x)$ there exists a unique function, denoted $x(\lambda)$, which is T-associated with x. Obviously, the sets $\delta_T(x)$ and $\rho_T(x)$ are open, hence $\gamma_T(x)$ and $\sigma_T(x)$ are closed in $\mathbf{C}_\infty$ for any $x \in \mathfrak{X}$.

We can introduce the following linear manifolds [11]:

$$\mathfrak{X}_T^0(F) = \{x \in \mathfrak{X};\ \gamma_T(x) \subseteq F\}, \qquad \mathfrak{X}_T(F) = \{x \in \mathfrak{X};\ \sigma_T(x) \subseteq F\},$$

where F is a set in $\mathbf{C}_\infty$.

An operator $T \in C(\mathfrak{X})$ has the single-valued extension property if $S_T = \emptyset$. In this case $\sigma_T(x) = \gamma_T(x)$, for any $x \in \mathfrak{X}$.

We shall say that a subspace $\mathfrak{Y} \subseteq \mathfrak{X}$ (i.e. a closed linear manifold) is *invariant with respect to* T if $\mathfrak{Y} \subseteq \mathfrak{D}_T$ and $T\mathfrak{Y} \subseteq \mathfrak{Y}$ [11]. Obviously by Banach's theorem, $T \mid \mathfrak{Y} \in B(\mathfrak{Y})$.

2. The single-valued extension property

This section has two main results. One of them is a characterization of the single-valued extension property which could be of interest in case of several variables, namely, the characterization which is obtained for one operator suggests a definition of the single-valued extension property for finite sets of operators, by using a suitable locally joint spectrum of an element. The other result is a theorem which implies that the single-valued extension property can be obtained by passing to a suitable quotient space.

PROPOSITION 2.1. *An operator $T \in C(\mathfrak{X})$ has the single-valued extension property if and only if $\mathfrak{X}_T^0(\emptyset) = \{0\}$.*

Proof. First let $T \in C(\mathfrak{X})$ with $\mathfrak{X}_T^0(\emptyset) = \{0\}$. Let $f : \omega \subseteq \mathbf{C}_\infty \to \mathfrak{D}_T$ be an analytic function satisfying $(\lambda I - T)f(\lambda) = 0$ for $\lambda \in \omega \cap \mathbf{C}$. Then for any $\lambda_0 (\neq \infty)$ we have $\gamma_T(f(\lambda_0)) = \emptyset$ (see [11]); thus $f(\lambda_0) = 0$ and hence $f(\lambda) \equiv 0$ for $\lambda \in \omega$. Conversely, let $T \in C(\mathfrak{X})$ have the single-valued extension property and let $x \in \mathfrak{X}_T^0(\emptyset)$, $x \neq 0$. Since $S_T = \emptyset$, there exists a unique T-associated function $x(\lambda)$ defined on the whole complex compactified plane. We have even $x(\infty) = 0$. Indeed, if $\lambda_n \to \infty$ then $x_n = x(\lambda_n)/\lambda_n$ converges to zero and $Tx_n = x(\lambda_n) - x/\lambda_n$ to $x(\infty)$. Because T is closed, we necessarily have $x(\infty) = 0$. Consequently $x(\lambda)$ is analytic in the whole complex plane and is zero at ∞. By the Liouville Theorem we have $x(\lambda) = 0$; thus $x = 0$. This contradiction proves our assertion.

LEMMA 2.1 *Let $\mathfrak{Y}$ be a closed subspace of $\mathfrak{X}$, $\tilde{\mathfrak{X}} = \mathfrak{X}/\mathfrak{Y}$ and $(f(\lambda))^\sim : \omega \to \tilde{\mathfrak{X}}$ an* *analytic function* (ω *open set in* $\mathbf{C}_\infty$). *Then for any* $\lambda_0 \in \omega$ *there exists an* $\mathfrak{X}$*-valued function* $f(\lambda; \lambda_0)$, *analytic in a neighbourhood of* λ_0, *such that* $f(\lambda; \lambda_0) \in (f(\lambda))^\sim$.

Proof. Let $\lambda_0 (\neq \infty)$ be an arbitrary point in ω. Since $(f(\lambda))^\sim$ is analytic in ω, we can write in a neighbourhood of λ_0,

$$(f(\lambda))^\sim = \sum_{k=0}^{\infty} \tilde{a}_k (\lambda - \lambda_0)^k$$

where $\tilde{a}_k \in \tilde{\mathfrak{X}}$ for all k. Then there exists an $L > 0$ such that for any $n \in \mathbf{N}$ there exists an $N(n)$ such that the *n-th* seminorm satisfies

$$|\tilde{a}_k|_n \leqq L^k \qquad \text{for all } k \geqq N(n).$$

If n is fixed and $N(n) \leqq k < N(n+1)$, we choose an $a_k \in \tilde{a}_k$ such that

$$|a_k|_{N(n)} \leqq (L+1)^k,$$

which is possible from the above relation and the definition of the topology on the quotient space. With no loss of generality the family of seminorms can be taken non-decreasing, and $N(n+1) \geqq N(n) \geqq n$, for all n. With these conditions the series

$$\sum_{k=0}^{\infty} a_k (\lambda - \lambda_0)^k$$

defines an analytic function with values in $\mathfrak{X}$. Indeed, for fixed n we have

$$\begin{aligned}
\Big| \sum_{k \geqq N(n)} a_k (\lambda - \lambda_0)^k \Big|_n &\leqq \sum_{k \geqq N(n)} |a_k|_n |\lambda - \lambda_0|^k \\
&= \sum_{p=0}^{\infty} \sum_{k=N(n+p)}^{N(n+p+1)-1} |a_k|_n |\lambda - \lambda_0|^k \\
&\leqq \sum_{p=0}^{\infty} \sum_{k=N(n+p)}^{N(n+p+1)-1} |a_k|_{N(n+p)} |\lambda - \lambda_0|^k \\
&\leqq \sum_{p=0}^{\infty} \sum_{k=N(n+p)}^{N(n+p+1)-1} (L+1)^k |\lambda - \lambda_0|^k = \sum_{k \geqq N(n)} (L+1)^k |\lambda - \lambda_0|^k
\end{aligned}$$

and the last expression converges to zero as $n \rightarrow \infty$ when $|\lambda - \lambda_0| < 1/L + 1$. Hence the function $f(\lambda) = \sum_{k=0}^{\infty} a_k (\lambda - \lambda_0)^k$ is analytic in a neighbourhood of λ_0. A similar argument can be used when $\lambda_0 = \infty \in \omega$ to obtain the same conclusion.

PROPOSITION 2.2 *Let* $T \in B(\mathfrak{X})$ *and* $\mathfrak{Y}$ *be an invariant subspace of* T. *If* $\tilde{T}$ *is the operator induced by* T *on the space* $\tilde{\mathfrak{X}} = \mathfrak{X}/\mathfrak{Y}$ *then* $S_{\tilde{T}} \subseteq \sigma(T \mid \mathfrak{Y}) \cup S_T$.

Proof. Indeed, if $(f(\lambda))^\sim$ is an analytic function on an open set

$$\omega \subseteq \rho(T \mid \mathfrak{Y}) \cap \complement S_T$$

such that $(\lambda \tilde{I} - \tilde{T})(f(\lambda))^\sim \equiv 0^\sim$ for $\lambda \in \omega$ then, by Lemma 2.1, we shall choose $f(\lambda; \lambda_0) \in (f(\lambda))^\sim$ to be analytic in a neighbourhood of any point $\lambda_0 \in \omega$. Thus

$$(\lambda I - T) f(\lambda; \lambda_0) = g(\lambda) \in \mathfrak{Y}$$

and we obviously have

$$(\lambda I - T)[f(\lambda; \lambda_0) - R(\lambda, T \mid \mathfrak{Y}) g(\lambda)] \equiv 0$$

for λ in a neighbourhood of $\lambda_0 \in \complement S_T$. Therefore

$$f(\lambda; \lambda_0) = R(\lambda, T \mid \mathfrak{Y}) g(\lambda) \in \mathfrak{Y}$$

and from this we have $(f(\lambda))^{\sim} = 0^{\sim}$. The point $\lambda_0 \in \omega$ being arbitrarily chosen, we have $(f(\lambda))^{\sim} \equiv 0^{\sim}$. Hence $\omega \subseteq \complement S_{\tilde{T}}$ for any

$$\omega \subseteq \rho(T \mid \mathfrak{Y}) \cap \complement S_T.$$

By minimality, we obtain $S_{\tilde{T}} \subseteq \sigma(T \mid \mathfrak{Y}) \cup S_T$.

PROPOSITION 2.3. *Let $T \in C(\mathfrak{X})$, $S_T \subseteq \mathbf{C}_\infty$ and let $\mathfrak{Y} = \mathfrak{X}T(M)$. If $\mathfrak{Y}$ is closed in $\mathfrak{X}$ and $\mathfrak{Y} \subseteq \mathfrak{D}_T$, then $\mathfrak{Y}$ is an invariant subspace of T and*

$$\sigma(T \mid \mathfrak{Y}) \equiv \overline{M \cap \sigma(T)}.$$

Proof. We can act as in the corresponding proposition of [11]. For any

$$\lambda \in \complement \overline{M \cap \sigma(T)}$$

there exists a neighbourhood V_λ with $V_\lambda \cap \overline{M \cap \sigma(T)} = \emptyset$.

We define the operators

$$A_\mu y = y(\mu) \qquad (y \in \mathfrak{Y},\ \mu \in V_\lambda \cap \mathbf{C})$$

and it will follow that $A_\mu = (\mu I \mid \mathfrak{Y} - T \mid \mathfrak{Y})^{-1}$. Now, V_λ can be chosen compact in $\mathbf{C}_\infty$; thus $y(V_\lambda)$ is bounded in $\mathfrak{Y}$ for any $y \in \mathfrak{Y}$. From this we obtain $\lambda \in \rho(T \mid \mathfrak{Y})$ and the proof is finished.

THEOREM 2.1. *Let $T \in B(\mathfrak{X})$, $S_T \subseteq M = \bar{M} \subseteq \mathbf{C}_\infty$, and suppose that $\mathfrak{Y} = \mathfrak{X}_T(M)$ is closed. If $\tilde{\mathfrak{X}} = \mathfrak{X}/\mathfrak{Y}$ and $\tilde{T}$ is the operator induced by T on $\tilde{\mathfrak{X}}$, then we have $\tilde{\mathfrak{X}}_{\tilde{T}}(M) = \{0^{\sim}\}$.*

Proof. Let $\tilde{x} \in \tilde{\mathfrak{X}}_{\tilde{T}}(M)$ and $\lambda_0 \in \complement M \cap \mathbf{C}$. From Proposition 2.2 and 2.3, we have

$$S_{\tilde{T}} \subseteq \sigma(T \mid \mathfrak{Y}) \cup S_T \subseteq M.$$

By Lemma 2.1, we can take $f(\lambda)$ analytic in a neighbourhood of λ_0 with values in $\mathfrak{X}$, such that $f(\lambda) \in (x(\lambda))^{\sim}$. Since $T \in B(\mathfrak{X})$ it follows that the function $h(\lambda) = (\lambda I - T) f(\lambda) - x$ is analytic in a neighbourhood of λ_0. Furthermore

$$(h(\lambda))^{\sim} = ((\lambda I - T) f(\lambda))^{\sim} - \tilde{x} = 0,$$

thus $h(\lambda) \in \mathfrak{Y}$ for any λ. Then, from the relation $\sigma(T \mid \mathfrak{Y}) \subseteq M$ it follows that the mapping

$$k(\lambda) = R(\lambda, T \mid \mathfrak{Y}) h(\lambda)$$

is analytic in a neighbourhood of λ_0. Therefore

$$x = (\lambda I - T)[f(\lambda) - k(\lambda)],$$

with $f(\lambda) - k(\lambda)$ analytic; since $\lambda_0 \in \complement S_T$, we must have $\lambda_0 \in \rho_T(x)$. Consequently $\sigma_T(x) \subseteq M$, and hence $x \in \mathfrak{X}_T(M)$. Therefore $\tilde{x} = 0^{\sim}$ and being arbitrariy chosen in $\tilde{\mathfrak{X}}_{\tilde{T}}(M)$ the assertion is proved.

COROLLARY 1. *With the conditions of the previous theorem the operator T has the single valued extension property.*

Indeed, $\tilde{\mathfrak{X}}^0_T(\emptyset) \equiv \tilde{\mathfrak{X}}_{\tilde{T}}(S_{\tilde{T}}) \subseteq \tilde{\mathfrak{X}}_{\tilde{T}}(M) = \{0^\sim\}$ and by Proposition 2.1 our assertion follows.

This result shows that "regularization in the sense of the single valued extension property" is achieved by using the induced operator on a suitable quotient space. We shall finish this section with a special result concerning the consequences of the compactness of local spectra. Let us remark that for a closed set $F \subseteq C_\infty$, the inclusion $F \subseteq C$ is equivalent with the compactness of F in $\mathbf{C}$. In what follows it will be convenient to put $\sup\{|\lambda|; \lambda \in \emptyset\} = 0$.

PROPOSITION 2.4. *If $T \in C(\mathfrak{X})$, then for any $x \in \mathfrak{X}$ with $\gamma_T(x) \subseteq \mathbf{C}$ it follows that $x \in \mathfrak{D}_{T^k}$ for all $k \geqq 1$ and*

$$\sup\{|\lambda|; \lambda \in \gamma_T(x)\} \leqq \sup_n \overline{\lim}_{k\to\infty} |T^k x|_n^{1/k} < \infty.$$

Proof. From the inclusion $\gamma_T(x) \subseteq \mathbf{C}$ it follows that there exists a T-associated function f_x of x which is analytic at ∞; therefore in a neighborhood of ∞ we have

$$f_x(\mu) = \sum_{k=0}^{\infty} x_k/\mu^{k+1}.$$

Indeed, as in the proof of Proposition 2.1, if $\lambda_n \to \infty$ then $y_n = f_x(\lambda_n)/x_n \to 0$ and $Ty_n \to f_x(\infty)$, therefore $f_x(\infty) = 0$. Then since T is closed, we may write

$$0 = \lim_{\lambda\to\infty} Tf_x(\lambda) = \lim_{\lambda\to\infty} [\lambda f_x(\lambda) - x] = x_0 - x;$$

therefore $x_0 = x$.

Since

$$\lim_{\lambda\to\infty} \lambda f_x(\lambda) = x \quad \text{and} \quad \lim_{\lambda\to\infty} \lambda Tf_x(\lambda) = \lim_{\lambda\to\infty} \lambda[\lambda f_x(\lambda) - x] = x_1,$$

we may conclude that $x \in \mathfrak{D}_T$ and $Tx = x_1$. By recurrence we obtain that for any $k \geqq 0$ we have $x \in \mathfrak{D}_{T^k}$ and $x_{k+1} = Tx_k = T^{k+1}x$. Hence our function can be written as

$$f_x(\lambda) = \sum_{k=0}^{\infty} T^k x/\mu^{k+1}$$

in a neighbourhood of ∞ and the series is convergent if

$$|\mu| > \sup_n \overline{\lim}_{k\to\infty} |T^k x|_n^{1/k} = \tau_x.$$

From this it is obvious that

$$\sup\{|\lambda|; \lambda \in \gamma_T(x)\} \leqq \tau_x.$$

PROPOSITION 2.5. *If $T \in C(\mathfrak{X})$, then for any $x \in \mathfrak{X}$ with $\sigma_T(x) \subseteq \mathbf{C}$ it follows that $x \in \mathfrak{D}_{T^k}$ for all $k \geqq 1$ and*

$$\sup\{|\lambda|; \lambda \in \sigma_T(x)\} = \sup_n \overline{\lim}_{k\to\infty} |T^k x|_n^{1/k} < \infty.$$

Obviously, the proof is similar. The last equality is valid because in $\rho_T(x)$ the T-associated function $x(\mu)$ has no singularities and its Taylor expansion must exist outside the disk with radius $\sup\{|\lambda|; \lambda \in \sigma_T(x)\}$.

3. Strongly residually decomposable operators

First of all we must recall the definition of residually decomposable operators [11].

Let $S \subseteq \mathbf{C}_\infty$ be a closed set. A family of open sets $\{G_j\}_{j=1}^n \cup \{G_S\}$ is an S-covering for the closed set $\Delta \subseteq \mathbf{C}_\infty$ if $\bigcup_{j=1}^n G_j \cup G_S \supseteq \Delta \cup S$ and $\bar{G}_j \cap S = \emptyset$ $(j = 1, \cdots, n)$.

By $\mathcal{J}_{T,F}$ (F closed in $\mathbf{C}_\infty$) we mean the family of all invariant subspaces $\mathfrak{Y}$ of T with the property $\sigma(T \mid \mathfrak{Y}) \subseteq F$.

An operator $T \in C(\mathfrak{X})$ is *S-residually decomposable* [11] if:

(δ_1) For every closed $F \subseteq \mathbf{C}_\infty$ with $F \cap S = \emptyset$ the family $\mathcal{J}_{T,F}$ has a least upper bound (with respect to the inclusion relation) denoted by $\mathfrak{X}_{T,F}$.

(δ_2) For every S-covering of $\sigma(T)$ there exist invariant subspaces $\{\mathfrak{X}_j\}_{j=1}^n$ of T such that

(δ_2') $\sigma(T \mid \mathfrak{X}_j) \subseteq G_j$ $(j = 1, \cdots, n)$ and

(δ_2'') any $x \in \mathfrak{X}$ has a decomposition of the form

$$x = x_1 + \cdots + x_n + x_S$$

where $x_j \in \mathfrak{X}_j$ $(j = 1, \cdots, n)$ and $\sigma_T(x_s) \subseteq \bar{G}_S$.

For such an operator $T \in C(\mathfrak{X})$ we know now that $S_T \subseteq S$ and some results concerning the relations among the spaces $\mathfrak{X}_{T,F}$, $\mathfrak{X}_T^0(F)$ and $\mathfrak{X}_T(F)$ [11]. But it seems that this definition is not sufficient to insure the existence and the uniqueness of a minimal closed set S_T^d with the property that T is S_T^d-residually decomposable. Before going on this direction we shall give some additional information.

PROPOSITION 3.1. *If $T \in C(\mathfrak{X})$ is S-residually decomposable and $S \subseteq \mathbf{C}$ then $T \in B(X)$.*

Proof. Let $\{G_1\} \cup \{G_S\}$ be an S-covering of $\sigma(T)$, with G_S relatively compact in $\mathbf{C}$. From the definition, there exists an invariant subspace $\mathfrak{X}_1 \subseteq \mathfrak{X}$ such that any $x \in \mathfrak{X}$ has a decomposition of the form $x = x_1 + x_S$, with $x_1 \in \mathfrak{X}_1$ and $\sigma_T(x_S) \equiv \bar{G}_S$. By our choice, the set $\bar{G}_S$ is compact in $\mathbf{C}$; thus, by Proposition 2.5, we have $x_S \in \mathfrak{D}_T$. It follows that $\mathfrak{D}_T = \mathfrak{X}$ and by Banach's theorem, we obtain $T \in B(\mathfrak{X})$.

DEFINITION 3.1. We shall say that an operator $T \in C(\mathfrak{X})$ which is S-residually decomposable has a localized spectrum if in the above definition the condition (δ_2'') is replaced by the stronger condition

(δ_2'')* any $x \in \mathfrak{X}$ has a decomposition of the form

$$x = x_1 + \cdots + x_n + x_S,$$

where $x_j \in \mathfrak{X}_j$, $\gamma_T(x_j) \subseteq \gamma_T(x)$ $(j = 1, \cdots, n)$ and $\sigma(x_S) \subseteq \bar{G}_S$.

(A similar condition was used in [1] to define a strong decomposability for bounded operators on Banach spaces)

Let us remark that if $\gamma_T(x) \cap S = \emptyset$ for a certain $x \in \mathfrak{X}$ then in the decomposition given by $(\delta_2'')^*$ we have

$$x = x_1 + \cdots + x_n + x_S \quad \text{with } x_S \in \mathfrak{X}_T^0(\emptyset).$$

In particular, if $S_T = \emptyset$ then $x_S = 0$ for such an element.

DEFINITION 3.2. We shall say that an operator $T \in C(X)$ is strongly residually decomposable if $S_T = \emptyset$ and for any closed F_1, F_2 such that $\mathfrak{X}_T(F_1)$ and $\mathfrak{X}_T(F_2)$ are in $\mathfrak{D}_T$ and are closed, it follows that $\mathfrak{X}_T(F_1 \cup F_2)$ is in $\mathfrak{D}_T$ and is closed.

In particular any decomposable operator [8] is strongly residually decomposable.

PROPOSITION 3.2. *If $T \in C(\mathfrak{X})$ is S-residually decomposable with a localized spectrum and if $S_T = \emptyset$, then for any closed $F \subseteq \mathbf{C}_\infty$ with $F \cap S = \emptyset$ we have the equality $\mathfrak{X}_{T,F} = \mathfrak{X}_T(F)$.*

Proof. If $F \subseteq G \subseteq \bar{G} \subseteq G_1$ with G, G_1 open and $\bar{G}_1 \cap S = \phi$, then $\{G_1\} \cup \{G_S\}$ is an S-covering of $\sigma(T)$, where $G_S = \complement \bar{G}$. Since the spectrum of T is a localized one, for any $x \in \mathfrak{X}_T(F)$ we have $\sigma_T(x) \cap S = \phi$ and $S_T = \emptyset$; using an above observation, we obtain $\mathfrak{X}_T(F) \subseteq \mathfrak{X}_1 \subseteq \mathfrak{X}_{T,\bar{G}_1}$, where $\mathfrak{X}_1$ is an invariant subspace with $\sigma(T \mid \mathfrak{X}_1) \subseteq G_1$ corresponding to the chosen covering, and $\mathfrak{X}_{T,\bar{G}_1}$ exists since $\bar{G}_1 \cap S = \emptyset$ (see (δ_1)). The family

$$\{\bar{G}_1 \,;\, \bar{G}_1 \supseteq F,\, \bar{G}_1 \cap S = \emptyset\}$$

is directed on the left by the inclusion relation. As in [11], we can prove that $\mathfrak{X}_{T,F} = \bigcap_{\bar{G}_1} \mathfrak{X}_{T,G_1} \supseteq \mathfrak{X}_T(F)$. On the other hand we have the inclusion $\mathfrak{X}_{T,F} \subseteq \mathfrak{X}_T^0(F) = \mathfrak{X}_T(F)$ (since $(\lambda I \mid \mathfrak{X}_{T,F} - T \mid \mathfrak{X}_{T,F})^{-1}$ is defined in $\complement F$), and this finishes our proof.

THEOREM 3.1. *Let $T \in C(\mathfrak{X})$ be a strongly residually decomposable operator. Then there exists a unique minimal closed set $S_T^d \subseteq \mathbf{C}_\infty$ such that T is S_T^d-residually decomposable and has a localized spectrum.*

Proof. The family of the closed sets $\{S\}$ with the property that T is S-residually decomposable and with localized spectrum is non-void since it obviously contains the set $\sigma(T)$. Let $\{S_\alpha\}$ be a totally ordered part of it and let $S_0 = \bigcap_\alpha S_\alpha$. We shall show that T is S_0-residually decomposable and with localized spectrum. Indeed, if $F \subseteq \mathbf{C}_\infty$ is a closed set with $F \cap S_0 = \emptyset$, then there exists an index α_0 such that $F \cap S_{\alpha_0} = \emptyset$. Since the operator T is S_{α_0}-residually decomposable, it follows that the family $\mathcal{J}_{T,F}$ has least upper bound in $\mathfrak{X}$. If $\{G_j\}_{j=1} \cup \{G_S\}$ is an S_0-covering of $\sigma(T)$, then there exists an index α_1 such that it is an S_{α_1}-covering of $\sigma(T)$. The remaining conditions that T be S_0-residually decomposable depend now only of the covering and they are obviously satisfied. By the Zorn lemma, there exists at least one minimal set; we shall show that there exists only one. For, let S_1 and S_2

two minimal elements and let $S = S_1 \cap S_2$. We shall prove that T is S-residually decomposable and with localized spectrum which will be a contradiction if $S_1 \neq S_2$.

Let $F \subseteq \mathbf{C}_\infty$ be a closed set with the property $F \cap S = \emptyset$. We shall write F as $F_1 \cup F_2$ with F_1, F_2 closed and with the property $F_j \cap S_j = \emptyset$ $(j = 1, 2)$. Let $F_j \subseteq H_j \subseteq \bar{H}_j \subseteq G_j$ with H_j and G_j open, $\bar{G}_j \cap S_j = \emptyset$, and the S_j-covering $\{G_j, G_{S_j}\}$ with $G_{S_j} = \complement H_j$ $(j = 1, 2)$. Let also $x \in \mathfrak{X}_T(F)$ be arbitrary. It has the decomposition

$$x = x_1 + x_{S_1}$$

with respect to the covering $\{G_1, G_{S_1}\}$ and since the spectrum of T is localized, we have

$$\sigma_T(x_{S_1}) \subseteq \bar{G}_{S_1} \cap \sigma_T(x) \subseteq \bar{G}_{S_1} \cap F.$$

We can write $x_{S_1} = y_2 + y_{S_2}$ with respect to the covering $\{G_2, G_{S_2}\}$. So using again the localization of the spectrum of T, we have

$$\sigma_T(y_{S_2}) \subseteq G_{S_2} \cap \sigma_T(x_{S_1}) \subseteq \bar{G}_{S_1} \cap \bar{G}_{S_2} \cap F \subseteq \complement H_1 \cap \complement H_2 \cap F = \emptyset.$$

By our assumption, we have $S_T = \emptyset$. Thus, on account of Proposition 2.1, it follows that $y_{S_2} = 0$. Consequently, the following inclusion is true:

$$\mathfrak{X}_T(F) \subseteq \mathfrak{X}_{T,\bar{G}_1} + \mathfrak{X}_{T,\bar{G}_2}.$$

By Proposition 2.2, we have the equalities

$$\mathfrak{X}_{T,\bar{G}_j} = \mathfrak{X}_T(\bar{G}_j) \qquad (j = 1, 2);$$

thus

$$\mathfrak{X}_{T,\bar{G}_1} + \mathfrak{X}_{T,\bar{G}_2} = \mathfrak{X}_T(\bar{G}_1) + \mathfrak{X}_T(\bar{G}_2) \subseteq \mathfrak{X}_T(\bar{G}_1 \cup \bar{G}_2).$$

Since T is strongly residually decomposable, the last linear manifold is closed and included in $\mathfrak{D}_T$, therefore the family $\mathfrak{J}_{T,\bar{G}_1 \cup \bar{G}_2}$ has a least upper bound [11]. Then we can write

$$\mathfrak{X}_T(F) = \bigcap_{G_1 \cup G_2 \supseteq F} \mathfrak{X}_T(\bar{G}_1 \cup \bar{G}_2)$$

and as in Proposition 2.2 (see also [11]), the space $\mathfrak{X}_{T,F}$ exists. In this manner the condition (δ_1) of our definition is verified.

Now let $\{G_j\}_{j=1}^n \cup \{G_S\}$ be an S-covering of $\sigma(T)$. By completion of G_S, we shall construct two open sets G_{S_1} and G_{S_2} such that $G_{S_j} \cap (S_k \setminus G_S) = \emptyset$ $(j \neq k)$. Then we shall refine the sets $G_j = \bigcup_{l=1}^{n_j} G_j^l$ such that an open set G_l intersects at most one of the sets S_1 and S_2 and if it intersects the set S_1 (respectively S_2), it is completely contained in G_{S_1} (respectively G_{S_2}). Now we can form an S_k-covering of $\sigma(T)$ using only those open sets G_j^l which do not intersect the set S_k and G_{S_k} $(k = 1, 2)$.

Let $\{H_j^k\}_{j=1}^{m_k} \cup \{G_{S_k}\}$ $(k = 1, 2)$ be the covering obtained and let $x \in \mathfrak{X}$. Then x has the following decomposition with respect to the first covering:

$$x = x_1' + \cdots + x_{m_1}' + x_{S_1}$$

with $\sigma_T(x_{S_1}) \subseteq \bar{G}_{S_1}$. The element x_{S_i} has another decomposition with respect to the second covering

$$x_{S_1} = y_1 + \cdots + y_{m_2} + y_{S_2}$$

with $\sigma_T(y_{S_2}) \subseteq \bar{G}_{S_2}$. Since the spectrum of T is a localized one, we have

$$\sigma_T(y_{S_2}) \subseteq \sigma_T(x_{S_1}) \cap \bar{G}_{S_2} \subseteq \bar{G}_{S_1} \cap \bar{G}_{S_2} = G_S$$

(the last equality can be obtained by a special construction of the sets G_{S_1} and G_{S_2} which is always possible).

Let us denote by $\{\mathfrak{X}_j^k\}$ $(j = 1, \cdots, m_k; k = 1, 2)$ the system of invariant subspaces obtained from both coverings. Since T is S_k-residually decomposable $(k = 1, 2)$ we can suppose that $\mathfrak{X}_j^k = \mathfrak{X}_{T,F_j^k}$ where $F_j^k \subseteq H_j^k$ are closed and $F_j^k \cap S_l = \emptyset$ $(k \neq l)$; therefore by Proposition 3.2, $\mathfrak{X}_{T,F_j} = \mathfrak{X}_T(F_j)$. But T is strongly residually decomposable, hence the $\mathfrak{X}_j = \mathfrak{X}_T(F_j)$ are closed in $\mathfrak{X}$ and included in $\mathfrak{D}_T$, where $F_j = \bigcup_{F_k^l \subseteq G_j} F_k^l$ $(j = 1, \cdots, n)$. Furthermore, by Proposition 2.3, we have $\sigma(T \mid \mathfrak{X}_j) \subseteq G_j$ $(j = 1, \cdots n)$ and the element $x \in \mathfrak{X}$ now has the decomposition

$$x = x_1 + \cdots + x_n + x_S$$

where the $x_j \in \mathfrak{X}_j$ are obtained by adding the corresponding elements and where $x_S = y_{S_1}$. Then T is S-residually decomposable and its spectrum is obviously localized. This is a contradiction if $S_1 \neq S_2$. Therefore there exists a unique minimal set S_T^d such that T is S_T^d-residually decomposable and its spectrum is localized.

DEFINITION 3.3. If $T \in C(\mathfrak{X})$ is strongly residually decomposable, the set S_T^d will be called the spectral residuum of T.

For a spectral operator $T \in B(\mathfrak{X})$ in the sense of Dunford [5] or Tulcea [9], one obtains easily that $S_T^d = \emptyset$; if the operator is not bounded [2], then $S_T^d = \{\infty\}$.

We note that there exist simple examples for which the spectral residuum is not a trivial one. For instance the (isometrical) "shift" on a separable Hilbert space is strongly residually decomposable but its spectral residuum is equal to its spectrum.

REFERENCES

1. C. APOSTOL, *Restriction and quotients of decomposable operators in Banach space*, Rev. Roumaine Math. Pures Appl., vol. 13 (1968), pp. 146–150.
2. W. G. BADE, *Unbounded spectral operators*, Pacific J. Math., vol. 4 (1954), pp. 373–392.
3. N. BOUBRAKI, Espaces vectoriels topologiques, Éléments de Mathématique, Hermann, Paris, 1955.
4. I. COLOJOARĂ AND C. FOIAS, *Theory of generalized spectral operators*, Grodon and Breach, New York, 1968.
5. N. DUNFORD, *Spectral operators*, Pacific J. Math., vol. 4 (1954), pp. 321–354.

6. ———, *A survey of the theory of spectral operators*, Bull. Amer. Math. Soc., vol. 64 (1958), pp. 217–274.
7. C. Foias, *Une application des distributions vectorielles à la théorie spectrale*, Bull. Sci. Math., vol. 84 (1960), pp. 147–158.
8. ———, *Spectral maximal spaces and decomposable operators in Banach spaces*, Arch. Math. (Basel), vol. 17 (1963), pp. 341–349.
9. C. T. Ionescu Tulcea, *Spectral operators on locally convex spaces*, Bull. Amer. Math. Soc., vol. 67 (1961), pp. 125–128.
10. F.-Y. Maeda, *Generalized spectral operators on locally convex spaces*, Pacific J. Math., vol. 13 (1963), pp. 177–192.
11. F. H. Vasilescu, *Residually decomposable operators in Banach spaces*, Tôhoku Math. J., vol. 21 (1969), pp. 509–522.
12. L. Waelbroeck, *Le calcul symbolique dans les algèbres commutatives*, J. Math. Pures Appl., 9e série, vol. 33 (1954), pp. 147–186.

Académie de la République Socialiste de Roumanie
Bucarest, Roumania