VECTOR LATTICES AND SEQUENCE SPACES[1]

BY
G. Crofts and J. J. Masterson

1. Introduction

In [6], Peressini and Sherbert study Köthe sequence spaces and mappings between these spaces with special emphasis on their order properties. Variations of these results appear in [5]. The purpose of this paper is to extend some of these results. Among other things, it will be shown by using techniques involving the Dedekind completion of a vector lattice, that condition "E is Dedekind complete" can be replaced by "E is Archimedean" in several places.

Let E be a vector lattice. We say the net $\{x_\alpha\}$ order converges to x in E (we write $x_\alpha \to^{\circ} x$) if there exists a net $\{y_\alpha\} \subset E$ such that $| x_\alpha - x | \leqq y_\alpha$ and $y_\alpha \downarrow 0$ (i.e., $\{y_\alpha\}$ is down-directed and $\inf(y_\alpha) = 0$). As in [6], we note the following significant restrictions which may be put on E.

(A) If A is a subset of E that has a supremum, then there is a countable subset A' of A such that $\sup A' = \sup$ A. (We say that E is *order separable.*)

(B) If $\{y_{n,\, m}\}$ is a sequence in E that order converges to y_n in E (for each $n = 1, 2, \cdots,$) and if $\{y_n\}$ order converges to y_0 there is an increasing sequence $\{m_n : n = 1, 2, \cdots,\}$ of positive integers such that $\{y_{n,\, m_n}\}$ order converges to y_0. (We say E has the *diagonal property.* Note that we use the definition in [5] rather than [6].)

(C) If $\{A_n\}$ is a sequence of non-majorized subsets of E, there exist finite subsets A'_n of A_n $(n = 1, 2, \cdots)$ such that

$$\{\sup A'_n : n = 1, 2, \cdots\}$$

is not a majorized set. (We say that E is *finitely unbounded.*)

2. The Dedekind completion

Let E be an Archimedean vector lattice throughout this section. Nakano has shown [3] that a necessary and sufficient condition that E be Archimedean is that it possess a Dedekind completion (or cut completion) i.e., that there exist a complete vector lattice $\hat{E}$ such that:

(D1) E is embedded as a subvector lattice in $\hat{E}$,

(D2) for each $u \in \hat{E}$,

$$u = \sup\{x : x \in E, x \leqq u\} = \inf\{y : y \in E, y \geqq u\}$$

In [1], it is shown that (D2) can be replaced by

(D2′) For $0 < u \in \hat{E}$, there are x, y in E such that $0 < x \leqq u < y$.

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As defined by either set of two conditions, $\hat{E}$ is unique up to isomorphism. It is also shown in the reference just cited, that if E is a subvector lattice of a complete vector lattice F, D is the ideal in F generated by E, and $0 < u \in D$ implies the existence of $x \in E$ such that $0 < x \leqq u$, then D is the Dedekind completion of E.

What will be shown in this section is that E inherits some properties from $\hat{E}$ and vice-versa. These facts will then be used to obtain results about vector lattices (in particular sequence spaces) free of the hypothesis that E be Dedekind complete.

We first note that if $\hat{E}$ is order separable, then E is clearly order separable.

Luxemburg and Zaanen prove [2, Theorem 11.5] that an Archimedean vector lattice L is order separable if and only if every non-empty subset D which is bounded above has a countable subset possessing the same set of upper bounds as D. We use this fact to show that $\hat{L}$ is order separable if L is.

Let $u \in \hat{L}$, $u = \sup V$ where V is a subset of $\hat{L}$. For each $v \in V$, let

$$A_v = \{a \in L : a \leqq v\}.$$

Then $A = \bigcup\{A_v : v \in V\}$ is a subset of L and $\sup A = u$. But, this implies that u is the smallest of the $\hat{E}$-upper bounds of A, hence $u \leqq b$ where b is any upper bound of A in E. By definition of the Dedekind Completion, then, $u = \inf B$ (in $\hat{E}$) where B is the set of upper bounds of A which lie in E. The result mentioned above [2, Theorem 11.5] gives us a sequence $\{a_n\} \subset A$ which has B as its set of E-upper bounds. Hence $u = \sup\{a_n\}$ (in $\hat{E}$). But for each n, there is $v_n \in V$ for which $a_n \leqq v_n$. $\{v_n\}$ is a countable subset of V and $\sup \{v_n\} = u$. $\hat{L}$ is thus order separable. We have shown

Remark. L is order separable if and only if $\hat{L}$ is order separable.

The following will be useful for our further discussion.

LEMMA 2.1. *Let E be an Archimedean order separable vector lattice. Then, if $\{v_n\} \subset \hat{E}$ and $v_n \downarrow 0$, there is a sequence $\{x_n\} \subset E$ such that $x_n \downarrow 0$ and $v_n \leqq x_n$, $n = 1, 2, \cdots$.*

Proof. Assume $\{v_n\} \subset \hat{E}$ and $v_n \downarrow 0$. Let $A_n = \{x \in X : x \geqq v_n\}$. Then $A = \bigcup_{n=1}^{\infty} A_n$ is directed downward and $\inf (A) = 0$. Since E is order separable, there exists $\{y_k\} \subset A$ such that $\inf (y_k) = 0$ and $y_k \geqq y_{k+1}$, $k = 0, 1, 2, \cdots$. By definition of A, there is a subsequence of the integers $1 = n_0 < n_1 < n_2 < n_3 \cdots$ such that $v_{n_k} \leqq y_k$, $k = 0, 1, 2, \cdots$. Now, choose $x_1 > v_1$, and let $x_m = y_k$ for $n_k \leqq m \leqq n_{k+1} - 1$. Then, $v_m \leqq v_{n_k} \leqq y_k = x_m$. Since $y_k \downarrow 0$, it is clear that $x_n \downarrow 0$ and $v_n \leqq x_n$ for each $n = 1, 2, \cdots$.

PROPOSITION 2.2. *Let E be an Archimedean order separable vector lattice. Given $\{x_n\} \subset E$, then $x_n \to^{\circ} 0$ (in E) if and only if $x_n \to^{\circ} 0$ (in $\hat{E}$).*

Proof. Clearly, $x_n \to^{\circ} 0$ (in E) implies $x_n \to^{\circ} 0$ (in $\hat{E}$). The converse is a direct application of 2.1.

PROPOSITION 2.3. *Let E be an Archimedean order separable vector lattice. Then, E has the diagonal property if and only if $\hat{E}$ does.*

Proof. The necessity is a routine application of 2.1.

Suppose now that E has the diagonal property. Let $v_n \to^\circ v_0$, and $v_{n,k} \to^\circ v_n$, $n = 1, 2, \cdots, \cdots$ where v_n, v_0 and $v_{n,k}$ are in $\hat{E}$.

By using 2.1 again, there are sequences $\{x_n\} \subset E$ and $\{x_{n,k}\} \subset E$, $(n = 1, 2, \cdots, k = 1, 2, \cdots)$ such that

$$| v_0 - v_n | \leqq x_n, \qquad | v_n - v_{n,k} | \leqq x_{n,k}$$

and $x_n \downarrow 0$, $x_{n,k} \downarrow 0$ as $k \to \infty$ for each $n = 1, 2, \cdots$. Let $y_{n,k} = \sup (x_{n,k}, x_n)$; then $y_{n,} \to^\circ x_n$ and $x_n \to^\circ 0$ (since $\inf_k (\sup (x_{n,k}, x_n)) = x_n$). Since E has the diagonal property, there is for each n an integer $k(n)$ such that $y_{n,k(n)} \to^\circ 0$. So, there exists $\{w_n\} \subset E$, $w_n \downarrow 0$ and $y_{n,k(n)} \leqq w_n$. But then,

$$| v_0 - v_{n,k(n)} | \leqq | v_0 - v_n | + | v_n - v_{n,k(n)} | \leqq x_n + y_{n,k(n)} \leqq x_n + w_n$$

and since $(x_n + w_n) \downarrow 0$, we have $v_{n,k(n)} \to^\circ v_n$.

The next result concerns property (C) in the introduction.

PROPOSITION 2.4. *Let E be an Archimedean vector lattice. Then E is finitely unbounded if and only if $\hat{E}$ is finitely unbounded.*

Proof. If $\hat{E}$ is finitely unbounded it follows from the definition that E inherits the property.

Suppose now that E is finitely unbounded. Let $U_n \subset \hat{E}$, $n = 1, 2, \cdots$, form a sequence of non-majorized sets in $\hat{E}$. Let

$$\tilde{A}_n = \{v \in \hat{E} : | v | \leqq u, \text{ some } u \in U_n\} \quad \text{and} \quad A_n = \tilde{A}_n \cap E.$$

Then, the sequence $\{A_n\}$ of subsets of E is not majorized in E. So, there exists for each n, a finite set $A'_n \subset A_n$ such that $\{\sup A'_n\}$ is not majorized in E. Let

$$A'_n = \{a_n^{(k)} : k = 1, 2, \cdots, p(n)\}.$$

Then there is $u_n^{(k)} \in U_n$ such that $a_n^{(k)} \leqq u_n^{(k)}$. So, letting

$$U'_n = \{u_n^{(k)} : k = 1, \cdots, p(n)\},$$

U'_n is a finite subset of U_n for each n and $\{\sup U'_n\}$ is not a majorized set in $\hat{E}$.

With this result, it is possible to remove the hypothesis of completeness in Kantorovich's characterization of a bounded set in a finitely unbounded vector lattice.

PROPOSITION 2.5. *Let E be a finitely unbounded Archimedean vector lattice. Then, $B \subset E$ is an order bounded set if and only if for each sequence $\{x_n\}$ in B and for any sequence $\{\alpha_n\}$ of scalars decreasing to zero, $\alpha_n x_n \to^\circ 0$. (We call this latter condition "property * ".)*

Proof. If B is an order-bounded set, property $*$ is easily shown to hold.

Suppose $B \subset E$ and B has property $*$. Let $\hat{B}$ be the solid hull of B in $\hat{E}$. If $\{u_n\}$ is a sequence in $\hat{B}$, then for each n, there is an $x_n \in B$ such that $| u_n | \leqq | x_n |$. If $\{\alpha_n\}$ is a sequence of scalars and $\alpha_n \downarrow 0$, then $\alpha_n x_n \rightarrow^{\circ} 0$, i.e., there is a sequence $\{y_n\} \subset E$ such that $y_n \downarrow 0$ and $| \alpha_n x_n | \leqq y_n$. But $| \alpha_n u_n | \leqq | \alpha_n x_n |$ shows that $\alpha_n u_n \rightarrow^{\circ} 0$. By 2.4, $\hat{E}$ is finitely unbounded and Kantorovich's result implies that $\hat{B}$ is order bounded in $\hat{E}$, so B is order bounded in $\hat{E}$, so B is order bounded in E.

We give an example to show that "order-separable" cannot be removed from the hypotheses of the first three propositions in Section 2.

Example 1. Let $\mathfrak{B}(X)$ be the Dedekind complete vector lattice of all bounded functions on the uncountable set X, with the usual operations defined. Let $\mathfrak{F}(X)$ be the subvector lattice of functions which assume a single value on all but a finite number of points in X. By checking conditions (D1) and (D2′) it is easily shown that $\mathfrak{B}(X)$ is the Dedekind completion of $\mathfrak{F}(X)$. Since $\mathfrak{B}(X)$ is easily seen to be not order separable, $\mathfrak{F}(X)$ is not order separable by the remark before 2.1.

(a) Let $X_1 = \{x_1, x_2, x_3, \cdots\}$ be a countable subset of X. Define a sequence $\langle v_n \rangle$ in $\mathfrak{B}(X)$ as follows:

$$\begin{aligned} v_n(x) &= 1 \quad \text{if } x = x_k,\ k \geqq n, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

$\langle v_n \rangle$ decreases and $\inf_n (v_n) = 0$, hence $v_n \downarrow 0$. But if $\langle f_n \rangle$ is any sequence in $\mathfrak{F}(X)$ such that $v_n \leqq f_n$, then each f_n assumes a constant value $\geqq 1$ on all but a finite subset of X. It is then *not* possible to have $f_n \downarrow 0$. This shows that Lemma 2.1 is false if "order-separable" is removed.

(b) Let $\langle g_n \rangle$ be a sequence in $\mathfrak{F}(x)$ defined as follows:

$$\begin{aligned} g_n(x) &= 1 \quad \text{if } x = x_n, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Then $0 \leqq g_n \leqq v_n$, where v_n is defined as in (a) above. It follows then that $g_n \rightarrow^{\circ} 0$ in $\mathfrak{B}(X)$. If, however, $\langle f_n \rangle$ is any sequence in $\mathfrak{F}(X)$ such that $0 \leqq g_n \leqq f_n$, $n = 1, 2, \cdots$ and $f_n \downarrow$, each f_n would have to take on values larger than or equal to 1 on an infinite set. As in the previous example, it would then be impossible to have $\inf_n (f_n) = 0$. So, $g_n \rightarrow^{\circ} 0$ in $\mathfrak{F}(X)$ could not hold. This shows that Proposition 2.2 does not hold when "order-separable" is removed.

(c) Let $\{f_n\} \subset \mathfrak{F}(X)$ be any sequence such that $f_n \downarrow 0$. Then, given any $\alpha > 0$, there is a positive integer n_0 such that $f_{n_0}(x) < \alpha$ for all x except possibly a finite set. If not, $f_n(x) \geqq \alpha$ for each n on some co-finite set. But then, there is an uncountable set $X_0 \subset X$ such that $f_n(x) \geqq \alpha$ for all $x \in X_0$ and all $n = 1, 2, 3, \cdots$ contradicting $f_n \downarrow 0$. Having such an n_0, there exists $n_1 > n_0$ such that $f_{n_1}(x) < \alpha$ for all x, again since $f_n \downarrow 0$.

We make $\mathfrak{F}(X)$ into a normed space by putting

$$\| f \| = \sup \{ | f(x) | : x \in X \}.$$

It follows from the above paragraph that if $g_n \to^\circ g$ in $\mathfrak{F}(X)$, then $\| g_n - g \| \to 0$. For, $g_n \to^\circ g$ implies there is $\{f_n\} \subset \mathfrak{F}(X)$ such that $| g_n - g | \leqq f_n$ and $f_n \downarrow 0$. We have shown above that the latter implies that $\| f_n \| \downarrow 0$, hence $\| g_n - g \| \to 0$. Since the function $e(X) \equiv 1$ is in $\mathfrak{F}(X)$, norm convergence clearly implies order convergence. We conclude, then, that norm convergence and order convergence are identical in $\mathfrak{F}(X)$. Since the topological convergence defined by the norm is diagonalizable, order convergence must be also. Hence $\mathfrak{F}(X)$ has the diagonal property. $\mathfrak{B}(X)$, however does not have the diagonal property as the following argument shows. Let $X_0 = \{x_1, x_2, \cdots\}$ be a sequence of distinct elements in X. Consider the elements $h_{n,m}$ in $\mathfrak{B}(X)$, given by

$$h_{n,m}(x) = n \quad \text{if } x = x_i, \, i > m,$$
$$= 0 \quad \text{otherwise}$$

Then $h_{n,m} \to^\circ 0$ in $\mathfrak{B}(X)$ for each n, however, for each sequence

$$\{m_n : n = 1, 2, \cdots\}$$

the sequence $\{h_{n,m_n}\}$ is unbounded in the order sense.

We conclude this section with an example of a vector lattice which is order separable, has the diagonal property, is finitely unbounded but is not Dedekind complete.

Example 2. Let k be any natural number. Define $l^1(k)$ to be the set of all sequences of real numbers which are coordinatewise products of two sequences defined as follows:

$$c_i = a_i b_i$$

where $\langle a_i \rangle$ is a periodic sequence of period k, $\langle b_i \rangle$ is a sequence in l^1 composed of constant blocks of length k, i.e.,

$$b_{mk+1} = b_{mk+2} = \cdots = b_{m(k+1)}, \qquad \text{each} \quad m = 0, 1, 2, \cdots.$$

$l^1(k)$ is a vector lattice with the above stated properties. Indeed, its Dedekind completion is l^1 as is readily verified. Verification that it has the stated properties reduces to noting that l^1 does and then using the results of this section relating $l^1(k)$ to $l^1(k) = l^1$.

3. Topological vector lattices

Let E be a vector lattice. A subset B of E is said to be *solid* if $y \in B$ whenever $x \in B$ and $| y | \leqq | x |$. Suppose that E is also a topological vector space.

Then E is called a *topological vector lattice* if the topology has a neighborhood base at zero consisting of solid sets. If in addition the topology is locally convex we will call E a *locally convex lattice.*

Let $\mathfrak{T}$ be a locally convex topology on E where E is an Archimedean vector lattice. Let ν be a base of neighborhoods of zero for $\mathfrak{T}$. For each $V \in \nu$, define $\hat{V}$ as the solid hull of V in $\hat{E}$. Denote by $\hat{\nu}$ the collection of all $\hat{V}$ for $V \in \nu$. Routine verification shows that $\hat{\nu}$ is a base of neighborhoods at zero for a locally convex topology on $\hat{E}$. Moreover, a base of solid neighborhoods at zero for E clearly generates a base of solid neighborhoods of 0 for $\hat{E}$. Hence if E is a locally convex lattice under $\mathfrak{T}$, then $\hat{E}$ is a locally convex lattice under $\hat{\mathfrak{T}}$. Examples show that a Hausdorff locally convex topology on E need not generate one such on $\hat{E}$. However, if E is locally convex lattice, the "solidity" of the neighborhood base generates a Hausdorff topology on $\hat{E}$. There are many interesting questions concerning the relationship of $(E, \mathfrak{T})$ to $(\hat{E}, \hat{\mathfrak{T}})$ but we will not attempt such a study here. We rather state one theorem in this connection which will be of value to us.

Proposition P [5, 1.21, p. 155]. *Suppose that $(E, \mathfrak{T})$ is a locally convex lattice with the property that $\{x_\alpha : \alpha \in I\}$ converges to 0 for $\mathfrak{T}$ whenever $\{x_\alpha\} \downarrow 0$. Then, there is a locally convex topology $\hat{\mathfrak{T}}$ on the cut-completion $\hat{E}$ of E such that $\hat{\mathfrak{T}}$ induces $\mathfrak{T}$ on E, $(E, \mathfrak{T})$ is dense in $(\hat{E}, \hat{\mathfrak{T}})$ and $(\hat{E}, \hat{\mathfrak{T}})$ is a locally convex lattice.*

(*Note.* The topology $\hat{\mathfrak{T}}$ in [5] is described by a family $\{\hat{P}_\beta : \beta \in B\}$ derived from the family $\{P_\beta : \beta \in B\}$ of seminorms which describe the topology $\mathfrak{T}$ on E. If we let $V_\beta = \{x \in E : P_\beta(x) < 1\}$, then under the hypotheses of this proposition, $\hat{V}_\beta = \{u \in E : \hat{P}_\beta(u) < 1\}$. The topology $\hat{\mathfrak{T}}$ described here then, coincides with our previous definition of the naturally generated topology on $\hat{E}$.)

Remark. It is also readily shown that if $(E, \mathfrak{T})$ is a metrizable space, $(\hat{E}, \hat{\mathfrak{T}})$ is also a metrizable space. Moreover, in Proposition P, we can replace the hypothesis "$\{x_\alpha\} \downarrow 0$ implies $\{x_\alpha\}$ $\mathfrak{T}$-converges to zero" by the following property which will be of use to us later, when E is order separable.

(P*) $\{x_n\}_{n=1}^{\infty} \downarrow 0$ implies that the sequence $\{x_n\}$ $\mathfrak{T}$-converges to 0 as $n \to \infty$.

The following is a slight generalization of the property "boundedly order-complete" in [5] and seems more natural in arbitrary (not necessarily Dedekind complete) vector lattices.

Definition. $(E, \mathfrak{T})$ is *monotone bounded* if every topologically bounded monotone increasing net is order bounded above.

The next result gives the precise relationship between the two properties.

Proposition 3.1. *If $(E, \mathfrak{T})$ is a monotone bounded locally convex lattice, then $(\hat{E}, \hat{\mathfrak{T}})$ is boundedly order complete.*

Proof. Let $\{u_\alpha : \alpha \in \mathfrak{A}\}$ be a net in $\hat{E}$ which is topologically bounded and $u_\alpha \uparrow$. We assume without loss of generality that $u_\alpha \geqq 0$, for each $\alpha \in \mathfrak{A}$. For each α, choose a net $\{x_{\alpha,\beta} : \beta \in B_\alpha\} \subset E^+$ so that $x_{\alpha,\beta} \uparrow u_\alpha$. Let $\{y_\gamma\}$ be the net of finite suprema of the set

$$\{x_{\alpha,\beta} : \alpha \in \mathfrak{A}, \beta \in B_\alpha\}.$$

Then $y_\gamma \uparrow$ and for each γ there exists α such that $y_\gamma \leqq u_\alpha$. Hence $\{y_\gamma\}$ is topologically bounded. (For any solid neighborhood V of 0 in $(E, \mathfrak{T})$ there is a positive scalar t such that $u_\alpha \in t\hat{V}$ for all α. But then $y_\gamma \in tV$ for all γ.) It follows then that $\{y_\gamma\}$ is order bounded in E from above by some $z \in E$. Hence $\{u_\alpha\}$ is also bounded by z in $\hat{E}$. So $\sup\{u_\alpha\}$ exists in $\hat{E}$.

PROPOSITION 3.2. *If $(E, \mathfrak{T}$ is an order separable locally convex lattice with property* P^* *then $(\hat{E}, \hat{\mathfrak{T}})$ also has property* P^*.

Proof. Suppose $\{u_n\} \downarrow 0$ in $\hat{E}$. By Lemma 2.1, there is a sequence $\{x_n\} \subset E$ such that $u_n \leqq x_n$ and $x_n \downarrow 0$. But then, $\{x_n\}\mathfrak{T}$-converges to 0 in E, and hence, considered as a sequence in $\hat{E}$, $\{x_n\}$ is $\hat{\mathfrak{T}}$-convergent to zero. It follows easily from the "solidity" of the neighborhood base for $\hat{\mathfrak{T}}$ that $u_n \leqq x_n$ implies $\{u_n\}$ also $\hat{\mathfrak{T}}$-converges to zero.

The above results will now be applied to show that "σ-order complete" may be replaced by "Archimedean" in Proposition 2.6 in [5], if E is assumed to be order separable. (cf., [5, p. 164]).

PROPOSITION 3.3. *If $(E, \mathfrak{T})$ is a metrizable, Archimedean order separable locally convex lattice with property* (P^*) *and if, in addition, it is monotone bounded, then E has the diagonal property.*

Proof: By Proposition (P) and the remark following the topology $\hat{\mathfrak{T}}$ generated on $\hat{E}$ makes $(\hat{E}, \hat{\mathfrak{T}})$ a locally convex lattice which is metrizable. From 3.1 and 3.2 we have that $(\hat{E}, \hat{\mathfrak{T}})$ is boundedly order complete and satisfies condition P^*. From [5], Proposition 2.6, we have that $\hat{E}$ has the diagonal property.

We conclude this section by giving a condition for sequence spaces that is equivalent to P^* and is more frequently found in the literature.

In being consistent with the usual notation we let φ denote the set of all finitely nonzero sequences of real numbers. Further, we define the n^{th} section, $x^{(n)}$ of x by $x_i^{(n)} = x_i$, for $i \leqq n$; $x_i^{(n)} = 0$, for $i > n$.

PROPOSITION 3.4. *Let λ be a sequence space with a topology $\mathfrak{T}$ such that $(\lambda, \mathfrak{T})$ is a locally convex lattice. $(\lambda, \mathfrak{T})$ satisfies property* P^* *if and only if φ is dense in $(\lambda, \mathfrak{T})$.*

Proof. Suppose φ is dense in $(\lambda, \mathfrak{T})$ and let $\{_nx\}_{n=1}^{\infty} \downarrow 0$ in λ.

We first show that

$$z^{(n)} \xrightarrow{\mathfrak{T}} z,$$

for each z in λ. Let ν be a fundamental system of solid convex neighborhoods of 0 for $\mathfrak{T}$, and $W \in \nu$. By assumption there is a $y \in \varphi$ with $y - z \in W$. There exists an n_0 such that $y_n = 0, n > n_0$. For each $n > n_0$,

$$|z^{(n)} - z| \leqq |y - z| \quad \text{and} \quad z^{(n)} - z \in W.$$

Let V be an arbitrary element in ν and choose $U \in \nu$ such that $U + U \subset V$.

By the above there exists an n_0 such that ${}_1x^{(n_0)} - {}_1x \in U$. Since U is solid and ${}_{n+1}x \leqq {}_nx$, we have ${}_mx^{(n_0)} - {}_mx \in U$, for each m.

The sequence $\{{}_nx\} \downarrow 0$, so $\{{}_nx\}$ coordinatewise converges to 0. Hence, there is an n_1 such that $n > n_1$ implies ${}_nx^{(n_0)} \in U$.

Letting $n > n_1$, we have ${}_nx = {}_nx^{(n_0)} + ({}_nx - {}_nx^{(n_0)}) \in U + U \subset V$, and $\{{}_nx\}$ converges to 0 in $\mathfrak{T}$.

Converseıy, suppose P* holds. Let $x \in \lambda$. $\{x - x^{(n)}\} \downarrow 0$, so

$$\{x - x^{(n)}\} \rightarrow 0 \text{ in } \mathfrak{T} \quad \text{or} \quad x^{(n)} \rightarrow x \text{ in } \mathfrak{T}.$$

Thus φ is dense in $(\lambda, \mathfrak{T})$.

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Virginia Polytechnic Institute and State University
Blacksburg, Virginia
Michigan State University
East Lansing, Michigan