SOME HOMOLOGY GROUPS OF WREATHE PRODUCTS[1]

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Let p be a prime and for each integer $n \geq 1$ denote by P_n the Sylow p-subgroup of the symmetric group of degree p^n. Thus P_n is a group of order p^k, where $k = 1 + p + \cdots + p^{n-1}$; in particular P_1 is the cyclic group of order p. P_n acts as a permutation group on p^n symbols and if these symbols form a basis of an elementary Abelian p-group A_n, then A_n is a ZP_n-module. The split extension of A_n by P_n is P_{n+1}:

$$P_{n+1} = A_n P_n .$$

In this note the groups $H_2(P_n, \mathbf{Z})$ and $H_1(P_n, A_n)$ will be computed. I wish to express my gratitude to L. Evens for a number of discussions which have helped me considerably in this work.

1. Statement of results

For $n = 1$, $H_2(P_1, \mathbf{Z}) = 0$ since P_1 is cyclic. For $n > 1$, P_n is the wreathe product of P_1 and P_{n-1}:

$$P_n = P_1 \wr P_{n-1} .$$

The calculation of $H_2(P_n, \mathbf{Z})$ will be achieved by computing the Schur multiplier of a wreathe product $G \wr H$, where G and H are arbitrary groups and G acts as in its regular representation. To state the result let T be the tensor square of the abelian group H/H':

$$T = H/H' \otimes H/H'.$$

Let K be the subgroup of T generated by all elements of the form

$$h_1 H' \otimes h_2 H' + h_2 H' \otimes h_1 H' \qquad (h_1, h_2 \in H).$$

Let G_1 denote a set of elements of G having the property that if $x \in G$ and $x^2 \neq 1$, then G_1 contains either x or x^{-1} but not both. Let G_2 be the set of involutions in G. Let $C(G; H)$ denote the direct sum of $|G_1|$ copies of T and $|G_2|$ copies of T/K.

THEOREM 1. *$H_2(G \wr H, \mathbf{Z})$ is the direct sum of $H_2(G, \mathbf{Z})$, $H_2(H, \mathbf{Z})$ and $C(G; H)$.*

Application of this with $G = P_1$, $H = P_{n-1}$ shows that $H_2(P_n, \mathbf{Z})$ is the direct sum of $H_2(P_{n-1}, \mathbf{Z})$ and $C(P_1; P_{n-1})$. As is well known, P_{n-1}/P'_{n-1} is elementary Abelian of order p^{n-1}, so in this case T is elementary Abelian of

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order $p^{(n-1)^2}$. For p odd, $|G_1| = \frac{1}{2}(p-1)$ and $|G_2| = 0$, so $C(P_1; P_{n-1})$ is elementary Abelian of order p^c, where $c = \frac{1}{2}(p-1)(n-1)^2$ For $p = 2$ $|G_1| = 0$, $|G_2| = 1$ and $|K| = 2^{(1/2)(n-2)(n-1)}$; hence $C(P_1; P_{n-1})$ is elementary Abelian of order $2^{(1/2)n(n-1)}$. The following is thus a consequence of Theorem 1.

COROLLARY. *$H_2(P_n, Z)$ is elementary Abelian of order p^m, where*

$$\begin{aligned} m &= \tfrac{1}{2}(p-1)(1^2 + 2^2 + \cdots + (n-1)^2) \quad (p \text{ odd}), \\ &= \tfrac{1}{6}n(n^2-1) \quad (p = 2). \end{aligned}$$

Another fact emerges from the calculation used to prove Theorem 1. This concerns a certain characteristic subgroup $Z(G)$ defined for any group G as follows. An element x of G lies in $Z(G)$ if and only if whenever ρ is an isomorphism of G onto T/U with U contained in the center of T, then $x\rho$ is contained in the center of T. If G is isomorphic to F/R, where F is free, $Z(G)$ corresponds to the group Y/R, where $Y/[R, F]$ is the center of $F/[R, F]$.

An element z lies in $Z(G)$ if G is generated by the roots of z (cf. [2, page 137]). It follows from this fact and the definition of the wreathe product that $Z(P_2)$ is the center of P_2 if p is odd.

THEOREM 2. *Suppose that G is a finite group and that H is a group for which $H' \cap Z(H) \neq 1$. Then $W' \cap Z(W) \neq 1$, where $W = G \wr H$.*

COROLLARY. *For p odd, $Z(P_n)$ is the center of P_n.*

This corollary is proved by induction on n. It is trivial for $n = 1$ and has been established for $n = 2$. For $n > 2$, $Z(P_{n-1})$ is the center of P_{n-1} by the inductive hypothesis. Since P_{n-1} is non-Abelian it follows that $P'_{n-1} \cap Z(P_{n-1}) \neq 1$. By Theorem 2, $P'_n \cap Z(P_n) \neq 1$. Thus $Z(P_n)$ is a non-trivial subgroup of the center of P_n. Since the center of P_n is of order p, the corollary is proved.

This corollary implies a theorem of L. Evens [1] which states that, for p odd, if G is a p-group and $G/\gamma_k(G)$ is isomorphic to P_n then $\gamma_k(G) = 1$.

The proof of Theorem 1 follows the method of Schur for the calculation of the multiplier. For the one-dimensional homology groups let A be a $\mathbf{Z}G$-module and let R be the kernel of the ZG-epimorphism of $A \otimes \mathbf{Z}G$ onto A which carries $a \otimes g$ into ag $(a \in A, g \in G)$. Since $H_1(G, A \otimes \mathbf{Z}G) = 0$ the exact homology sequence gives the isomorphism

$$H_1(G, A) = R \cap [A \otimes ZG, G]/[R, G].$$

It is possible to approach this isomorphism from a more group-theoretical viewpoint which brings out the analogy with the method of Schur. To do this the following will be proved; in this the restriction that A be Abelian is dropped. Thus suppose that G, A are groups and that G acts on A; that is, for each $g \in G$ an automorphism $a \to a^g$ of A is defined and $(a^{g_1})^{g_2} = a^{g_1 g_2}$. The free product $G * A$ of G and A will be considered, and the embeddings of A, G in $G * A$ will be denoted respectively by i, j.

THEOREM 3. *Let S be the kernel of the epimorphism of $P = G * A$ onto the split extension of A by G. Then $S/[S, P]$ is generated by the elements*

$$\bar{d}(g, a) = (a^g i)^{-1}(gj)^{-1}(ai)(gj)[S, P],$$

where g, a run through G, A respectively. The definining relations of the Abelian group $S/[S, P]$ are

$$\bar{d}(g, a_1)\bar{d}(g, a_2) = \bar{d}(g, a_1 a_2), \qquad \bar{d}(g_1 g_2, a) = \bar{d}(g_1, a)\bar{d}(g_2, a^{g_1}).$$

When A is Abelian, that is, when A is a $\mathbf{Z}G$-module, A is written additively and ag is written for a^g. In this case the theorem states that $S/[S, P]$ is the group $C_1(G, A)/B_1(G, A)$. Thus there is a homomorphism α of $S/[S, P]$ into A such that $\bar{d}(g, a)\alpha = a(1 - g)$ and the kernel of α is $H_1(G, A)$. Let β be the epimorphism of P onto the direct product of G and A, and let D be the kernel of β. Since β carries S into A, $[S, P] \leq D$. Hence β induces α on $S/[S, P]$ and the kernel of α is $S \cap D/[S, P]$.

COROLLARY 1. $H_1(G, A) \cong S \cap D/[S, P]$.

Let H be the subgroup $(Ai)S$ of P. Then H/H' is a $\mathbf{Z}G$-module and it is deduced from the universal properties of the free and tensor products that there is a $\mathbf{Z}G$-isomorphism beeween H/H' and $A \otimes ZG$ in which $(gj)^{-1}(ai)\cdot(gj)H'$ and a $a \otimes g$ correspond ($a \in A$, $g \in G$). In this isomorphism S/H' corresponds to the kernel R of the $\mathbf{Z}G$-epimorphism of $A \otimes \mathbf{Z}G$ into A which carries $a \otimes g$ into ag, and $[S, P]/H'$ corresponds to $[R, G]$. Theorem 3 thus has the following consequence.

COROLLARY 2. *Suppose that A is a $\mathbf{Z}G$-module and that R is the kernel of the $\mathbf{Z}G$-homomorphism of $A \otimes \mathbf{Z}G$ into A which carries $a \otimes g$ into ag ($a \in A$, $g \in G$). Then $R/[R, G]$ is generated by the elements*

$$c(a, g) = a \otimes g - ag \otimes 1 + [R, G].$$

The defining relations of the abelian group $R/[R, G]$ are

$$c(a_1, g) + c(a_2, g) = c(a_1 + a_2, g), \qquad c(a, g_1 g_2) = c(a, g_1) + c(ag_1, g_2)$$

Thus an isomorphism exists between $R/[R, G]$ and $C_1(G, A)/B_1(G, A)$, and considerations similar to those following Theorem 3 show that in this isomorphism

$$R \cap [A \otimes ZG, G]/[R, G]$$

corresponds to $H_1(G, A)$. This last isomorphism is the same as the one obtained from the exact homology sequence, though at first sight it looks a little different. It should be observed that in view of the second of these relations, if G is generated by X, $R/[R, G]$ is generated by the $c(a, x)$ with $x \in X$.

Corollary 2 can be used to calculate the first homology group whenever sufficiently simple defining relations of G are known. For example suppose that

A is a $\mathbf{Z}H$-module for some group H. Then if G is any group $A \otimes \mathbf{Z}G$ has the structure of a $\mathbf{Z}(G \wr H)$-module.

THEOREM 4. *$H_1(G \wr H, A \otimes \mathbf{Z}G)$ is the direct sum of $H_1(H, A)$ and $|G| - 1$ copies of $A/[A, H] \otimes H/H'$.*

The computation of $H_1(P_n, A_n)$ follows easily. For $n = 1$, $A_1 = \mathbf{Z}/p\mathbf{Z} \otimes \mathbf{Z}P_1$, so $H_1(P_1, A_1) = 0$. For $n > 1$, A_n may be taken to be $A_{n-1} \otimes \mathbf{Z}P_1$, since $P_1 \wr P_{n-1}$ acts faithfully on this. Thus Theorem 4 shows that $H_1(P_n, A_n)$ is the direct sum of $H_1(P_{n-1}, A_{n-1})$ and $p - 1$ copies of

$$A_{n-1}/[A_{n-1}, P_{n-1}] \otimes P_{n-1}/P'_{n-1}.$$

However, $A_{n-1}/[A_{n-1}, P_{n-1}]$ is cyclic of order p and P_{n-1}/P'_{n-1} is elementary Abelian of order p^{n-1}. The following result is therefore obtained.

COROLLARY. *$H_1(P_n, A_n)$ is elementary Abelian of order p^k, where $k = \frac{1}{2}n(n-1)(p-1)$.*

It thus only remains to prove Theorems 1–4.

2. Proofs of Theorems 1 and 2

We begin by expressing the groups $C(G; H)$ and $G \wr H$ in terms of generators and relations.

LEMMA 1. *$C(G; H)$ is the Abelian group generated by a set of symbols $r^g(h_1, h_2)$, where h_1, h_2 run through H and g runs through $G - \{1\}$, with defining relations*

(1) $$r^g(h_1 h_2, h_3) = r^g(h_1, h_2) + r^g(h_2, h_3),$$

(2) $$r^g(h_1, h_2 h_3) = r^g(h_1, h_2) + r^g(h_1, h_3)$$

(3) $$r^g(h_1, h_2) + r^{g^{-1}}(h_2, h_1) = 0$$

where h_1, h_2, h_3 run through H and g runs through $G - \{1\}$.

Let U be the Abelian group with these generators and relations. For $x \in G_1$ let V_x be the Abelian group generated by $r^x(h_1, h_2)$ and $r^{x^{-1}}(h_1, h_2)$ with defining relations (1), (2), (3), where g runs through $\{x, x^{-1}\}$. For $y \in G_2$ let W_y be the Abelian group generated by $r^y(h_1, h_2)$ with defining relations (1), (2), (3), where $g = g^{-1} = y$. Clearly U is the direct sum of the groups V_x and W_y, as x runs through G_1 and y runs through G_2. In view of the relation (3), V_x is generated by the $r^x(h_1, h_2)$ alone. Elimination of $r^{x^{-1}}(h_1, h_2)$ from the defining relations of V_x shows that V_x is generated by the $r^x(h_1, h_2)$ with defining relations (1), (2), where $g = x$. Thus V_x is isomorphic to T. As for W_y, the defining relations show that there is an epimorphism of T onto W_y which carries $h_1 H' \otimes h_2 H'$ onto $r^y(h_1, h_2)$; the kernel is generated by the elements corresponding to the left hand side of (3) and is therefore K. Thus W_y is isomorphic to T/K and U is isomorphic to $C(G; H)$.

For arbitrary groups G and H the wreathe product $G \wr H$ is a split extension by G of the direct product B of $|G|$ copies of H. G acts transitively and regularly on these copies of H; thus if we identify one of them with H, the copies are precisely the transforms H^g as g runs through G. B is the direct product of the H^g and each element x of B may be written uniquely in the form

$$x = \prod_{g \in G} x_g^g,$$

where $x_g \in H$ and all but a finite number of x_g are equal to 1. The element x_g will be called the *g-component* of x. The action of G on B is described by the statement that the g_1-component of $x^g\,(g \in G)$ is $x_{g_1 g^{-1}}$.

It follows that $G \wr H$ is generated by G and H. A set of defining relations of $G \wr H$ is furnished by the multiplication tables of G and H, together with relations expressing the commutativity of elements in H^{g_1} and H^{g_2} for distinct elements g_1, g_2 of G. To state this more formally let F be a free group with basis consisting of a set of symbols $u(g)$, $v(h)$, where g runs through $G - \{1\}$ and h runs through $H - \{1\}$. Put $u(1) = v(1) = 1$. Let

$$b(g_1, g_2) = u(g_1 g_2)^{-1} u(g_1) u(g_2) \qquad (g_1, g_2 \in G), \tag{4}$$

$$c(h_1, h_2) = v(h_1 h_2)^{-1} v(h_1) v(h_2) \qquad h_1, h_2 \in H), \tag{5}$$

$$d^g(h_1, h_2) = [v(h_1)^{u(g)}, v(h_2)] \quad (h_1, h_2 \in H, g \in G - \{1\}). \tag{6}$$

Let R be the normal closure in F of the elements $b(g_1, g_2)$, $c(h_1, h_2)$, $d^g(h_1, h_2)$. Then there is an isomorphism between $G \wr H$ and F/R in which g, h correspond to $u(g)R$, $v(h)R$. The Schur multiplier of $G \wr H$ is $R \cap F'/[R, F]$.

The group $R/[R, F]$ will be investigated first. Let

$$\bar{b}(g_1, g_2) = b(g_1, g_2)[R, F],\ \bar{c}(h_1, h_2) = c(h_1, h_2)[R, F],$$
$$\bar{d}^g(h_1, h_2) = d^g(h_1, h_2)[R, F].$$

$R/[R, F]$ is of course generated by $\bar{b}(g_1, g_2)$, $\bar{c}(h_1, h_2)$ and $\bar{d}^g(h_1, h_2)$. These elements satisfy the following relations.

$$\bar{b}(g, 1) = \bar{b}(1, g) = 1, \tag{7}$$

$\bar{b}(g_2, g_3)\bar{b}(g_1, g_2 g_3) = \bar{b}(g_1 g_2, g_3)\bar{b}(g_1, g_2)$ for any elements g_1, g_2, g_3 of G.

$$\bar{c}(h, 1) = \bar{c}(1, h) = 1, \tag{8}$$

$\bar{c}(h_2, h_3)\bar{c}(h_1, h_2 h_3) = \bar{c}(h_1 h_2, h_3)\bar{c}(h_1, h_2)$ for any elements h_1, h_2, h_3 of H.

For $g \in G - \{1\}$ and any elements h_1, h_2, h_3 of H,

$$\begin{aligned} \bar{d}^g(h_1 h_2, h_3) &= \bar{d}^g(h_1, h_3)\bar{d}^g(h_2, h_3), \\ \bar{d}^g(h_1, h_2 h_3) &= \bar{d}^g(h_1, h_2)\bar{d}^g(h_1, h_3). \end{aligned} \tag{9}$$

For $g \in G - \{1\}$ and any elements h_1, h_2 of H,

$$\bar{d}^g(h_1, h_2)\bar{d}^{g^{-1}}(h_2, h_1) = 1. \tag{10}$$

The relations (7) and (8) are of course simply the usual expression of associativity in an extension. The first of the relations (9) is proved as follows.

$$\begin{aligned}\bar{d}^g(h_1 h_2, h_3) &= [v(h_1 h_2)^{u(g)}, v(h_3)][R, F] \\ &= [(v(h_1)v(h_2)c(h_1, h_2)^{-1})^{u(g)}, v(h_3)][R, F] \quad \text{by (5)} \\ &= [(v(h_1)^{u(g)}v(h_2)^{u(g)}, v(h_3)][R, F],\end{aligned}$$

since $c(h_1, h_2) \in R$. Since $[v(h_1)^{u(g)}, v(h_3)]$ is central modulo $[R, F]$, it follows that

$$\begin{aligned}\bar{d}^g(h_1 h_2, h_3) &= [v(h_1)^{u(g)}, v(h_3)][v(h_2)^{u(g)}, v(h_3)][R, F] \\ &= \bar{d}^g(h_1, h_3)\bar{d}^g(h_2, h_3).\end{aligned}$$

The second relation (9) is proved similarly. As for (10),

$$\begin{aligned}\bar{d}^g(h_1, h_2)^{-1} &= [v(h_2), v(h_1)^{u(g)}][R, F] \\ &= [v(h_2)^{u(g)^{-1}}, v(h_1)]^{u(g)}[R, F].\end{aligned}$$

Since $u(g)^{-1} \equiv u(g^{-1})$ modulo R,

$$\bar{d}^g(h_1, h_2)^{-1} = [v(h_2)^{u(g^{-1})}, v(h_1)]^{u(g)}[R, F] = \bar{d}^{g^{-1}}(h_2, h_1),$$

as asserted.

Theorem 2 will now be proved. Thus suppose that $Z(H) \cap H'$ contains an element $z \neq 1$. Since G is finite, an element

$$w = \prod_{g \in G} v(z)^{u(g)}$$

may be defined. The order in the product is arbitrary but fixed. Thus

$$w^{u(g_2)} = \prod_{g_1 \in G} v(z)^{u(g_1)u(g_2)} = \prod_{g_1 \in G} v(z)^{u(g_1 g_2)b(g_1, g_2)}$$

by (4), so

$$w^{u(g_2)} \equiv \prod_{g_1 \in G} v(z)^{u(g_1 g_2)} \quad \text{modulo } [R, F],$$

since $b(g_1, g_2) \in R$. The product on the right hand side is the same as w except for the order of the factors. Restoration of the original order involves the introduction of certain commutators of the form

$$[v(z)^{u(g_1 g_2)}, v(z)^{u(g_3 g_2)}].$$

But this commutator is conjugate to $d^g(z, z)$ modulo $[R, F]$. Since $z \in H'$, the relation (9) shows that $d^g(z, h) \in [R, F]$ for any $h \in H$. Hence

$$w^{u(g_2)} \equiv w \quad \text{modulo } [R, F].$$

Again for $h \in H$,

$$\begin{aligned}w^{v(h)} &= \prod_{g \in G} v(z)^{u(g)v(h)} \\ &= \prod_{g \in G} v(z)^{u(g)}[v(z)^{u(g)}, v(h)] \\ &= v(z)^{v(h)} \prod_{g \in G - \{1\}} v(z)^{u(g)} d^g(z, h) \\ &\equiv v(z)^{v(h)} \prod_{g \in G - \{1\}} v(z)^{u(g)} \quad \text{modulo } [R, F],\end{aligned}$$

as before. If T is the group generated by R and all $v(h)$ $(h \in H)$, there is an epimorphism of $T/[R, F]$ onto H and the kernel $R/[R, F]$ is central. It follows since $z \in Z(H)$ that $v(z)$ lies in the center of T modulo $[R, F]$, so

$$v(z)^{v(h)} \equiv v(z) \quad \text{modulo } [R, F].$$

Hence

$$w^{v(h)} \equiv w \quad \text{modulo } [R, F].$$

It has therefore been proved that w lies in the center of F modulo $[R, F]$. Hence the element of $W = G \wr H$ corresponding to w, namely

$$t = \prod_{g \in G} z^g,$$

lies in $Z(W)$. Since $z \in H'$, $t \in W'$, so $W' \cap Z(W) \neq 1$. This completes the proof of Theorem 2.

Returning to the general case, Theorem 1 rests upon the following lemma.

LEMMA 2. *$R/[R, F]$ is the Abelian group generated by $\bar{b}(g_1, g_2)$, $\bar{c}(h_1, h_2)$ and $\bar{d}^g(h_1, h_2)$ with defining relations* (7)–(10).

To prove Lemma 2, choose a well-ordering $\leq$ of G; this ordering need have no relation to the group structure of G.

Let A be the additively written Abelian group generated by elements $\beta(g_1, g_2)$, $\gamma(h_1, h_2)$ and $\delta^g(h_1, h_2)$ with defining relations.

$$\beta(g, 1) = \beta(1, g) = \gamma(h, 1) = \gamma(1, h) = 0,$$

$$\beta(g_2, g_3) + \beta(g_1, g_2 g_3) = \beta(g_1 g_2, g_3) + \beta(g_1, g_2),$$

$$\gamma(h_2, h_3) + \gamma(h_1, h_2 h_3) = \gamma(h_1 h_2, h_3) + \gamma(h_1, h_2),$$

$$\delta^g(h_1 h_2, h_3) = \delta^g(h_1, h_3) + \delta^g(h_2, h_3),$$

$$\delta^g(h_1, h_2 h_3) = \delta^g(h_1, h_2) + \delta^g(h_1, h_3),$$

$$\delta^g(h_1, h_2) + \delta^{g^{-1}}(h_2, h_1) = 0,$$

where the h_i run through H, the g_i through G and g through $G - \{1\}$. By (7)–(10) there is an epimorphism φ of A onto $R/[R, F]$ such that

$$\beta(g_1, g_2)\varphi = \bar{b}(g_1, g_2), \quad \gamma(h_1, h_2)\varphi = \bar{c}(h_1, h_2), \quad \delta^g(h_1, h_2)\varphi = \bar{d}^g(h_1, h_2).$$

The assertion of Lemma 2 is that φ is a monomorphism; this will be proved by constructing a mapping ψ of $R/[R, F]$ into A such that $\varphi\psi$ is the identity mapping on A.

First a factor set of $G \wr H$ in A will be constructed. In doing this elements of B will be denoted by x, y, z and the g-components of x, y, z are denoted by x_g, y_g, z_g respectively. Mappings σ, π of $B \times B$ into A are defined by the formulae

$$\sigma(x, y) = \sum_{g \in G} \gamma(x_g, y_g), \tag{11}$$

$$\pi(x, y) = \sum_{g_1 < g_2} \delta^{g_2 g_1^{-1}}(x_{g_2}, y_{g_1}). \tag{12}$$

(A summation sign with inequalities underneath it involving $g_1, g_2, \cdots$ means that summation is to be carried out over all elements $g_1, g_2, \cdots$ of G for which the inequalities hold). From the defining relations of A the following relations are easily deduced.

$$\sigma(y, z) + \sigma(x, yz) = \sigma(xy, z) + \sigma(x, y), \tag{13}$$

$$\sigma(x^g, y^g) = \sigma(x, y), \tag{14}$$

$$\pi(xy, z) = \pi(x, z) + \pi(y, z), \tag{15}$$

$$\pi(x, yz) = \pi(x, y) + \pi(x, z). \tag{16}$$

Next for each $g \in G$, a mapping τ_g of B into A is defined by the formula

$$\tau_g(x) = \sum_{g_1<g_2, g_1g>g_2g} \delta^{g_1g_2^{-1}}(x_{g_1}, x_{g_2}). \tag{17}$$

(Note that τ_1 is the zero mapping.) The relation

$$\tau_g(xy) - \tau_g(x) - \tau_g(y) = \pi(x^g, y^g) - \pi(x, y) \tag{18}$$

holds for all $x \in B$, $y \in B$. For the defining relations of A applied to the left-hand side yield

$$\sum_{g_1<g_2, g_1g>g_2g} \{\delta^{g_1g_2^{-1}}(x_{g_1}, y_{g_2}) + \delta^{g_1g_2^{-1}}(y_{g_1}, x_{g_2})\}.$$

Upon application of the last of the defining relations of A and (12) to the second term this becomes

$$\sum_{g_1<g_2, g_1g>g_2g} \delta^{g_1g_2^{-1}}(x_{g_1}, y_{g_2}) - \pi(x, y) + \sum_{g_1<g_2, g_1g<g_2g} \delta^{g_2g_1^{-1}}(x_{g_2}, y_{g_1}).$$

Interchanging g_1 and g_2 in the last term,

$$\begin{aligned}\tau_g(xy) - \tau_g(x) - \tau_g(y) &= -\pi(x, y) + \sum_{g_1g>g_2g} \delta^{g_1g_2^{-1}}(x_{g_1}, y_{g_2}) \\ &= -\pi(x, y) + \sum_{g_1>g_2} \delta^{g_1g^{-1}}(x_{g_1g^{-1}}, y_{g_2g^{-1}}) \\ &= -\pi(x, y) + \pi(x^g, y^g),\end{aligned}$$

which is (18). Also if $g \in G$ and $g' \in G$,

$$\tau_{g'}(x^g) = \tau_{gg'}(x) - \tau_g(x). \tag{19}$$

To prove this the summands in the definition of $\tau_{g'}(x^g)$ are to be split into two halves defined by $g_1 < g_2$ and $g_1 > g_2$; in the latter half interchange g_1 and g_2 and apply the last of the defining relations of A. Subtraction of $\tau_{gg'}(x)$ from the resulting expression readily yields (19).

The desired factor set may now be constructed. For $w_i \in W = G \wr H$, write $w_i = g_i x_i$ with $g_i \in G$, $x_i \in B$. A mapping α of $W \times W$ into A is defined by the formula

$$\alpha(w_1, w_2) = \tau_{g_2}(x_1) + \pi(x_1^{g_2}, x_2) + \sigma(x_1^{g_2}, x_2) + \beta(g_1, g_2).$$

It follows immediately from (13)–(16) and (18)–(19) that

$$\alpha(w_2, w_3) - \alpha(w_1 w_2, w_3) + \alpha(w_1, w_2 w_3) - \alpha(w_1, w_2) = 0.$$

Let Γ be the central extension of A by W with this factor set. Thus there is a epimorphism θ of Γ onto W and a mapping ω of W into Γ such that A is the kernel of θ, $\omega\theta$ is the identity mapping and

$$\omega(w_1)\omega(w_2) = \omega(w_1 w_2)\alpha(w_1, w_2)$$

for all w_1 and w_2 in W. In particular for g_1, g_2 in G and h_1, h_2 in H, it follows from (11), (12) and (17) that

$$\omega(g_1)\omega(g_2) = \omega(g_1 g_2)\beta(g_1, g_2) \tag{20}$$

$$\omega(h_1)\omega(h_2) = \omega(h_1 h_2)\gamma(h_1, h_2) \tag{21}$$

Also if $g \in G - \{1\}$,

$$\omega(h_1^g)\omega(h_2) = \omega(h_1^g h_2)\pi(h_1^g, h_2), \qquad \omega(h_2)\omega(h_1^g) = \omega(h_1^g h_2)\pi(h_2, h_1^g),$$

whence

$$[\omega(h_1^g), \omega(h_2)] = \pi(h_1^g, h_2) - \pi(h_2, h_1^g).$$

It follows that

$$[\omega(h_1)^{\omega(g)}, \omega(h_2)] = \delta^g(h_1, h_2). \tag{22}$$

From (20)–(22) it is seen that A is contained in the group generated by all $\omega(g)$, $\omega(h)$ as g, h run through G, H respectively. Hence Γ is generated by these elements. Therefore since F is free, there is an epimorphism χ of F onto Γ such that $u(g)\chi = \omega(g)$, $v(h)\chi = \omega(h)$. By comparing (4)–(6) with (20)–(22) it is seen that

$$b(g_1, g_2)\chi = \beta(g_1, g_2), \quad c(h_1, h_2)\chi = \gamma(h_1, h_2), \quad d^g(h_1, h_2)\chi = \delta^g(h_1, h_2).$$

Since A lies in the center of Γ, χ carries R onto A, and $[R, F]$ is contained in the kernel of χ. Hence χ induces an epimorphism ψ of $R/[R, F]$ onto A, and ψ is given by

$$\bar{b}(g_1, g_2)\psi = \beta(g_1, g_2), \quad \bar{c}(h_1, h_2)\psi = \gamma(h_1, h_2), \quad \bar{d}^g(h_1, h_2)\psi = \delta^g(h_1, h_2).$$

Hence $\varphi\psi$ is the identity mapping, and Lemma 2 is proved.

Lemma 2 shows that $R/[R, F]$ is the direct sum of three groups $\bar{B}$, $\bar{C}$ and $\bar{D}$. $\bar{B}$ is generated by the elements $\bar{b}(g_1, g_2)$ and has defining relations (7); thus $\bar{B}$ is isomorphic to $C_2(G, \mathbf{Z})/B_2(G, \mathbf{Z})$ and the boundary operator corresponds to the homomorphism ν_1 of $\bar{B}$ into F/F' which carries $\bar{b}(g_1, g_2)$ into $u(g_2)u(g_1 g_2)^{-1}u(g_1)F'$. Similarly $\bar{C}$ is generated by the elements $\bar{c}(h_1, h_2)$ and has defining relations (8); thus $\bar{C}$ is isomorphic to $C_2(H, \mathbf{Z})/B_2(H, \mathbf{Z})$ and the boundary operator corresponds to the homomorphism ν_2 of $\bar{C}$ into F/F' which carries $\bar{c}(h_1, h_2)$ into $v(h_2)v(h_1 h_2)^{-1}v(h_1)F'$. Finally $\bar{D}$ is generated by the elements $\bar{d}^g(h_1, h_2)$ and has defining relations (9) and (10); thus by Lemma 1, $\bar{D}$ is isomorphic to $C(G; H)$.

To complete the proof of Theorem 1 let ν be the natural homomorphism of F onto F/F'. Of course ν induces a homomorphism $\bar{\nu}$ of $R/[R, F]$ into F/F', and the kernel of $\bar{\nu}$ is the desired group $R \cap F'/[R, F]$. By (4), (5) and (6), the restriction of $\bar{\nu}$ to $\bar{B}$ is ν_1, the restriction of $\bar{\nu}$ to $\bar{C}$ is ν_2, and the restriction of $\bar{\nu}$ to $\bar{D}$ is zero. Since the images of ν_1 and ν_2 intersect in 1, the kernel of $\bar{\nu}$ is the direct sum of the kernel of ν_1, the kernel of ν_2 and $\bar{D}$. So $R \cap F'/[R,F]$ is isomorphic to the direct sum of $H_2(G, Z]$, $H_2(H, \mathbf{Z})$ and $C(G; H)$.

Theorem 1 is therefore proved.

3. Proof of Theorem 3

The proof of Theorem 3 is along the same lines as that of Lemma 2. It will be recalled that the group G acts on the group A, that P is the free product $G * A$ of G and A and that S is the kernel of the epimorphism of P onto the split extention of A by G. Denote by i, j respectively the embeddings of A, G in P and for $g \in G$, $a \in A$ define

$$d(g, a) = (a^g i)^{-1}(gj)^{-1}(ai)(gj).$$

Then $d(g, a) \in S$. It is easy to check the following relations:

$$d(g, a')^{ai} = d(g, a'a^{g^{-1}})d(g, a^g)^{-1},$$

$$d(g', a)^{gj} = d(g, a^{g'})^{-1}(d(g'g, a).$$

It follows from these three relations that every element of P is of the form $(gj)(ai)\,d$, where d is a product of the $d(g, a)$ and their inverses. Hence S is generated by the $d(g, a)$. If

$$\bar{d}(g, a) = d(g, a)[S, P],$$

the above relations become

$$\bar{d}(g, a_1 a_2) = \bar{d}(g, a_1)\bar{d}(g, a_2),$$

$$\bar{d}(g_1 g_2, a) = \bar{d}(g_2, a^{g_1})\bar{d}(g_1, a).$$

Let C be the Abelian group generated by a set of symbols $\delta(g, a)$ ($g \in G$, $a \in A$) with defining relations

$$\delta(g, a_1 a_2) = \delta(g, a_1)\delta(g, a_2),$$

$$\delta(g_1 g_2, a) = \delta(g_2, a^{g_1})\delta(g_1, a).$$

Then there is an epimorphism φ of C onto $S/[S, P]$ such that

$$\delta(g, a)\varphi = \bar{d}(g, a).$$

The assertion of Theorem 3 is that φ is a monomorphism; this will be proved by constructing a mapping ψ of $S/[S, P]$ into C such that $\varphi\psi$ is the identity mapping on C.

The split extension of A by G will be denoted by K and the element k_i of K

will be written $g_i a_i$ with $g_i \in G$, $a_i \in A$. A mapping α of $K \times K$ into C is defined by the formula

$$\alpha(k_1, k_2) = \delta(g_2, a_1).$$

Then

$$\alpha(k_2, k_3)\alpha(k_1 k_2, k_3)^{-1}\alpha(k_1, k_2 k_3)\alpha(k_1, k_2)^{-1}$$
$$= \delta(g_3, a_2)\delta(g_3, a_1^{g_2}a_2)^{-1}\delta(g_2 g_3, a_1)\delta(g_2, a_1)^{-1}$$
$$= 1.$$

Hence α is a factor set and there exists a corresponding central extension Γ of C by K. Thus there is an epimorphism θ of Γ onto K and a mapping ω of K into Γ such that C is the kernel of θ, $\omega\theta$ is the identity mapping and

$$\omega(k_1)\omega(k_2) = \omega(k_1 k_2)\alpha(k_1, k_2)$$

for all k_1 and k_2 in K. In particular

$$\omega(g)\omega(a^g) = \omega(ga^g)\alpha(g, a^g) = \omega(ag)\delta(1, 1) = \omega(ag),$$

and

$$\omega(a)\omega(g) = \omega(ag)\alpha(a, g) = \omega(ag)\delta(g, a),$$

so that

$$\omega(a^g)^{-1}\omega(g)^{-1}\omega(a)\omega(g) = \delta(g, a);$$

Also $\omega(g_1)\omega(g_2) = \omega(g_1 g_2)$ and $\omega(a_1)\omega(a_2) = \omega(a_1 a_2)$. Hence there is a homomorphism χ of P into Γ such that $(ai)\chi = \omega(a)$ and $(gj)\chi = \omega(g)$ for $a \in A$, $g \in G$. Thus

$$d(g, a)\chi = \omega(a^g)^{-1}\omega(g)^{-1}\omega(a)\omega(g) = \delta(g, a).$$

Hence χ carries S onto C, and since C is contained in the center of Γ, $[S, P]$ is contained in the kernel of χ. Hence χ induces an epimorphism ψ of $S/[S, P]$ onto C, and ψ is given by

$$\bar{d}(g, a)\psi = \delta(g, a).$$

Hence $\varphi\psi$ is the identity mapping and Theorem 3 is proved.

4. Proof of Theorem 4

Suppose that G, H are groups and that A is a ZH-module. Then the split extension AH of A by H and the wreathe product $K = G \wr AH$ may be formed. K may then be regarded as the split extension of $B = A \otimes ZG$ by $W = G \wr H$, the action of W on B being given by

$$(a \otimes 1)h = ah \otimes 1, \qquad (a \otimes g_1)g_2 = a \otimes g_1 g_2,$$

where $a \in A$, $h \in H$, $g_1 \in G$, $g_2 \in G$; further if $g \in G - \{1\}$,

$$(a \otimes g)h = a \otimes g.$$

Thus B is a $\mathbf{Z}W$-module. Let R be the kernel of the $\mathbf{Z}W$-homomorphism of $B \otimes \mathbf{Z}W$ onto B which carries $b \otimes w$ into bw. By a remark following Theorem

3, Corollary 2, $R/[R, W]$ is generated by the elements

$$\bar{b}_{g'}(a, g) = b_{g'}(a, g) + [R, W] \quad \text{and} \quad \bar{c}_g(a, h) = c_g(a, h) + [R, W],$$

where

$$b_{g'}(a, g) = (a \otimes g') \otimes g - (a \otimes g')g \otimes 1,$$

$$c_g(a, h) = (a \otimes g) \otimes h - (a \otimes g)h \otimes 1;$$

here $a \in A$, $g \in G$, $g' \in G$, $h \in H$. Let $b(a, g) = b_1(a, g)$, $\bar{b}(a, g) = \bar{b}_1(a, g)$. Then it is easy to verify that

$$b_{g'}(a, g) = b(a, g'g) - b(a, g')g.$$

Hence

$$\bar{b}_{g'}(a, g) = \bar{b}(a, g'g) - \bar{b}(a, g'),$$

so $R/[R, W]$ is generated by the $\bar{b}(a, g)$ and the $\bar{c}_g(a, h)$. The following relations hold.

(1) $\bar{b}$ is linear in a.
(2) For all $g \in G$, $\bar{c}_g$ is linear in a.
(3) $\bar{b}(a, 1) = 0$.
(4) For $a \in A$ and $h_i \in H$, $\bar{c}_1(a, h_1) + \bar{c}_1(ah_1, h_2) = \bar{c}_1(a, h_1 h_2)$.
(5) For $g \in C - \{1\}$, $\bar{c}_g$ is homomorphic in h.
(6) For $g \in G - \{1\}$, $c_g(a, h) = 0$ if $a \in [A, H]$.

Of these (1), (2), (3) are obvious and (4), (5) follow easily from the definition of $\bar{c}_g$. To prove (6) it is necessary to show that

$$(a(1 - h) \otimes g) \otimes (1 - h') \in [R, W]$$

for any $a \in A$, $h \in H$, $h' \in H$ and $g \in G - \{1\}$. If $b = a \otimes g$, $bh^g = ah \otimes g$, so it must be shown that

$$b(1 - h^g) \otimes (1 - h') \in [R, W].$$

But the left side is easily seen to be equal to

$$(b \otimes h^g - bh^g \otimes 1)(1 - h') - (b \otimes h' - bh' \otimes 1)(1 - h^g),$$

since $hh'^g = h'^g h$ and $bh' = b$.

Lemma 3. *The relations* (1)–(6) *constitute a system of defining relations of* $R/[R, W]$.

To prove this let C be an additively written Abelian group generated by symbols $\beta(a, g)$ and $\gamma_g(a, h)$, where a, g, h run through A, G, H respectively with defining relations

$$\beta(a_1 + a_2, g) = \beta(a_1, g) + \beta(a_2, g),$$

$$\gamma_g(a_1 + a_2, h) = \gamma_g(a_1, h) + \gamma_g(a_2, h),$$

$$\beta(a, 1) = 0,$$

$$\gamma_1(a, h_1 h_2) = \gamma_1(a, h_1) + \gamma_1(ah_1, h_2),$$
$$\gamma_g(a, h_1 h_2) = \gamma_g(a, h_1) + \gamma_g(a, h_2) \quad (g \neq 1),$$
$$\gamma_g(a, h) = 0 \quad (a \in [A, H], g \neq 1).$$

Let Γ be the direct sum of the abelian groups B and C; the elements of Γ will be written as ordered pairs (b, c). Mappings ξ and η of $B \times G$ and $B \times H$ into C respectively, both linear in B, may be defined satisfying

$$\xi(a \otimes g, g') = \beta(a, gg') - \beta(a, g), \qquad \eta(a \otimes g, h) = \gamma_g(a, h),$$

on account of the first two defining relations of C. Hence for $g \in G$ and $h \in H$ endomorphisms $\bar{g}$, $\bar{h}$ of Γ may be defined as follows:

$$(b, c)\bar{g} = (bg, c + \xi(b, g)), \qquad (b, c)\bar{h} = (bh, c + \eta(b, h)).$$

It is easily deduced from the definition and linearity of ξ that

$$\xi(b, g_1 g_2) = \xi(b, g_1) + \xi(bg_1, g_2),$$

and hence $\bar{g}_1 \bar{g}_2 = \overline{g_1 g_2}$; also $\bar{1}_G$ is the identity mapping. Again the fourth and fifth defining relations of C imply that

$$\eta(b, h_1 h_2) = \eta(b, h_1) + \eta(bh_1, h_2),$$

whence $\bar{h}_1 \bar{h}_2 = \overline{h_1 h_2}$; also $\bar{1}_H$ is the identity mapping. The relations

$$\xi(bh, g) - \xi(b, g) = \xi(bhh'^{g^{-1}}, g) - \xi(bh'^{g^{-1}}, g),$$
$$\eta(bg^{-1}, h) + \eta(bh^g, h') = \eta(b, h') + \eta(bh'g^{-1}, h)$$

also hold for $g \neq 1$, but this verification is slightly more tedious. In proving both it may be assumed that $b = a \otimes g'$ in view of the linearity of ξ and η. The first relation is clear, since if $g' = 1$, $bh'^{g^{-1}} = b$ and $bhh'^{g^{-1}} = bh$, whereas if $g' \neq 1$, $bh = b$. Similarly the second relation reduces to $\eta(bg^{-1}, h) = \eta(bh'g^{-1}, h)$ if $g' = 1$, or to $\eta(bh^g, h') = \eta(b, h')$ if $g' \neq 1$; the second is trivial unless $g' = g$, so both reduce to $\eta(ah \otimes g, h') = \eta(a \otimes g, h')$, which follows from the last of the defining relations of C. Thus the relations are proved, and from them it is easy to see that for $g \neq 1$, $\bar{h}^{\bar{g}}$ and $\bar{h}'$ commute. Hence Γ is a $\mathbf{Z}W$-module, and

$$(b, c)g = (bg, c + \xi(b, g)), \qquad (b, c)h = (bh, c + \eta(b, h)).$$

In particular the projection of Γ onto B is a $\mathbf{Z}W$-homomorphism, and since $\beta(a, 1) = 0$.

$$(a \otimes 1, 0)g - (a \otimes g, 0) = (0, \beta(a, g)).$$

Also

$$(a \otimes g, 0)h - ((a \otimes g)h, 0) = (0, \gamma_g(a, h)).$$

There is an Abelian group homomorphism χ of $B \otimes \mathbf{Z}W$ into Γ such that $(b \otimes w)\chi = (b, 0)w$ for all $b \in B$, $w \in W$. This is clearly a $\mathbf{Z}W$-homomorphism.

The kernel of the composite of χ with the projection of Γ onto B is R, so the first component of any element of $R\chi$ is 0. Since $(0, c)w = (0, c)$, $[R, W]$ is contained in the kernel of χ. Hence χ induces a homomorphism ψ of $R/[R, F]$ into C, given by

$$\bar{b}(a, g)\psi = \beta(a, g), \qquad \bar{c}_g(a, h)\psi = \gamma_g(a, h).$$

But on account of (1)–(6), there is an epimorphism φ of C onto $R/[R, W]$ such that $\varphi\psi$ is the identity mapping on C. Hence φ is an isomorphism and Lemma 3 is proved.

Lemma 3 shows that $R/[R, W]$ is the direct sum of the group $\bar{B}$ generated by all $\bar{b}(a, g)$ and the groups $\bar{C}_g$ $(g \in G)$ generated by the $\bar{c}_g(a, h)$. The Abelian group $\bar{B}$ has defining relations (1) and (3). For $g \neq 1$, $\bar{C}_g$ has defining relations (2), (5) and (6) and is therefore isomorphic to $A/[A, H] \otimes H/H'$. Finally $\bar{C}_1$, having defining relations (2) and (4), is isomorphic to $C_1(H, A)/B_1(H, A)$.

Let ν be the additive epimorphism of $B \otimes ZW$ onto B which carries $b \otimes w$ into b; thus $[B \otimes ZW, W]$ is the kernel of ν. The homomorphism $\bar{\nu}$ of $R/[R, W]$ into B induced by ν is given by

$$\bar{b}(a, g)\bar{\nu} = a \otimes (1 - g), \qquad \bar{c}_1(a, h)\bar{\nu} = a(1 - h) \otimes 1,$$

$$\bar{c}_g(a, h)\bar{\nu} = 0 \quad (g \neq 1).$$

If μ is the additive endomorphism of $A \otimes ZG$ onto A which carries $a \otimes g$ into a, μ is zero on $\bar{B}\bar{\nu}$ but μ is faithful on $\bar{C}_1\bar{\nu}$. Hence $\bar{B}\bar{\nu} \cap \bar{C}_1\bar{\nu} = 0$. Thus the kernel $R \cap [B \otimes ZW, W]/[R, W]$ of $\bar{\nu}$ is the direct sum of the kernel S_1 of the restriction of $\bar{\nu}$ to $\bar{B}$ and the groups $\bar{C}_g$ $(g \neq 1)$. S_1 is of course isomorphic to $H_1(H, A)$, and it is clear that $S_2 = 0$. Hence $R \cap [B \otimes ZW, W]/[R, W]$ is the direct sum of $H_1(H, A)$ and $|G| - 1$ copies of $A/[A, H] \otimes H/H'$. Since the former group is isomorphic to $H_1(W, B)$, Theorem 4 is proved.

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