LOCALLY COMPACT COMMUTATIVE ARTINIAN RINGS

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Here we shall continue a study, initiated in [11], of metrizable, locally compact, finite-dimensional algebras over discrete fields. Every finite-dimensional algebra with identity is an artinian ring; and every commutative local artinian ring of zero or prime characteristic is a finite-dimensional algebra over a field.

If A is a finite-dimensional topological algebra over a topological field K, then A contains a smallest open ideal. Indeed, as A is finite dimensional, there is a minimal open ideal $\mathfrak{o}$; if $\mathfrak{o}_1$ is any open ideal, then $\mathfrak{o} \cap \mathfrak{o}_1$ is an open ideal contained in and hence identical with $\mathfrak{o}$, so $\mathfrak{o}_1 \supseteq \mathfrak{o}$; thus $\mathfrak{o}$ is the smallest open ideal. The topology of A is, of course, completely determined by that of $\mathfrak{o}$. Our principal purpose in §1 is to show that if A is, in addition, indiscrete (i.e., not discrete), metrizable, and locally compact, then $\mathfrak{o}$ may be given a finite-dimensional vector space structure over an indiscrete locally compact field F so that the topology $\mathfrak{o}$ inherits from A is the unique Hausdorff topology on $\mathfrak{o}$ making it a topological vector space over F; the scalar multiplication of the F-space $\mathfrak{o}$ is, furthermore, related in certain natural ways with that of the K-algebra A. From this it follows that K itself is (algebraically) a subfield of finite codegree of an indiscrete locally compact field (if L is a subfield of a field E, we shall call the degree of E over L the *codegree* of the subfield L of E).

In §1, no restriction is made on multiplication. Since a vector space may be regarded as an algebra whose multiplication is defined by $xy = 0$ for all vectors x, y, we may regard finite-dimensional, locally compact, metrizable vector spaces over discrete fields as examples of the objects under investigation.

In §2 we specialize to local artinian rings of zero or prime characteristic. May the action of F on $\mathfrak{o}$, or at least on part of $\mathfrak{o}$, be identified with that of a subfield of A? The answer is no, generally, for all of $\mathfrak{o}$, but yes for closed principal ideals contained in $\mathfrak{o}$ (however, it can happen that $\mathfrak{o}$ is not a principal ideal but that every nonzero principal ideal contained in $\mathfrak{o}$ is dense in $\mathfrak{o}$). This result yields a completely satisfactory account of locally compact, metrizable, special principal ideal rings [12, p. 245] of zero or prime characteristic.

1. Locally compact, metrizable, finite-dimensional algebras

If K is a field equipped with the discrete topology, a K-algebra A equipped with a topology is a topological K-algebra if and only if A is a topological ring and, for each $\lambda \in K$, $x \mapsto \lambda x$ is continuous at zero [11, p. 383]. We shall

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denote by $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{Q}_q$ respectively the locally compact fields of real numbers complex numbers, and q-adic numbers (where q is a prime); $\mathbf{Q}$ denotes the field of rationals, and $\mathbf{Z}_p((X))$ denotes the locally compact power series field over the prime field $\mathbf{Z}_p$ of p elements (where p is a prime). We recall that the topology of an indiscrete locally compact field F is given by an absolute value [4, Proposition 1, p. 156]; consequently, as F is complete, a finite-dimensional vector space E over F admits only one Hausdorff topology making it a topological vector space over F; every multilinear transformation from $E \times \cdots \times E$ (and, in particular, every linear transformation from E) into a topological vector space over F is continuous; moreover, E is metrizable and locally compact [6, pp. 17–21].

DEFINITION. Let A be a finite-dimensional algebra [vector space] over a field K, let $\mathfrak{a}$ be an ideal [subspace] of A, and let F be an indiscrete locally compact field. An F-vector space structure on $\mathfrak{a}$ is *subordinate* to the K-algebra A [the K-vector space A] if the F-vector space $\mathfrak{a}$ and the K-vector space $\mathfrak{a}$ have the same addition, if $\mathfrak{a}$ is finite dimensional over F, and if the F-scalar multiplication $\cdot$ satisfies conditions 1°, 2°, and 3° [conditions 1° and 2°]:

1° For each $\alpha \in F$ there exists a sequence $(\lambda_n)_{n \geq 1}$ in K such that for every $x \in \mathfrak{a}$, $\lambda_n x \mapsto \alpha . x$ for the unique Hausdorff topology on $\mathfrak{a}$ making it a topological vector space over F.

2° $\alpha.(\lambda x) = \lambda(\alpha.x)$ for all $\alpha \in F$, $\lambda \in K$, $x \in \mathfrak{a}$.

3° $\alpha.(xy) = (\alpha.x)y$ and $\alpha.(yx) = y(\alpha.x)$ for all $\alpha \in F$, $x \in \mathfrak{a}$, $y \in A$.

Let A be a finite-dimensional algebra over K, and suppose that there is an F-vector space structure on an ideal $\mathfrak{o}$ subordinate to the K-algebra A, where F is an indiscrete locally compact field. The unique Hausdorff topology on $\mathfrak{o}$ making it a topological vector space over F induces a topology on A, which we shall call the topology *defined* by that subordinate vector space structure, obtained by declaring the filter of neighborhoods of zero in $\mathfrak{o}$ a fundamental system of neighborhoods of zero in A. Equipped with this topology, A is a topological algebra over the discrete field K. Indeed, the filter base clearly has the required properties to define a topology making A a topological group under addition, and $\mathfrak{o}$ is an open ideal for that topology. By 3°, multiplication on $\mathfrak{o} \times \mathfrak{o}$ is F-bilinear and hence is continuous; therefore as $\mathfrak{o}$ is open, multiplication on $A \times A$ is continuous at $(0, 0)$. Let $\lambda \in K$ and $y \in A$; the functions $x \mapsto \lambda x$, $x \mapsto yx$, and $x \mapsto xy$ on $\mathfrak{o}$ are F-linear by 2° and 3° and hence are continuous; therefore as $\mathfrak{o}$ is open, the functions $x \mapsto \lambda x$, $x \mapsto yx$, and $x \mapsto xy$ on A are continuous at zero. Thus A is a topological algebra over the discrete field K. By 1°, every closed K-subspace of A that is contained in $\mathfrak{o}$ is an F-subspace. An open ideal of A contained in $\mathfrak{o}$ is closed and hence is an F-subspace; since $\mathfrak{o}$ contains no proper open F-subspaces, therefore, $\mathfrak{o}$ is the smallest open ideal of A. Since $\mathfrak{o}$ is locally compact and metrizable, so is A. We have therefore proved the following theorem:

THEOREM 1. *Let A be a finite-dimensional algebra over a field K, and let $\mathfrak{o}$ be an ideal having a vector space structure over an indiscrete locally compact field F subordinate to the K-algebra A. Equipped with the topology defined by that subordinate vector space structure, A is a locally compact, metrizable algebra over the discrete field K, $\mathfrak{o}$ inherits from A the unique Hausdorff topology making it a topological vector space over F, $\mathfrak{o}$ is the smallest open ideal of A, and every closed K-subspace of A contained in $\mathfrak{o}$ is also an F-subspace.*

If $x \notin \mathfrak{o}$, then $Kx \cap \mathfrak{o} = (0)$ and hence Kx is a discrete subspace. Therefore if $\mathfrak{o}$ is a proper ideal, the only topology on K making A a topological algebra over K is the discrete topology. But we may ask if there are other topologies on K weaker than the discrete topology making $\mathfrak{o}$ or, more generally, a closed subspace of $\mathfrak{o}$ a topological vector space over K.

THEOREM 2. *Let A be a finite-dimensional algebra over a field K, let $\mathfrak{a}$ be a nonzero subspace that has a vector space structure over an indiscrete locally compact field F subordinate to the K-vector space A, and let $\mathfrak{a}$ be equipped with the unique Hausdorff topology making it a topological vector space over F. There is a weakest topology $\mathfrak{J}(\mathfrak{a})$ on K making K a topological field such that $\mathfrak{a}$ is a topological vector space over $(K, \mathfrak{J}(\mathfrak{a}))$. The completion $K^\wedge$ of K for $\mathfrak{J}(\mathfrak{a})$ is a commutative locally compact ring whose identity element is contained in a subfield F' topologically isomorphic to F. Moreover, $K^\wedge$ is a finite-dimensional algebra over both K and F'. The set of invertible elements of $K^\wedge$ is open, and $u \mapsto u^{-1}$ is continuous on that set. Finally, if every nonzero K-subspace of $\mathfrak{a}$ is dense in $\mathfrak{a}$, then $K^\wedge$ is an indiscrete locally compact field.*

Proof. For each $\lambda \in K$, let $L_\lambda : x \mapsto \lambda x$, $x \in \mathfrak{a}$. By 2° of the definition, $L_\lambda \in \mathrm{End}_F(\mathfrak{a})$, the F-algebra of all F-linear operators on $\mathfrak{a}$. As K is a field, $L : \lambda \mapsto L_\lambda$ is an isomorphism from K onto a subfield K' of $\mathrm{End}_F(\mathfrak{a})$. Let $b_1, \cdots, b_n$ be a basis of the F-vector space $\mathfrak{a}$. The topology of F is given by an absolute value, and the topology of $\mathfrak{a}$ is defined, for example, by the norm

$$\| \textstyle\sum_{i=1}^n \alpha_i.b_i \| = \max \{ |\alpha_i| : 1 \leq i \leq n \}.$$

As $\mathfrak{a}$ is finite dimensional over F, the F-algebra $\mathrm{End}_F(\mathfrak{a})$ has a unique Hausdorff topology making it a topological algebra over F; $\mathrm{End}_F(\mathfrak{a})$ is a locally compact Banach algebra over F and its topology is given, for example, by either of the norms

$$\| u \|_1 = \sup \{ \| u(x) \| : \| x \| \leq 1 \}, \qquad \| u \|_2 = \sup \{ \| u(b_i) \| : 1 \leq i \leq n \}.$$

Let $\mathfrak{J}(\mathfrak{a})$ be the topology on K making L a homeomorphism from K onto the subfield K' of $\mathrm{End}_F(\mathfrak{a})$. Since $\mathrm{End}_F(\mathfrak{a})$ is a Banach algebra, $(K, \mathfrak{J}(\mathfrak{a}))$ is a topological ring, and its completion $K^\wedge$ may be canonically identified with the closure of K' in $\mathrm{End}_F(\mathfrak{a})$. Consequently, $K^\wedge$ is a commutative, locally compact, metrizable ring.

For each $\alpha \in F$, let $M_\alpha : x \mapsto \alpha.x$, $x \in \mathfrak{a}$. Then $\alpha \mapsto M_\alpha$ is a topological isomorphism from F onto a subfield F' of $\mathrm{End}_F(\mathfrak{a})$. Since the topology of $\mathrm{End}_F(\mathfrak{a})$ is given by $\| \ \|_2$, $F' \subseteq K^\wedge$ by 1° of the definition. Therefore $K^\wedge$ is not discrete, and $K^\wedge$ is an F-subalgebra of $\mathrm{End}_F(\mathfrak{a})$. Hence if $u \in K^\wedge$ and if u is invertible in $\mathrm{End}_F(\mathfrak{a})$, then u is a cancellable and hence invertible element of $K^\wedge$ as $K^\wedge$ is a finite-dimensional F-algebra. Consequently by familiar theorems about Banach algebras [5, Proposition 13, p. 75], the set of invertible elements of $K^\wedge$ is open in $K^\wedge$, and $u \mapsto u^{-1}$ is continuous on that set. In particular, $(K, \mathfrak{J}(\mathfrak{a}))$ is a topological field.

Since $\| \ \|_1$ gives the topology of $\mathrm{End}_F(\mathfrak{a})$, $(u, x) \mapsto u(x)$ is continuous from $\mathrm{End}_F(\mathfrak{a}) \times \mathfrak{a}$ into $\mathfrak{a}$, and consequently $(\lambda, x) \mapsto \lambda x = L_\lambda(x)$ is continuous from $K \times \mathfrak{a}$ into $\mathfrak{a}$. Thus $\mathfrak{a}$ is a topological vector space over $(K, \mathfrak{J}(\mathfrak{a}))$. As $\| \ \|_2$ defines the topology of $\mathrm{End}_F(\mathfrak{a})$, the sets V_ϵ defined by

$$V_\epsilon = \{\lambda \in K : \| \lambda b_i \| \leq \varepsilon, 1 \leq i \leq n\},$$

where $\varepsilon > 0$, form a fundamental system of neighborhoods of zero for $\mathfrak{J}(\mathfrak{a})$. But clearly V_ϵ is a neighborhood of zero for any topology on K making $\lambda \to \lambda b_i$ continuous at zero for all $i \in [1, n]$. Thus $\mathfrak{J}(\mathfrak{a})$ is the weakest topology on K making $\mathfrak{a}$ a topological vector space over K.

As $K^\wedge$ is commutative, for each $\lambda \in K$, $u \in K^\wedge$,

$$u(\lambda x) = (u \circ L_\lambda)(x) = (L_\lambda \circ u)(x) = \lambda u(x)$$

for all $x \in \mathfrak{a}$, and thus u is K-linear. Consequently, $K^\wedge$ is also a subalgebra of the K-algebra $\mathrm{End}_K(\mathfrak{a})$, since $\lambda u = L_\lambda \circ u \in K^\wedge$ for all $\lambda \in K$, $u \in K^\wedge$. Therefore $K^\wedge$ is a finite-dimensional algebra over K.

Finally, suppose that every nonzero K-subspace of $\mathfrak{a}$ is dense in $\mathfrak{a}$, and let u be a nonzero element of $K^\wedge$. The kernel N of u is then closed as u is F-linear, and N is a K-subspace of $\mathfrak{a}$ as u is K-linear. Thus $N = (0)$, so u is invertible in $\mathrm{End}_F(\mathfrak{a})$ and hence in $K^\wedge$. Therefore as $u \mapsto u^{-1}$ is continuous on the set of nonzero elements of $K^\wedge$, $K^\wedge$ is an indiscrete locally compact field.

THEOREM 3. *If A is a finite-dimensional algebra over a field K and if $\mathfrak{a}$ is a nonzero subspace having a vector space structure over an indiscrete locally compact field F subordinate to the K-vector space A, then K is (algebraically) a subfield of finite codegree of an indiscrete locally compact field $K^\wedge$ that also contains a subfield F_1 of finite codegree topologically isomorphic to F.*

Proof. Let n be the smallest number that is the dimension of a nonzero closed K-subspace of $\mathfrak{a}$, where $\mathfrak{a}$ is equipped with the unique Hausdorff topology making it a topological vector space over F, and let $\mathfrak{b}$ be a closed K-subspace of $\mathfrak{a}$ of dimension n. By the final statement of Theorem 1, $\mathfrak{b}$ is a subspace of the F-vector space $\mathfrak{a}$, and by definition of n, every nonzero K-subspace of $\mathfrak{b}$ is dense in $\mathfrak{b}$. The assertion now follows from the final assertion of Theorem 2 applied to $\mathfrak{b}$.

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following theorem, which is an extension of the well-known theorem that the fields $\mathbf{Q}_p$ and $\mathbf{Q}_q$ are not isomorphic if p and q are distinct primes [7, Corollary 91.4, p. 201].

THEOREM 4. *Let p and q be distinct primes. A field E does not contain subfields Q_p and Q_q of finite codegree respectively isomorphic to $\mathbf{Q}_p$ and $\mathbf{Q}_q$.*

Proof. Assume that E does contain such subfields. Suppose first that p and q are odd. Let n be a quadratic nonresidue of q. As $(p) + (q) = \mathbf{Z}$, there exist integers k and j such that $kp - jq = n - 1$. Let $a = kp + 1 = jq + n$. By Dirichlet's theorem on primes in arithmetic sequence, there is an infinite sequence $(r_i)_{i \geq 1}$ of distinct primes of the form $a + mpq$, whence $r_i \equiv a \equiv 1 \pmod{p}$, $r_i \equiv a \equiv n \pmod{q}$. By [7, Theorem 91.1, p. 198], $X^2 - r_i$ has a root $\sqrt{r_i}$ in Q_p but is irreducible over Q_q. Consequently, $E \supseteq Q_q(\sqrt{r_1}, \sqrt{r_2}, \cdots)$. But an easy argument establishes that if r and s are distinct primes and if $X^2 - r$ and $X^2 - s$ are irreducible over a field F of characteristic zero, then $X^2 - r$ is irreducible over $F(\sqrt{s})$. Consequently, $X^2 - r_{i+1}$ is irreducible over $Q_q(\sqrt{r_1}, \cdots, \sqrt{r_i})$, so by induction

$$[E : Q_q] \geq [Q_q(\sqrt{r_1}, \cdots, \sqrt{r_i}) : Q_q] = 2^i$$

for all i, a contradiction of the hypothesis that $[E : Q_q] < \infty$.

Finally, suppose that p is odd and $q = 2$. Let n be a quadratic nonresidue of p. As $(8) + (p) = \mathbf{Z}$, there exist integers k and j such that $8k - pj = n - 1$. Let $a = 1 + 8k = pj + n$. By Dirichlet's theorem there is an infinite sequence $(r_i)_{i \geq 1}$ of distinct primes of the form $a + 8mp$, whence $r_i \equiv a \equiv 1 \pmod{8}$, $r_i \equiv a \equiv n \pmod{p}$. Let c_i be even such that $r_i = 4c_i + 1$. For any $x \in E$, $x^2 + x - c_i = 0$ if and only if $(2x + 1)^2 = r_i$. Consequently, as $X^2 + X - c_i$ has a root in Q_2 by [7, Theorem 91.3, p. 200], $X^2 - r_i$ has a root in E but is irreducible over Q_p by [7, Theorem 91.1, p. 198]. Arguing as above, we obtain the desired contradiction.

THEOREM 5. *Let A be an indiscrete, finite-dimensional, locally compact, metrizable algebra over a discrete field K, and let $\mathfrak{o}$ be the smallest open ideal of A. Then $\mathfrak{o}$ is either connected or totally disconnected, and*

$$\mathfrak{o} = \{x \in A : \textit{either } x = 0 \textit{ or } Kx \textit{ is indiscrete}\}.$$

The topology of A is the topology defined by a vector space structure on $\mathfrak{o}$ subordinate to the K-algebra A over either $\mathbf{Z}_p((X))$ for some prime p, $\mathbf{Q}_q$ for some prime q, or $\mathbf{R}$, according as K has characteristic p, K has characteristic zero and $\mathfrak{o}$ is totally disconnected, or $\mathfrak{o}$ is connected.

Proof. Case 1. The characteristic of K is a prime p. By [11, Theorem 9], $\mathfrak{o}$ is totally disconnected. By the proof of [11, Theorem 11],

$$\mathfrak{o} = \{x \in A : \text{either } x = 0 \text{ or } Kx \text{ is indiscrete}\},$$

and $\mathfrak{o}$ is the topological direct sum of ideals $\mathfrak{o}_1, \cdots, \mathfrak{o}_n$ of A where for each

$i \in [1, n]$, $\mathfrak{o}_i$ has a vector space structure over an indiscrete locally compact field F_i subordinate to the K-algebra A and the topology $\mathfrak{o}_i$ inherits from A is the unique Hausdorff topology on $\mathfrak{o}_i$ making it a topological vector space over F_i; moreover, there exists $\lambda \in K$ transcendental over the prime subfield P of K such that F_i is the completion of the field $P(\lambda)$ for the h_i-adic valuation, the prime polynomials $h_1, \cdots, h_n$ over P being distinct. Each F_i is topologically isomorphic to a finite extension of the locally compact field $\mathbf{Z}_p((X))$; let σ_i be a topological isomorphism from $\mathbf{Z}_p((X))$ onto a subfield of F_i of finite codegree, and let ${}_{(i)}$ denote the scalar multiplication of the F_i-vector space $\mathfrak{o}_i$. It is easy to verify that, with the topology inherited from A, $\mathfrak{o}$ becomes a finite-dimensional topological vector space over $\mathbf{Z}_p((X))$ under the scalar multiplication $\cdot$ defined by

$$\alpha.(x_1 + \cdots + x_n) = \sigma_1(\alpha)_{(1)}\, x_1 + \cdots + \sigma_n(\alpha)_{(n)}\, x_n$$

for all $\alpha \in \mathbf{Z}_p((X))$, $x_1 \in \mathfrak{o}_1, \cdots, x_n \in \mathfrak{o}_n$, and that 2° and 3° of the definition hold. To show that 1° of the definition holds, let $\alpha \in \mathbf{Z}_p((X))$; it suffices to show that for each $m \geq 0$ there exists $\mu_m \in P(\lambda)$ such that $v_i(\sigma_i(\alpha) - \mu_m) \geq m$ for all $i \in [1, n]$, where v_i is the extension to F_i of the h_i-adic valuation on $P(\lambda)$, since then

$$\mu_m(x_1 + \cdots + x_n) = \textstyle\sum_{i=1}^{n} \mu_m\, x_i \to \sum_{i=1}^{n} \sigma_i(\alpha)_{(i)}\, x_i = \alpha.\,(x_1 + \cdots + x_n)$$

for all $x_1 \in \mathfrak{o}_1, \cdots, x_n \in \mathfrak{o}_n$. The existence of such a $\mu_m \in P(\lambda)$ follows, however, from the density of $P(\lambda)$ in F_i and the Approximation Theorem [4 Theorem 1, p. 134] applied to the h_i-adic valuations on $P(\lambda)$.

Case 2. The characteristic of K is zero, and $\mathfrak{o}$ is totally disconnected. By [11, Theorem 11],

$$\mathfrak{o} = \{x \in A : \text{either } x = 0 \text{ or } Kx \text{ is indiscrete}\},$$

and $\mathfrak{o}$ is the topological direct sum of nonzero ideals $\mathfrak{o}_1, \cdots, \mathfrak{o}_n$ of A where for each $i \in [1, n]$, $\mathfrak{o}_i$ has a vector space structure over $\mathbf{Q}_{q(i)}$ for some prime $q(i)$ subordinate to the K-algebra A, and the topology $\mathfrak{o}_i$ inherits from A is the unique Hausdorff topology on $\mathfrak{o}_i$ making it a topological vector space over $\mathbf{Q}_{q(i)}$. Let $q = q(1)$; we shall show that each $q(i) = q$. Suppose, on the contrary, that $q(i) = p \neq q$ for some i. By Theorem 3, K is a subfield of finite codegree of a field L that also contains a subfield Q_q of finite codegree isomorphic to $\mathbf{Q}_q$, and K is also a subfield of finite codegree of a field L' that also contains a subfield of finite codegree isomorphic to $\mathbf{Q}_p$. Let Ω be the algebraic closure of L. By Steinitz's Theorem [2, Theorem 1, p. 90] there is a K-isomorphism from L' onto a subfield L'' of Ω; thus K is a subfield of L'' of finite codegree, and L'' also contains a subfield Q_p of finite codegree isomorphic to $\mathbf{Q}_p$. Let E be the subfield $L(L'')$ of Ω. By [2, Proposition 4, p. 79],

$$[E:Q_q] = [L(L''):L][L:Q_q] \leq [L''\ K][L:Q_q] < \infty,$$

$$[E:Q_p] = [L''(L):L''][L'':Q_p] \leq [L:K][L'':Q_p] < \infty,$$

in contradiction to Theorem 4. Thus as $q(i) = q$ for all i, we may define a continuous scalar multiplication on $\mathfrak{o}$ over $\mathbf{Q}_q$ as in Case 1. The verification of 1° of the definition in this case is trivial, since $\mathbf{Q}$ is dense in $\mathbf{Q}_q$.

Case 3. $\mathfrak{o}$ is connected. The assertion follows from [11, Theorem 13]. Indeed, as $\mathbf{Q}$ is dense in $\mathbf{R}$, 1° of the definition is trivially verified; by a continuity argument, 3° of the definition follows from 1°.

To complete the proof of the theorem, we need to show that if the characteristic of K is zero, then $\mathfrak{o}$ is either connected or totally disconnected. Let $\mathfrak{c}$ be the connected component of zero, a closed ideal of A contained in $\mathfrak{o}$ as $\mathfrak{o}$ is both open and closed. Suppose that $\mathfrak{o}$ is not connected, i.e., that $\mathfrak{c} \neq \mathfrak{o}$. By the definition of $\mathfrak{o}$, $\mathfrak{c}$ is not open. Thus $A/\mathfrak{c}$ is a finite-dimensional, indiscrete, locally compact, metrizable, totally disconnected algebra over the discrete field K. By Case 2 and Theorem 3, K is a subfield of finite codegree of a field L that also contains a subfield Q_q of finite codegree isomorphic to $\mathbf{Q}_q$ for some prime q. An algebraic closure C of L is then an algebraic closure, of both K and Q_q; by Eisenstein's Theorem $X^n - q$ is irreducible over $\mathbf{Q}_q$ for all $n \geq 1$; hence C is an infinite-dimensional extension of Q_q and hence also of K. Suppose further that $\mathfrak{c} \neq (0)$. By Case 3 and Theorem 3, we may regard K as a subfield of finite codegree of a finite extension of $\mathbf{R}$ and hence as a subfield of finite codegree of $\mathbf{C}$. Thus $\mathbf{C}$ is an algebraic closure of K and $[\mathbf{C}:K] < \infty$, in contradiction to the fact that C is an infinite-dimensional extension of K [2, Theorem 2, p. 91]. Consequently, either $\mathfrak{c} = \mathfrak{o}$ or $\mathfrak{c} = (0)$, i.e., either $\mathfrak{o}$ is connected or $\mathfrak{o}$ is totally disconnected.

THEOREM 6. *If K is a field, there is an indiscrete, finite-dimensional, locally compact metrizable algebra over the discrete field K if and only if K is (algebraically) a subfield of finite codegree of an indiscrete locally compact field.*

Proof. The condition is necessary by Theorems 3 and 5. Conversely, since locally compact fields are metrizable, an indiscrete locally compact field that is a finite extension of K is a locally compact, metrizable, finite-dimensional algebra over the discrete field K.

In view of Theorem 6, it would be of interest to know more about subfields of finite codegree of locally compact fields. For example, does $\mathbf{Q}_q$ contain proper subfields of finite codegree? The field $\mathbf{Z}_p((X))$ contains many subfields of finite codegree, e.g., $\mathbf{Z}_p((X^m))$; does it contain a subfield of finite codegree that is not isomorphic to $\mathbf{Z}_p((X))$? By [1, Theorem 4], a proper subfield of $\mathbf{C}$ of finite codegree is a real-closed field, and consequently the field obtained by adjoining i to it is $\mathbf{C}$ [3, Theorem 3, p. 39]. As shown in [1, Theorem 11], there exist (even archimedean-ordered) real-closed subfields of $\mathbf{C}$ of codegree 2 that are not isomorphic to $\mathbf{R}$.

To illuminate the concepts introduced, we present examples of a locally compact finite-dimensional Cohen algebra whose topology is defined by a vector space structure on its maximal ideal. (A *local* ring is a commutative ring with identity having only one maximal ideal; a *Cohen algebra* over a field is a local algebra whose maximal ideal has codimension 1.)

Example 1. Let σ be a discontinuous automorphism of $\mathbf{C}$ such that $\sigma(i) = i$ (i.e., any automorphism other than the identity automorphism satisfying $\sigma(i) = i$). Let

$$A = \mathbf{C}[X, Y]/(X^2) + (XY) + (Y^2).$$

Thus A has a vector space basis 1, a, b, where $a^2 = ab = b^2 = 0$. The maximal ideal $\mathfrak{m} = (a, b)$ of A admits a vector space structure over $\mathbf{C}$ subordinate to the $\mathbf{C}$-algebra A by the scalar multiplication

$$\alpha.(\lambda a + \mu b) = \alpha\lambda a + \sigma^{-1}(\alpha)\mu b.$$

Since $\sigma(i) = i$ and $\mathbf{Q}(i)$ is dense in $\mathbf{C}$, 1° of the definition holds, and the remaining verifications are easy. For all λ, $\mu \in \mathbf{C}$, $\lambda a = \lambda.a$ and $\mu b = \sigma(\mu).b$. Thus (a) and (b) are closed, $\mathfrak{J}((a))$ is the usual topology $\mathfrak{J}$ of $\mathbf{C}$, and $\mathfrak{J}((b))$ is the image $\sigma(\mathfrak{J})$ of $\mathfrak{J}$ under σ. Every principal ideal of A contained in $\mathfrak{m}$ other than (a) or (b) is dense in $\mathfrak{m}$ and hence is not closed. Finally, $\mathfrak{J}(\mathfrak{m}) = \sup(\mathfrak{J}, \sigma(\mathfrak{J}))$, and hence the completion of $\mathbf{C}$ for $\mathfrak{J}(\mathfrak{m})$ is topologically isomorphic to the locally compact ring $\mathbf{C} \times \mathbf{C}$.

Example 2. Let σ be as in Example 1, and let

$$A = \mathbf{R}[X, Y]/(X^2) + (XY) + (Y^2).$$

Again, A has a vector space basis 1, a, b satisfying $a^2 = ab = b^2 = 0$. The maximal ideal $\mathfrak{m} = (a, b)$ of A admits a two-dimensional vector space structure over $\mathbf{R}$ subordinate to the $\mathbf{R}$-algebra A by the scalar multiplication

$$\alpha.(\lambda a + \mu b) = (r\lambda - s\mu)a + (r\mu + s\lambda)b,$$

where $\sigma(\alpha) = r + is$. For any $\lambda \in \mathbf{R}$, $\lambda a = \alpha.a + \beta.b$ and $\lambda b = -\beta.a + \alpha.b$ where $\sigma(\alpha + i\beta) = \lambda$. From this it follows easily that every one-dimensional subspace of $\mathfrak{m}$ is dense; in particular, no nonzero proper principal ideal of A is closed. Using the above equalities, one may readily show that the completion of $\mathbf{R}$ for $\mathfrak{J}(\mathfrak{m})$ is topologically isomorphic to $\mathbf{C}$.

2. Locally compact, metrizable, local artinian rings of zero or prime characteristic

A *coefficient field* of a local ring A with maximal ideal $\mathfrak{m}$ is any subfield K of A such that the restriction to K of the canonical epimorphism $x \mapsto x + \mathfrak{m}$ from A onto $A/\mathfrak{m}$ is an isomorphism from K onto $A/\mathfrak{m}$. If K_1 and K_2 are coefficient subfields of A, there is a canonical isomorphism from K_1 onto K_2, namely, the composite of the mappings $K_1 \rightarrow A/\mathfrak{m} \rightarrow K_2$. This isomorphism is the unique isomorphism σ from K_1 into K_2 such that $x - \sigma(x) \in \mathfrak{m}$ for each $x \in K_1$.

Let A be a local artinian ring of either zero or prime characteristic. The maximal ideal $\mathfrak{m}$ of A is nilpotent. Consequently, if A has characteristic zero, so does $A/\mathfrak{m}$. Obviously, if A has prime characteristic p, so does $A/\mathfrak{m}$.

Hence by I. S. Cohen's theorem [13, Theorem 27, p. 304], A has a coefficient field; if K is any coefficient field and if $\mathfrak{m} = (a_1, \cdots, a_n)$, then $f \mapsto f(a_1, \cdots, a_n)$ is an epimorphism from $K[X_1, \cdots, X_n]$ onto A whose kernel contains a power of each X_i; thus A is a finite-dimensional Cohen algebra over K. The results of §1 are therefore applicable to locally compact, metrizable, local artinian rings of zero or prime characteristic.

Any commutative artinian ring is the direct sum of finitely many ideals, each a local artinian ring [12, Theorem 3, p. 205]. Since the associated projection on each summand is multiplication by an idempotent, if A is a topological commutative artinian ring under some topology, A is the topological direct sum of local artinian rings. Thus a study of locally compact, metrizable, local artinian rings of zero or prime characteristic really includes the study of all locally compact, metrizable, commutative artinian rings each element of which has either infinite or squarefree additive order.

Let A be a locally compact, metrizable, local artinian ring of zero or prime characteristic, and let $\mathfrak{o}$ be the smallest open ideal of A. If $\mathfrak{o} = A$, then A is a locally compact algebra over an indiscrete locally compact field [10, Theorem 7]. Otherwise, $K \cap \mathfrak{o} = (0)$ for any coefficient field K, and consequently every coefficient field is discrete. In view of Theorem 5, however, it is natural to ask if some coefficient field K can be topologized by an indiscrete locally compact topology compatible with its field structure so that the topology $\mathfrak{o}$ inherits from A is the unique Hausdorff topology on $\mathfrak{o}$ making it a topological vector space over K (where scalar multiplication is the restriction to $K \times \mathfrak{o}$ of the given multiplication on A). If this is the case, then every ideal of A contained in $\mathfrak{o}$ is a K-subspace and hence is complete and thus closed. The following example shows, however, that even if every ideal is closed, there need not exist such a coefficient field.

Example 3. Let A be the $\mathbf{R}$-algebra $\mathbf{R}[X, Y]/(X^3) + (Y^2) + (X^2Y)$. Then A has a vector space basis $1, a, a^2, b, ab$, where $a^3 = b^2 = a^2b = 0$. The maximal ideal $\mathfrak{m}$ of A has the vector space basis a, a^2, b, ab, and $\mathfrak{m}^2$ has the vector space basis a^2, ab. For each $x \in A$, let $L_x : y \to xy, y \in \mathfrak{m}$. Then $L : x \to L_x$ is an epimorphism from A onto a subring $\mathfrak{A}$ of $\mathrm{End}_{\mathbf{R}}(\mathfrak{m})$ with kernel $\mathfrak{m}^2$. Let U be the linear operator on the $\mathbf{R}$-vector space $\mathfrak{m}$ satisfying $U(a) = a^2$, $U(a^2) = U(b) = U(ab) = 0$. Then $U(a^2) = aU(a)$, $U(ab) = aU(b) = bU(a)$; from this it follows easily that $U(xy) = xU(y)$ for all $x \in A$, $y \in \mathfrak{m}$. Consequently $U \circ L_x = L_x \circ U$ for all $x \in A$, so $\mathfrak{A}[U]$ is a commutative subalgebra of the $\mathbf{R}$-algebra $\mathrm{End}_{\mathbf{R}}(\mathfrak{m})$. As $U^2 = 0$,

$$\mathfrak{A}[U] = \{L_x + L_y \circ U : x, y \in A\}.$$

Clearly $\mathfrak{A}[U]$ is a local ring whose maximal ideal $\mathfrak{M}$ is $\{L_x + L_y \circ U : x \in \mathfrak{m}, y \in A\}$; $\mathfrak{M}^2 = (0)$ as $L_w = 0$ for all $w \in \mathfrak{m}^2$ and $L_z \circ U = 0$ for all $z \in \mathfrak{m}$. It is easy to verify that $L_\pi + U$ is transcendental over the prime field $Q = \{L_\lambda : \lambda \in \mathbf{Q}\}$ of $\mathfrak{A}[U]$. Hence as $\mathfrak{M}^2 = (0)$, there is a coefficient field R_1 in $\mathfrak{A}[U]$ containing $L_\pi + U$ [8, Zusatz 1, p. 169]. Let σ be the canonical isomorphism from $L(\mathbf{R})$

onto R_1, and let $\varphi : \alpha \to \sigma(L_\alpha)$. Thus $L_\alpha - \varphi(\alpha) \in \mathfrak{M}$ for all $\alpha \in \mathbf{R}$. We convert $\mathfrak{m}$ into a vector space over $\mathbf{R}$ by defining

$$\alpha . z = \varphi(\alpha)(z)$$

for all $\alpha \in \mathbf{R}$, $z \in \mathfrak{m}$. Since $\{V \in \mathrm{End}_{\mathbf{R}}(\mathfrak{m}) : V(xy) = xV(y) \text{ for all } x \in A, y \in \mathfrak{m}\}$ is a subring containing $L(\mathbf{R})$ and U, it contains $\mathfrak{A}[U]$ and hence R_1. Thus for all $z \in A$, $w \in \mathfrak{m}$.

$$\alpha . zw = z(\alpha . w).$$

If α, $\lambda \in \mathbf{R}$, then

$$\alpha .(\lambda z) = \varphi(\alpha)(\lambda z) = \lambda(\varphi(\alpha)(z)) = \lambda(\alpha . z)$$

for all $z \in \mathfrak{m}$. As $\alpha . z = \alpha z$ for all $\alpha \in \mathbf{Q}$, 1° of the definition holds. Thus . defines a vector space structure on $\mathfrak{m}$ subordinate to the K-algebra A; we topologize A with the topology defined by this subordinate vector space structure.

For each $z \in \mathfrak{m}$, let $z = \lambda_z a + \mu_z b + z'$, where $z' \in \mathfrak{m}^2$, and for each $\alpha \in \mathbf{R}$, let $x_\alpha \in \mathfrak{m}$, $y_\alpha \in A$ be such that $\varphi(\alpha) - L_\alpha = L_{x_\alpha} + L_{y_\alpha} \circ U$. Then

$$(1) \qquad \alpha . z - \alpha z = x_\alpha(\lambda_z a + \mu_z b) + y_\alpha \lambda_z a^2 \in \mathfrak{m}^2;$$

if $\lambda_z = 0$,

$$(2) \qquad \alpha . z - \alpha z = x_\alpha \mu_z b \in \mathbf{R}ab;$$

and if $z \in \mathfrak{m}^2$, i.e., if $\lambda_z = \mu_z = 0$,

$$(3) \qquad \alpha . z - \alpha z = 0.$$

Consequently, if $z \in \mathfrak{m}^2$, $Az = \mathbf{R}z = \mathbf{R}.z$ by (3). Therefore $\mathfrak{m}^2 = \mathbf{R}.ab + \mathbf{R}.a^2$. If $\lambda_z = 0$ and $\mu_z \neq 0$, then $Az \supseteq \mathbf{R}ab$, and hence by (2),

$$Az = \mathbf{R}z + \mathbf{R}ab = \mathbf{R}.z + \mathbf{R}ab = \mathbf{R}.z + \mathbf{R}.ab.$$

If $\lambda_z \neq 0$, then $Az \supseteq \mathbf{R}ab$ and hence also $Az \supseteq \mathbf{R}a^2 + \mathbf{R}ab = \mathfrak{m}^2$, so by (1),

$$Az = \mathbf{R}z + \mathfrak{m}^2 = \mathbf{R}.z + \mathbf{R}.ab + \mathbf{R}.a^2.$$

Thus every proper principal ideal of A is a subspace for the scalar multiplication . and hence is closed. Every proper ideal is a sum of proper principal ideals, hence is a subspace for the scalar multiplication . , and thus is closed. Therefore all ideals of A are closed.

Suppose that there exists a coefficient field R_0 equipped with an indiscrete locally compact topology compatible with its field structure so that the topology $\mathfrak{m}$ inherits from A is the unique Hausdorff topology making it a topological vector space over R_0. For each nonzero $z \in \mathfrak{m}$, $R_0 z$ is closed as it is an R_0-subspace, and $\mathbf{R}.z$ is closed; as the set of all rational multiples of z is dense in both sets, therefore, $R_0 z = \mathbf{R}.z$. Thus there is a surjection σ_z from $\mathbf{R}$ onto R_0 such that $\sigma_z(\alpha)z = \alpha . z$ for all $\alpha \in \mathbf{R}$. If $r \in R_0$, there is a unique scalar $\lambda_r \in \mathbf{R}$ such that $r - \lambda_r 1 \in \mathfrak{m}$. Consequently, a and b are linearly independent over

R_0, for if $ra + sb = 0$, then $\lambda_r\, a + \lambda_s\, b \in \mathfrak{m}^2 = \mathbf{R}ab + \mathbf{R}a^2$, so $\lambda_r = \lambda_s = 0$, whence $r \in R_0 \cap \mathfrak{m} = (0)$ and similarly $s = 0$. Therefore for all $\alpha \in \mathbf{R}$,

$$\begin{aligned}&\sigma_a(\alpha)a + \sigma_b(\alpha)b \\ &\qquad = \alpha . a + \alpha . b = \alpha . (a + b) = \sigma_{a+b}(\alpha)(a + b) = \sigma_{a+b}(\alpha)a + \sigma_{a+b}(\alpha)b,\end{aligned}$$

whence $\sigma_a(\alpha) = \sigma_{a+b}(\alpha) = \sigma_b(\alpha)$. Let $c \in A$ be such that $c + \pi = \sigma_a(\pi)$, whence also $c + \pi = \sigma_b(\pi)$. Then

$$(c + \pi)a = \sigma_a(\pi)a = \pi . a = \varphi(\pi)(a) = (L_\pi + U)(a) = \pi a + U(a)$$

and similarly $(c + \pi)b = \sigma_b(\pi)b = \pi b + U(b)$. Therefore $ca = U(a) = a^2$, $cb = U(b) = 0$. It is easy to verify, however, that no $c \in A$ satisfies $ca = a^2$, $cb = 0$.

If we restrict ourselves to closed principal ideals contained in the smallest open ideal of A, however, we obtain a positive result:

THEOREM 7. *Let A be a locally compact, metrizable, local artinian ring of zero or prime characteristic, let $\mathfrak{m}$ be the maximal ideal of A, let $\mathfrak{o}$ be the smallest open ideal of A, and let c be a nonzero element of A. Then there is a topology on a coefficient field K of A making it an indiscrete locally compact field such that the topology Ac inherits from A is the unique Hausdorff topology making it a topological vector space over K if and only if $c \in \mathfrak{o}$ and Ac is closed.*

Proof. Necessity. If c did not belong to $\mathfrak{o}$, then $Ac \cap \mathfrak{o}$ would be a proper open subspace of the topological ring Ac; however, a topological vector space over an indiscrete locally compact field has no proper open subspaces; therefore $c \in \mathfrak{o}$. We observed earlier that our hypothesis implies that Ac is complete and hence closed.

Sufficiency. By Theorem 5 there is a vector space structure on $\mathfrak{o}$ over an indiscrete locally compact field F that is subordinate to the L-algebra A, where L is a given coefficient field. Let $R = \{x \in A : xc \in F.c\}$. Clearly $1 \in R$. For each $x \in R$, let α_x be the unique scalar in F such that $xc = \alpha_x . c$. If $x, y \in R$, then

$$(x - y)c = (\alpha_x - \alpha_y).c,$$

$$xyc = x(\alpha_y.c) = \alpha_y.(xc) = \alpha_y.(\alpha_x.c) = \alpha_y\, \alpha_x.c = \alpha_x\, \alpha_y.c.$$

Hence R is a ring, and $x \mapsto \alpha_x$ is a homomorphism from R into F. Moreover, if $x \in R$ and if $x \notin \mathfrak{m}$, then $x^{-1} \in R$; indeed, $xc \neq 0$ as x is invertible, so $\alpha_x \neq 0$, and thus

$$\alpha_x.(x^{-1}c) = x^{-1}(\alpha_x.c) = x^{-1}(xc) = c = \alpha_x.(\alpha_x^{-1}.c),$$

whence $x^{-1}c = \alpha_x^{-1}.c$. Therefore R is a local ring whose maximal ideal is $R \cap \mathfrak{m}$, a nilpotent ideal. Consequently, R is a local artinian ring [10, p. 147] of zero or prime characteristic, and hence by I. S. Cohen's theorem, R contains a subfield F_0 that is canonically isomorphic to $R/(R \cap \mathfrak{m})$. The restriction of $x \mapsto \alpha_x$ to F_0 is thus a monomorphism from F_0 into F. To show that it is actually surjective, let $\alpha \in F$. Then as Ac is closed, $\alpha.c \in Ac$ by the final statement of Theorem 1, so $\alpha.c = xc$ for some $x \in A$; consequently $x \in R$, and hence there exists $y \in F_0$ such that $x - y \in \mathfrak{m}$. Let s be the smallest natural number such that $Ac \cap \mathfrak{m}^s = (0)$. Then there exists $z \in A$ such that $cz \neq 0$ and $cz \in \mathfrak{m}^{s-1}$. Therefore $(x - y)cz \in Ac \cap \mathfrak{m}^s = (0)$, so

$$0 = xcz - ycz = (\alpha.c)z - (\alpha_y.c)z = (\alpha - \alpha_y).cz,$$

whence $\alpha = \alpha_y$. Moreover,

$$\alpha_x.y = xy$$

for all $x \in F_0$, $y \in Ac$. Indeed, let $y = zc$; then

$$xy = zxc = z(\alpha_x.c) = \alpha_x.(zc) = \alpha_x.y.$$

We topologize F_0 so that $x \mapsto \alpha_x$ is a homeomorphism from F_0 onto F. Then F_0 is an indiscrete locally compact field, and the topology Ac inherits from A makes Ac a finite-dimensional topological algebra over F_0, where scalar multiplication is simply the restriction to $F_0 \times Ac$ of the given multiplication on A.

Composing the canonical monomorphisms $F_0 \to A/\mathfrak{m} \to L$, we obtain a monomorphism σ from F_0 into L such that $\sigma(x) - x \in \mathfrak{m}$ for all $x \in F_0$. Let $L_0 = \sigma(F_0)$. We wish to show that A is finite dimensional over F_0; we begin by showing that Ac is finite dimensional over L_0. First, $Ac \cap \mathfrak{m}^{s-1}$ is finite dimensional over L_0, for if $z \in Ac \cap \mathfrak{m}^{s-1}$ and if $x \in F_0$, then $(\sigma(x) - x)z \in Ac \cap \mathfrak{m}^s = (0)$, whence $\sigma(x)z = xz$. Thus

$$\dim_{L_0} (Ac \cap \mathfrak{m}^{s-1}) = \dim_{F_0} (Ac \cap \mathfrak{m}^{s-1}) < \infty.$$

Suppose that $Ac \cap \mathfrak{m}^t$ is finite dimensional over L_0, where $0 < t \leq s - 1$, and let $b_1, \cdots, b_m$ be a basis of $Ac \cap \mathfrak{m}^t$ over L_0; we shall show that $Ac \cap \mathfrak{m}^{t-1}$ is finite dimensional over L_0. Let $a_1, \cdots, a_n$ be a basis of the F_0-space $Ac \cap \mathfrak{m}^{t-1}$, and let $z \in Ac \cap \mathfrak{m}^{t-1}$. Then $z = x_1 a_1 + \cdots + x_n a_n$ where $x_1, \cdots, x_n \in F_0$. Moreover,

$$(\sigma(x_1) - x_1)a_1 + \cdots + (\sigma(x_n) - x_n)a_n \in Ac \cap \mathfrak{m}^t,$$

so

$$\sigma(x_1)a_1 + \cdots + \sigma(x_n)a_n = z + y_1 b_1 + \cdots + y_m b_m$$

where $y_1, \cdots, y_m \in L_0$. Thus $a_1, \cdots, a_n, b_1, \cdots, b_m$ is a set of generators for the L_0-space $Ac \cap \mathfrak{m}^{t-1}$. By induction, therefore, Ac is finite dimensional over L_0. Consequently as

$$\dim_{L_0} Ac = (\dim_L Ac)[L:L_0]$$

and as Ac is a nonzero ideal, we conclude that $[L:L_0] < \infty$, whence

$$\dim_{L_0} A = (\dim_L A)[L:L_0] < \infty.$$

Let r be the index of nilpotency of $\mathfrak{m}$. An argument similar to the one just made establishes that $\mathfrak{m}^{r-1}$ is finite dimensional over F_0, and that if m^k is finite dimensional over F_0 where $0 < k \leq r - 1$, then $\mathfrak{m}^{k-1}$ is finite dimensional over F_0. By induction, therefore, we conclude that A is finite dimensional over F_0.

As A is finite dimensional over F_0, A admits a unique Hausdorff topology $\mathfrak{J}'$ making it a topological vector space over the indiscrete locally compact field F_0; as Ac admits only one Hausdorff topology making it a topological vector space over F_0, the topologies Ac inherits from $\mathfrak{J}'$ and the original topology $\mathfrak{J}$ on A are identical. Multiplication is continuous on A for $\mathfrak{J}'$ since A is finite dimensional; thus $(A, \mathfrak{J}')$ is a locally compact local ring that contains an indiscrete locally compact subfield. By [9, Lemma 5], A contains a coefficient field K that is an indiscrete locally compact field for the topology inherited from $\mathfrak{J}'$. Thus Ac, with the topology inherited from $\mathfrak{J}$ (i.e., from $\mathfrak{J}'$) is a topological algebra over the indiscrete locally compact coefficient field K.

COROLLARY 1. *If A is a locally compact, metrizable, local artinian ring of zero or prime characteristic and if there is a nonzero, closed principal ideal contained in the smallest open ideal of A, then the coefficient fields of A are (algebraically) isomorphic to an indiscrete locally compact field.*

A *special* principal ideal ring is a local, artinian, principal ideal ring [12, p. 245]. Every principal ideal ring is the direct sum of ideals, each either a special principal ideal ring or a principal ideal domain [12, Theorem 33, p. 245]. The following corollary includes, in particular, a description of all locally compact, metrizable, special principal ideal rings of zero or prime characteristic.

COROLLARY 2. *If A is an indiscrete, locally compact, metrizable, local artinian ring of zero or prime characteristic, and if the smallest open ideal $\mathfrak{o}$ of A is a principal ideal, then there is a topology on some coefficient field K of A making it an indiscrete locally compact field such that the topology $\mathfrak{o}$ inherits from A is the unique Hausdorff topology making $\mathfrak{o}$ a topological vector space over K (where scalar multiplication is the restriction to $K \times \mathfrak{o}$ of the given multiplication).*

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