THE NORMAL INDEX OF MAXIMAL SUBGROUPS IN FINITE GROUPS

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In [4] Deskins defined the normal index of a maximal subgroup M in a finite group G as the order of a chief factor H/K of G where H is minimal in the set of normal supplements to M in G. We let $\eta(G:M)$ denote this number. The following two results relating to normal index were announced by Deskins [4].

(A) The finite group G is solvable if and only if for each maximal subgroup M of G, $\eta(G:M)$ is a power of a prime.

(B) The finite group G is solvable if and only if $\eta(G:M) = [G:M]$ for each maximal subgroup M of G.

In this note we obtain (B) as a corollary to a theorem on p-solvability. We also show that if G has at least one solvable maximal subgroup M such that $\eta(G:M) = [G:M]$, then G is solvable. The authors would like to thank Professor Deskins for some comments helpful in the preparation of this paper. All groups are assumed to be finite.

We begin with a lemma stated by Deskins [4, 2.1] and proved here for the sake of completeness.

LEMMA 1. *$\eta(G:M)$ is uniquely determined by M.*

Proof. We wish to show that if H_1 and H_2 are minimal in the set of normal supplements to M in G and K_1 and K_2 are maximal G-subgroups of H_1 and H_2 respectively, then $|H_1/K_1| = |H_2/K_2|$. The proof is by induction on $|G|$. By the minimality of H_i, $K_i \leq M$, $i = 1, 2$, so if $K_1 \cap K_2 \neq \langle 1 \rangle$, the result follows by induction. Thus we may suppose that $K_1 \cap K_2 = \langle 1 \rangle$. We note that

$$H_1 \cap K_2 \lhd G \quad \text{and} \quad H_1 \cap K_2 \leq H_1 \cap M$$

so $H_1 \cap K_2 \leq K_1$. Thus $H_1 \cap K_2 \leq K_1 \cap K_2 = \langle 1 \rangle$. Similarly, $H_2 \cap K_1 = \langle 1 \rangle$. In $G/K_1 K_2$, $H_1 K_2/K_1 K_2$ is minimal in the set of normal supplements to $M/K_1 K_2$. Certainly $H_1 K_2/K_1 K_2$ is a supplement, so suppose $X/K_1 K_2$ is a normal supplement to $M/K_1 K_2$ with $H_1 K_2/K_1 K_2 \geq X/K_1 K_2$. Then

$$(X \cap H_1)M = (X \cap H_1)K_2 M = (XK_2 \cap H_1 K_2)M = (X \cap H_1 K_2)M = XM = G.$$

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However by the minimality of H_1, $X \geq H_1$, so $X = H_1K_2$. Similarly H_2K_1/K_1K_2 is a minimal normal supplement to M/K_1K_2 in G/K_1K_2. If $K_1K_2 \neq \langle 1\rangle$, the lemma follows by induction, hence we may assume $K_1K_2 = \langle 1\rangle$. So H_1 and H_2 are minimal normal subgroups of G. Let L denote the core of M in G. If $L = \langle 1\rangle$, then by Corollary 2 of [1, p. 120], $|H_1| = |H_2|$. If $L \neq \langle 1\rangle$, consider $\eta(G/L:M/L)$. We claim that $\eta(G/L:M/L) = |H_1L/L|$. To show this it suffices to show that H_1L/L is a minimal normal supplement to M/L in G/L. Suppose $X/L \leq H_1L/L$ with $X \lhd G$ and $M/L\ X/L = G/L$. Then

$$(X \cap H_1)M = (X \cap H_1)LM = (XL \cap H_1L)M = XM = G.$$

However, by the minimality of H_1, $X \cap H_1 = H_1$, so that $X = H_1L$. Similarly H_2L/L is a minimal normal supplement to M/L in G/L. By induction $|H_1L/L| = |H_2L/L|$. However, since H_1 and H_2 are minimal normal subgroups, $H_1 \cap L = H_2 \cap L = \langle 1\rangle$. So

$$|H_1| = |H_1/(H_1 \cap L)| = |H_1L/L| = |H_2L/L| = |H_2/(H_2 \cap L)| = |H_2|,$$

and the lemma is proved.

Lemma 2. *If $N \lhd G$ and $N \leq M$, then $\eta(G/N:M/N) = \eta(G:M)$.*

Proof. Let $(X/N)/(Y/N)$ be a chief factor of G/N, where X is minimal with respect to $X/N\ M/N = G/N$. Then by Lemma 1, $\eta(G/N:M/N) = |X/Y|$. Let $H \leq X$ be a minimal normal supplement to M in G. $HN \leq X$, $HN \lhd G$, and $(HN)M = G$, so by the minimality of X, $HN = X$. Since $Y \geq N$, $HY = X$. Let H/K be a chief factor of G with $H \cap Y \leq K$. Then $Y \leq KY < X$, and $KY \lhd G$ so $KY = Y$ and $K = H \cap Y$. This implies that $|H/K| = |X/Y|$. By Lemma 1, $\eta(G:M) = |H/K|$, and $\eta(G/N:M/N) = |X/Y|$.

For notational purposes, let n_p denote the p-part of n. More precisely if p is a prime and $n = p^{\alpha}m$ with $(p, m) = 1$, then $n_p = p^{\alpha}$. The motivation for Theorem 1 is the result (B) mentioned in the introduction.

Theorem 1. *The finite group G is p-solvable if and only if*

$$(\eta(G:M))_p = [G:M]_p$$

for each maximal subgroup M of G.

Proof. Deny and let G be a counterexample of minimal order. Then G must satisfy the following.

(1) G is neither a p-group nor a p'-group, where p' denotes the complement of p in the set of all primes.

(2) G is not simple.

If G is simple then for each maximal subgroup M of G $\eta(G:M) = |G|$. However if M contains a Sylow p-subgroup of G, $[G:M]_p = 1$.

(3) G has a unique minimal normal subgroup H, and G/H is p-solvable.

Note that p-solvability is preserved by direct products and is inherited by subgroups. By the minimality of G and Lemma 2, every proper homomorphic image of G is p-solvable. Thus if H and K are two minimal normal subgroups of G, G/H and G/K are p-solvable and so $G/(H \cap K)$ is p-solvable.

(4) $p \mid |H|$.

If $p \nmid |H|$, then H is p-solvable, but by (3) G/H is p-solvable, sothat G is p-solvable.

(5) The Frattini subgroup, $\phi(G)$, is trivial. This follows by (3) and the fact that $\phi(G)$ is nilpotent.

(6) $H \leq \phi_p(G)$, where $\phi_p(G)$ is the intersection of all maximal subgroups of G with index relatively prime to p.

If L does not contain H then $\eta(G:L) = |H|$, so that $(\eta(G:L))_p = |H|_p$, however by hypothesis $\eta(G:L)_p = [G:L]_p$.

By Theorem 2 of [5], $\phi_p(G)$ is solvable. Since G/H is p-solvable G is p-solvable, this contradiction shows that G does not exist.

The converse follows easily. Suppose G is p-solvable and M is a maximal subgroup of G. Let $L = \text{core}\ (M)$. G/L is p-solvable so if $L \neq \langle 1 \rangle$, by induction,

$$(\eta(G/L:M/L))_p = [G/L:M/L]_p\,.$$

By Lemma 2, $(\eta(G/L:M/L))_p = \eta(G:M)_p$. If $L = \langle 1 \rangle$, then $\eta(G:M)_p = |H|_p$ where H is a minimal normal subgroup of G. (Note that G is not simple.) Since H is a minimal normal subgroup of a p-solvable group H is a p-group or a p'-group. If H is a p' group, then $[G:M]_p = |H|_p = 1$. If H is a p-group then H is abelian and $H \cap M \lhd G$. However, M is corefree so $H \cap M = 1$ and $[G:M] = |H|$.

COROLLARY. *The finite group G is solvable if and only if $\eta(G:M) = [G:M]$ for each maximal subgroup M of G.*

Proof. If $\eta(G:M) = [G:M]$ for each M, then in particular $(\eta(G:M))_p = [G:M]_p$ for each p. Thus G is p-solvable for each prime p, hence G is solvable. The converse is obvious.

Since $\eta(G:M)$ is the order of a chief factor of G, if G is simple then $\eta(G:M) = |G|$ for each maximal subgroup M of G. Thus if we force subgroups of equal normal index to be related in some way the structure of G is restricted somewhat as is indicated by Theorem 2.

THEOREM 2. *If all nonnormal maximal subgroups of equal normal index are conjugate in G, then G is solvable.*

Proof. Suppose that the theorem is false and let G be a counter-example of minimal order. Then G must satisfy the following.

(1) G is not simple.

If G is simple then all maximal subgroups in G are conjugate. By Lemma 2 of [3] G is cyclic. This contradiction implies that G is not simple.

(2) G has a unique minimal normal subgroup H, and furthermore G/H is solvable.

By (1), G is not simple so let H be a minimal normal subgroup of G. Then G/H inherits the conjugacy property, so that by the minimality of G, G/H is solvable. If there were two distinct minimal normal subgroups, then G would be solvable.

(3) Any two maximal subgroups which do not contain H are conjugate.

Let M_1 and M_2 be two maximal subgroups not containing H. Then by (2) M_1 and M_2 are selfnormalizing. Moreover, since H is the unique minimal normal subgroup of G, $\eta(G:M_1) = \eta(G:M_2) = |H|$. By hypothesis M_1 and M_2 are conjugate.

(4) $\phi(G) = 1$.

If not, then $H \leq \phi(G)$ so that $G/\phi(G)$ is solvable. But then G is solvable.

(5) Let M be a maximal subgroup which does not contain H, and let q be a prime divisor of $[G:M]$. Then $H \leq \phi_q(G)$.

Let L be a maximal subgroup of G with $([G:L], q) = 1$. Then L is not conjugate to M, so by (3), $L \supseteq H$.

By Theorem 2 of [5] H is solvable. Then G/H and H are solvable, which is a contradiction showing that G does not exist.

We now localize our conditions on index and normal index to one maximal subgroup of G. We obtain some results under the assumption that G possesses a solvable maximal subgroup.

THEOREM 3. *If G has a solvable maximal subgroup M with prime power normal index, then G is solvable.*

Proof. Assume that the theorem is false, and let G be a minimal counterexample. Let M be a solvable maximal subgroup of G with $\eta(G:M) = p^{\alpha}$, where p is a prime. Since $\eta(G:M) = p^{\alpha}$, G is not a simple group. Let N be a minimal normal subgroup of G. We consider two cases.

Case 1. $N \subseteq M$. Then $\eta(G/N:M/N) = \eta(G:M) = p^{\alpha}$ by Lemma 2. Since M is solvable, M/N and N are solvable. By the minimality of G, G/N is solvable. Thus G is solvable. This is a contradiction.

Case 2. $N \nsubseteq M$. Then $G = MN$ and $G/N \cong M/N \cap M$ so that G/N is solvable. Since $\eta(G:M) = |N|$ it follows that N is a p-group. Thus G is solvable. This contradiction shows that G does not exist, hence the theorem follows.

We now present the theorem mentioned in the introduction of the present paper.

THEOREM 4. *If G has a solvable maximal subgroup M such that $\eta(G:M) = [G:M]$, then G is solvable.*

Proof. Deny and let G be a counterexample of minimal order. Then G must satisfy the following.

(1) M is corefree.

If not, let $L = \text{core}(M)$. By Lemma 2, $\eta(G:M) = \eta(G/L:M/L)$. By the minimality of G, G/L is solvable. However L is solvable, and so G is solvable which is a contradiction.

(2) G is not simple.

If G is simple, $\eta(G:M) = |G|$ which implies that $M = \langle 1 \rangle$. But then G is cyclic contrary to the fact G is not solvable.

(3) Let K be a minimal normal subgroup of G. Then $G = MK$, $M \cap K = \langle 1 \rangle$.

By (1), $K \nsubseteq M$ so $\eta(G:M) = |K|$. Then $[G:M] = |K|$ so that $G = MK$ and $M \cap K = \langle 1 \rangle$.

Now let L be a minimal normal subgroup of M. Let

$$K_1 = C_K(L) = \{k \in K \mid l^{-1}kl = k \text{ for all } l \in L\}.$$

Obviously K_1 is a subgroup of K.

(4) $K_1 = \langle 1 \rangle$.

First note that K_1 is M-invariant. For let $g \in M$, $k \in K_1$ and $l \in L$. Then $lgkg^{-1}l^{-1} = gl_1kl_1^{-1}g^{-1}$, for some $l_1 \in L$. This follows by the normality of L in M. So $gkg^{-1}l^{-1} = gl_1kl_1^{-1}g^{-1} = gkg^{-1}$. That is, $gkg^{-1} \in K_1$, so that K_1 is M-invariant. However, since M is maximal in G, the only M-invariant subgroup of K are K and $\langle 1 \rangle$. If $K_1 = K$, we have $L \lhd G$ contrary to (1), thus $K_1 = \langle 1 \rangle$.

(5) $(|L|, |K|) = 1$.

If not, let $|L| = p^{\alpha}$ and let P be a Sylow p-subgroup of LK containing L. Then $P \cap K$ is a nontrivial normal subgroup of P so that $Z(P) \cap K \neq \langle 1 \rangle$. But, by (4), $Z(P) \cap K \subseteq C_K(L) = \langle 1 \rangle$. Therefore $P \cap K = 1$ and (5) follows.

(6) For each prime q dividing $|K|$, L leaves precisely one Sylow q-subgroup of K invariant.

This follows by Theorem 2.2 of [6, p. 224] and the fact that $C_K(L) = \langle 1 \rangle$.

(7) M leaves a Sylow subgroup of K invariant.

Let Q be an L-invariant Sylow subgroup of K. Let $g \in M$, $l \in L$. As in (4), $l^{-1}g^{-1}Qgl = g^{-1}l_1^{-1}Ql_1g = g^{-1}Qg$. So $g^{-1}Qg$ is an L-invariant Sylow subgroup of K. By (6) $g^{-1}Qg = Q$. Thus Q is an M-invariant Sylow subgroup of K.

Now K has no proper M-invariant subgroups, so $Q = K$ and so K is a solvable. Thus G/K and K are solvable so that G is solvable which is a contradiction, showing that G does not exist.

Considering S_4, the symmetric group on 4 symbols, we see that Theorem 4 cannot be substantially improved by replacing the solvability of M by nilpotence.

An attempt to localize Theorem 1 fails, as can be seen in the following example: Let $G = A_5 \times Z_5$, where A_5 is the simple group of order 60 and Z_5 is of order 5. Let $M = A_4 \times Z_5$. Then M is 5-solvable, indeed, M is 5-closed and 5-nilpotent. $[\eta(G:M))]_5 = [G:M]_5 = 5$, but G is not 5-solvable. We do obtain a result in this direction.

Recall that the group is p-closed if it has a normal Sylow p-subgroup.

THEOREM 5. *Suppose that G has a corefree maximal subgroup M such that M is p-closed, p a prime which divided $|M|$. Further, suppose that*

$$(\eta(G:M))_p = [G:M]_p .$$

Then G is p-solvable and the p-length of G is 1.

Proof. Assume that the theorem is false and let G be a counter-example of minimal order. As in the proof of Theorem 1, G is not simple. Let P be a p-Sylow subgroup of M. Then P is a normal subgroup of M, and since M is corefree it follows that P is a p-Sylow subgroup of G. Let K be a minimal normal subgroup of G. Then $G = MK$ and $\eta(G:M) = |K|$. Since $[G:M]_p = 1$, it follows that K is a p'-group. We also note that $G/K \cong M/K \cap M$ so that G/K is p-closed. This shows that G is p-solvable and $l_p(G) = 1$. Since G can not exist, the theorem follows.

The finite group G is supersolvable if and only if $\eta(G:M) = [G:M] = p$, p a prime, for each maximal subgroup M of G. This fact follows from results (A) and (B) of Deskins [4] mentioned earlier in the present note and by Theorems 7.2.8 and 9.3.8 of [7]. Hence, we can use the concept of normal index to characterize supersolvable groups.

Recall that a proper normal subgroup H of G is called a generalized Frattini subgroup of G if $G = N_G(P)$ for each normal subgroup L of G and each Sylow p-subgroup P of L such that $G = HN_G(P)$ (see [2]). Now let G be a supersolvable group. Then the Fitting subgroup $F(G)$ of G is not a generalized Frattini subgroup of G (see [2]) because of Corollary 3.6.1 of [2], hence $\phi(G)$ is properly contained in $F(G)$ by Corollary 3.1.1 of [2]. Therefore, there exists a maximal subgroup M of G such that $F(G) \leq M$. We note that M is supersolvable and $\eta(G:M) = [G:M] = p$, p is a prime. We now show that the converse to the above facts about supersolvable groups is also true.

THEOREM 6. *If G contains a supersolvable maximal subgroup M such that $\eta(G:M)$ is a prime and the Fitting subgroup, $F(G)$, is not contained in M, then G is supersolvable.*

Proof. Because of Theorem 3, G is solvable. Assume that $\phi(G) \neq 1$. Then $M/\phi(G)$ is a supersolvable maximal subgroup of $G/\phi(G)$ and

$$\eta(G/\phi(G):M/\phi(G)) = \eta(G:M)$$

by Lemma 2. By Theorem 7.4.8 of [7] it follows that

$$F(G/\phi(G)) = F(G)/\phi(G) \leq M/\phi(G).$$

By induction, $G/\phi(G)$ is supersolvable, hence G is supersolvable by Theorem 9.3.8 of [7]. Thus, we can assume that $\phi(G) = 1$. By Theorem 7.4.15 of [7], $F(G)$ is a direct product of all minimal normal subgroups of G. Since $F(G) \leq M$, there exists a minimal normal subgroup K of G not contained in M. Therefore, $G = MK$ and $M \cap K = 1$. From this it follows that $\eta(G{:}M) = |K|$, and the order of K is a prime. Since G/K is supersolvable, G is supersolvable.

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