POINTWISE SUPREMA OF ORDER-PRESERVING PERMUTATIONS

BY

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1. Introduction

The group $A(\Omega)$ of all order-preserving permutations of a chain Ω becomes a lattice-ordered group (l-group) when ordered pointwise, i.e., $f \leq g$ if and only if $\beta f \leq \beta g$ for all $\beta \in \Omega$. The supremum $f \vee g$ is obtained pointwise: $\beta(f \vee g) = \beta f \vee \beta g$ for all $\beta \in \Omega$. If an infinite collection $\{s_i \mid i \in I\}$ of elements of $A(\Omega)$ has a sup s, the sup is said to be *pointwise* if $\beta s = \sup \{\beta s_i \mid i \in I\}$ for all $\beta \in \Omega$. Questions involving complete distributivity led to the conjecture that sups in $A(\Omega)$ must always be pointwise. We shall find that this conjecture is correct under rather weak assumptions on Ω. In particular, the Holland representation [5] can be used to embed an arbitrary abstract l-group H in an $A(\Omega)$ in which sups are pointwise. Moreover, sups are "almost" pointwise in arbitrary $A(\Omega)$'s, and this leads to several results about complete distributivity. All of these results are based on a theorem by Lloyd [6], which states that for $\omega \in \Omega$, the stabilizer subgroup $\{g \in A(\Omega) \mid \omega g = \omega\}$ is closed under arbitrary sups.

2. G-static o-blocks

Let Ω be a chain. A permutation g of Ω is said to preserve order if $\alpha \leq \beta$ implies $\alpha g \leq \beta g$ for all $\alpha, \beta \in \Omega$. The group $A(\Omega)$ of all order-preserving permutations (o-permutations) of Ω, ordered pointwise, is an l-group. $A(\Omega)$ is not assumed to be transitive. For background information about l-groups, see [1].

In order to discuss pointwise suprema, we must first consider some preliminary concepts. We define Ω to be *static* if $A(\Omega) = \{1\}$, i.e., if there is no o-permutation of Ω other than the identity map. This concept was considered by Chang and Ehrenfeucht [3], who constructed a static chain Q' by enumerating the rationals Q by a function φ and then replacing each rational number $\varphi(n)$ by a chain of n elements. We note in passing that this process in fact allows one to embed any chain in a static chain by first embedding it in a chain Σ which is dense in itself (e.g., in an α-set) and then replacing each point in Σ by a different ordinal number.

Now let G be any subgroup of $A(\Omega)$. Under the order inherited from $A(\Omega)$, G is a partially ordered group. We shall call the pair (G, Ω) an *o-permutation group*. $\Delta \subseteq \Omega$ is convex if $\delta_1 \leq \omega \leq \delta_2$, with $\delta_1, \delta_2 \in \Delta$, implies $\omega \in \Delta$. By an *o-block* of G, we mean a segment (non-empty convex subset) Δ of Ω such that for any $g \in G$, either $\Delta G = \Delta$ or $\Delta G \cap \Delta = \square$. ($\square$ will denote the empty set.) An o-block is *trivial* if it contains only one point. An

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o-block Δ is *G-static* if $\Delta g = \Delta$ $(g \in G)$ implies that $\delta g = \delta$ for each $\delta \in \Delta$. Thus a segment Δ of Ω is a G-static o-block if $\Delta g \cap \Delta \neq \square$ $(g \in G)$ implies that $\delta g = \delta$ for each $\delta \in \Delta$. If $G = A(\Omega)$, then a G-static o-block is static under the earlier definition. Of course, if G is transitive, there are no non-trivial G-static o-blocks.

LEMMA 1. *Suppose that Δ is a G-static o-block and that $g \in G$. Then Δg is also a G-static o-block; and if Δh $(h \in G)$ meets Δg, then $\delta h = \delta g$ for each $\delta \in \Delta$.*

If Δ and J are chains, we shall write $(\Delta \times J)^{\leftarrow}$ for the lexicographic product ordered from the right, i.e., $(\delta_1, j_1) \leq (\delta_2, j_2)$ if and only if $j_1 < j_2$ or $j_1 = j_2$ and $\delta_1 \leq \delta_2$. From now on, J will denote the integers. $\bar{\Omega}$ will denote the completion of Ω by Dedekind cuts (without end points unless these end points belong to Ω). We shall consider Ω to be a subchain of $\bar{\Omega}$. Each $g \in G$ can be extended to $\bar{\Omega}$ by defining $\bar{\omega} g$ $(\bar{\omega} \in \bar{\Omega})$ to be $\sup \{\beta g \mid \beta \in \Omega, \beta \leq \bar{\omega}\}$.

THEOREM 2. *Let G be any subgroup of $A(\Omega)$, and let $\alpha \in \Omega$. Then there are two mutually exclusive possibilities:*

(1) *α is contained in a unique maximal G-static o-block*

(2) *α is contained in a segment of Ω of the form $(\Gamma \times J)^{\leftarrow}$, where Γ is a maximal G-static o-block; and G permutes these "copies" of Γ transitively.*

Proof. By Zorn's lemma, there is at least one maximal G-static o-block containing α. Suppose there are at least two, Γ and Δ, with $\inf \Gamma < \inf \Delta$ (and thus $\sup \Gamma < \sup \Delta$). Let $\bar{\gamma} = \inf \Gamma \in \bar{\Omega}$. There must exist $g \in G$ such that $\bar{\gamma} g \in \bar{\Delta} \backslash \bar{\Gamma}$, for otherwise $\Gamma \cup \Delta$ would be a G-static o-block, violating the maximality of Γ. Now since Δ is G-static, $\bar{\gamma} h = \bar{\gamma} g$ for every $h \in G$ such that $\bar{\gamma} h \in \bar{\Delta}$. Hence $[\bar{\gamma}, \bar{\gamma} g) \subseteq \Omega$ is a G-static o-block; so that by the maximality of Γ, $\bar{\gamma} g = \sup \Gamma$. The rest of (2) follows. Finally, if (2) holds, then for any $\gamma \in \Gamma$, if we let γ' be the corresponding point (Lemma 1) in the copy of Γ lying immediately above Γ, then $[\gamma, \gamma')$ is also a maximal G-static o-block. Hence (1) and (2) are mutually exclusive. Both cases occur with $G = A(\Omega)$: case (1) with $\Omega = (\{0, 1\} \times Q)^{\leftarrow}$, and case (2) with $\Omega = (Q' \times J)^{\leftarrow}$.

Let $\tilde{\Omega}$ denote the collection of maximal G-static o-blocks of type (1), together with certain of those of type (2), selected as explained below in order to effect a partition of Ω. If Γ is of type (2), and if $\Lambda = \bigcup \{\Gamma g \mid g \in G\}$ is the subset of Ω covered by translates of Γ, then if another maximal G-static o-block Π (necessarily also of type (2)) meets Λ, we have $\bigcup \{\Pi g \mid g \in G\} = \Lambda$. From each such Λ, we select one Γ and its translates for inclusion in $\tilde{\Omega}$. For any non-empty subsets Γ and Δ of Ω, we define $\Gamma < \Delta$ if and only if $\gamma < \delta$ for all $\gamma \in \Gamma$, $\delta \in \Delta$. This definition totally orders $\tilde{\Omega}$. Each $g \in G$ induces an o-permutation $\tilde{g}$ of $\tilde{\Omega}$, defined by $\Gamma \tilde{g} = \Gamma g$. The resulting o-permutation group will be denoted by $(\tilde{G}, \tilde{\Omega})$. Note that $(\tilde{G}, \tilde{\Omega})$ is independent of the choices of o-blocks of type (2) used to form $\tilde{\Omega}$.

THEOREM 3. *Let G be any subgroup of $A(\Omega)$. The map $g \rightarrow \tilde{g}$ is an o-isomorphism from G onto $\tilde{G}$; and $(\tilde{G}, \tilde{\Omega})$ has no non-trivial $\tilde{G}$-static o-blocks.*

Proof. The fact that each $\Gamma \in \tilde{\Omega}$ is G-static implies that the map $g \to \tilde{g}$ is one-to-one, that g is positive if $\tilde{g}$ is positive, and that $(\tilde{G}, \tilde{\Omega})$ has no non-trivial $\tilde{G}$-static o-blocks. The rest is clear.

Unfortunately, $G = A(\Omega)$ does not imply that $\tilde{G} = A(\tilde{\Omega})$. For example, let Ω be Q with some one point replaced by a two element chain. However, we can get around this obstacle by broadening the class of groups under consideration in such a way that the theorems in the later sections remain true for this larger class.

The *convexification* Conv (Π) of $\Pi \subseteq \Omega$ is defined to be

$$\{\beta \in \Omega \mid \pi_1 \leq \beta \leq \pi_2 \text{ for some } \pi_1, \pi_2 \in \Pi\}.$$

For $g \in A(\Omega)$ and $\gamma \in \Omega$ such that $\gamma g \neq \gamma$, the *interval of support* of g which contains γ is Conv $\{\gamma g^n \mid n \text{ an integer}\}$. An interval of support Δ of g is either *positive* ($\delta g > \delta$ for all $\delta \in \Delta$) or *negative* ($\delta g < \delta$ for all $\delta \in \Delta$). The ensuing theorems hold not only for $A(\Omega)$, but also for those l-subgroups G of $A(\Omega)$ (i.e., subgroups which are also sublattices) which share with $A(\Omega)$ the following property: If Δ is an interval of support of $g \in G$, so that $\Delta g = \Delta$, then there exists $h \in G$ such that $\beta h = \beta g$ if $\beta \in \Delta$, but $\beta h = \beta$ if $\beta \notin \Delta$. Intuitively, h is obtained by *depressing* g outside Δ. Groups having this property will be called *depressible.* Depressibility was one part of a more elaborate definition (of *full* subgroups) used for entirely different purposes in [4].

PROPOSITION 4. *Convex l-subgroups of $A(\Omega)$ are depressible.*

Proof. Let G be a convex l-subgroup of $A(\Omega)$ and let Δ be a positive interval of support of some $g \in G$. (The proof for Δ negative is similar). Obtain h by depressing g outside Δ. Then $1 \leq h \leq g \vee 1 \in G$, so by the convexity of G, $h \in G$.

PROPOSITION 5. *If G is a depressible l-subgroup of $A(\Omega)$, then $\tilde{G}$ is a depressible l-subgroup of $A(\tilde{\Omega})$.*

Proof. Let $\tilde{\Delta} \subseteq \tilde{\Omega}$ be an interval of support of $\tilde{g} \in \tilde{G}$. Then each $\Gamma \in \tilde{\Delta}$ is a segment of Ω, and $\Delta = \bigcup\{\Gamma \mid \Gamma \in \tilde{\Delta}\} \subseteq \Omega$ is an interval of support of $g \in G$. Obtain h by depressing g outside Δ. Then $\tilde{h} \in \tilde{G}$ is what one obtains by depressing $\tilde{g}$ outside $\tilde{\Delta}$.

3. Pointwise suprema

If G is an l-subgroup of $A(\Omega)$, and if in G, $s = \sup\{s_i \mid i \in I\}$, with I an infinite index set, we shall say that the sup is *pointwise at* $\beta \in \Omega$ if $\beta s = \sup\{\beta s_i \mid i \in I\}$. We shall say that *sups are pointwise in* (G, Ω) if whenever $s = \sup\{s_i \mid i \in I\}$, the sup is pointwise at each $\beta \in \Omega$. If sups are pointwise in $A(\Omega)$, then one can determine whether a collection $\{s_i \mid i \in I\}$ has a sup in $A(\Omega)$ simply by checking whether the map $s^*:\Omega \to \bar{\Omega}$, defined by

$$\beta s^* = \sup\{\beta s_i \mid i \in I\},$$

is an o-permutation of Ω. (s^* always preserves order, but it need not map Ω into Ω, nor need it be one-to-one or onto.) It has been conjectured informally by Paul Conrad that for every chain Ω, sups are pointwise in $A(\Omega)$. This is very nearly correct. However, if $\Omega = (\{0, 1\} \times Q)^{\leftarrow}$ it is easy to express the identity map on Ω as $\sup \{s_i \mid i \in I\}$, while arranging that $\sup \{(1, 0)s_i \mid i \in I\} = (0, 0)$. Despite this example, we shall show that sups are "almost" pointwise in $A(\Omega)$.

Suppose that G is an l-subgroup of $A(\Omega)$. Then for $\omega \in \Omega$, the stabilizer subgroup $G_\omega = \{g \in G \mid \omega g = \omega\}$ is an l-subgroup of G. Lloyd proved, in [6, Theorem 2] that if $G = A(\Omega)$, the stabilizer subgroups G_ω are *closed*, i.e., if $s = \sup \{s_i \mid i \in I\}$, with $s \in G$ and each $s_i \in G_\omega$, then $s \in G_\omega$ (i.e., the sup is pointwise at ω). Lloyd's proof can be applied without change to show that the stabilizer subgroups G_ω are closed for every depressible l-subgroup G of $A(\Omega)$.

LEMMA 6. *Let G be an l-subgroup of $A(\Omega)$ and let $\beta \in \Omega$ such that G_β is closed. Suppose that $s = \sup \{s_i \mid i \in I\}$, and let $\bar{\beta} = \sup \{\beta s_i \mid i \in I\}$. Then there is no $g \in G$ such that $\bar{\beta} \leq \beta g < \beta s$.*

Proof. Suppose that there is such a g. We may assume with no loss of generality that $g < s$. Let $h_i = (s_i \vee g)g^{-1}$. Then

$$sg^{-1} = \sup \{h_i \mid i \in I\}.$$

But $h_i \in G_\beta$ for all $i \in I$, while $sg^{-1} \notin G_\beta$, contradicting the assumption that G_β is closed.

An l-subgroup H of an l-group L is *complete* if whenever $h \in H$ is the sup in H of a collection $\{h_i \mid i \in I\}$ of elements of H, then h is also the *sup* in L of $\{h_i \mid i \in I\}$.

THEOREM 7. *Suppose that G is a transitive l-subgroup of $A(\Omega)$. Then the following are equivalent:*

(1) *Sups in G are pointwise.*

(2) *One (and hence every) stabilizer subgroup G_ω is closed. (This condition holds if G is depressible).*

(3) *G is complete in $A(\Omega)$.*

Proof. (2) implies (1) by the lemma. Now suppose (1) holds. If in G, $s = \sup \{s_i \mid i \in I\}$, then since the sup is pointwise, $t \geq s_i$ for all $i \in I$ implies $t \geq s$ even for $t \in A(\Omega)$, so $s = \sup \{s_i \mid i \in I\}$ in $A(\Omega)$. Thus (1) implies (3). Finally, if (3) holds, then since the stabilizer subgroups of $A(\Omega)$ are closed, those of G are also closed.

MAIN THEOREM 8. *Suppose that G is a depressible l-subgroup of $A(\Omega)$ and that in G we have $s = \sup \{s_i \mid i \in I\}$, but that the sup is not pointwise at $\beta \in \Omega$. Let*

$$\bar{\beta} = \sup \{\beta s_i \mid i \in I\} \in \bar{\Omega} \quad and \quad \Gamma = \{\gamma \in \Omega \mid \sup \{\gamma s_i\} = \bar{\beta}\}.$$

Then Γ *is a non-trivial maximal* G*-static o-block of type* (1). *Moreover, in* $\tilde{\Omega}$, $\Gamma\tilde{s}_i < \Gamma\tilde{s}$ *for all* $i \in I$, *and* $\Gamma\tilde{s} = \sup\{\Gamma\tilde{s}_i \mid i \in I\}$.

Proof. $\beta s_i < \bar{\beta}$ for all $i \in I$, for $\beta s_i = \bar{\beta}$ would contradict Lemma 6, since G_β is closed by depressibility. Since Γ has $\bar{\beta}s^{-1}$ as a lower bound, we may let $\bar{\gamma} = \inf \Gamma$. Pick any $\gamma \in \Gamma$ such that $\gamma > \bar{\gamma}$ (unless $\Gamma = \{\bar{\gamma}\}$, in which case we pick $\gamma = \bar{\gamma}$). If $\Gamma = \{\bar{\gamma}\}$, then $\sup\{\gamma s_i\} < \gamma s$ since the sup is not pointwise at $\gamma = \beta$. If $\gamma > \bar{\gamma}$, then again $\sup\{\gamma s_i\} = \bar{\beta} \leq \bar{\gamma}s < \gamma s$. Let $\Delta = \Delta\gamma = [\bar{\beta}, \gamma s)$. By Lemma 6, there is no $g \in G$ such that $\gamma g \in \Delta$. Hence if Δf meets Δ, $f \in G$, then $\Delta f \subseteq \Delta$, for otherwise either $\gamma s f^{-1} \in \Delta$ or $\gamma s_i f^{-1} \in \Delta$ for some $i \in I$. It follows that Δ is an o-block of G. Moreover, Δ is G-static. For if $\Delta h = \Delta$, but $\delta h \neq \delta$ for some $\delta \in \Delta$, then we would obtain $k \in G$ (which may be assumed positive) by depressing h outside Δ, and note that $s_i \leq sk^{-1}$ for all $i \in I$ (since whenever $\eta sk^{-1} \neq \eta s$, then $\eta s < \gamma s$, so $\eta s_i \leq \bar{\beta} \leq \eta sk^{-1}$). This would contradict the assumption that $s = \sup\{s_i\}$.

As noted above, $\bar{\gamma}s \geq \bar{\beta}$. Suppose by way of contradiction that $\bar{\gamma}s > \bar{\beta}$. Pick $\eta \in \Omega$ such that $\bar{\beta}s^{-1} \leq \eta < \bar{\gamma}$. Since $\eta < \bar{\gamma}$, $\sup\{\eta s_i\} < \bar{\beta}$; and since $\bar{\beta}s^{-1} \leq \eta$, $\bar{\beta} \leq \eta s$. By the first part of the proof, $\Pi = [\sup\{\eta s_i\}, \eta s)$ is a G-static o-block. Since $\sup\{\beta s_i\} = \bar{\beta}$ and since as noted above, $\beta s_i < \bar{\beta}$ for all i, β can be moved by various s_i's to more than one point in Π, violating Lemma 1. Therefore $\bar{\gamma}s = \bar{\beta}$. It follows that Γ is non-trivial, for if $\Gamma = \{\bar{\gamma}\}$, then as noted above, $\bar{\beta} = \sup\{\bar{\gamma}s_i\} < \bar{\gamma}s$.

Since for $\bar{\gamma} < \gamma \in \Gamma$, $\Delta_\gamma = [\bar{\beta}, \gamma s)$ is a G-static o-block, $\Delta\gamma\, s^{-1} = [\bar{\gamma}, \gamma)$ is also a G-static o-block. Hence Γ is the union of a tower of G-static o-blocks and thus is itself such an o-block. Moreover, as noted in the previous paragraph, no G-static o-block Π can contain points both above and below $\bar{\beta}$, so no G-static o-block Σ containing Γ can extend below $\bar{\gamma} = \bar{\beta}s^{-1}$; nor can Σ extend above $\sup \Gamma$, for then by the definition of Γ, there would be an s_i such that Σs_i meets Σs without equality obtaining, violating Lemma 1. Therefore Γ is a maximal G-static o-block. Γs is of type (1), so Γ is of type (1). Since $\beta \in \Gamma$ and $\beta s_i < \bar{\beta}$ for all $i \in I$, $\Gamma\tilde{s}_i < \Gamma s$. Since $\bar{\gamma}s = \bar{\beta} = \sup\{\beta s_i\}$, $\Gamma\tilde{s} = \sup\{\Gamma\tilde{s}_i\}$. This concludes the proof.

COROLLARY 9. *If a depressible* l*-subgroup* G *of* $A(\Omega)$ *has no non-trivial* G*-static o-blocks, then sups are pointwise in* G.

If $\varphi: G \to H$ is an l-isomorphism of an l-group G into an l-group H, then φ is *complete* if whenever $g = \sup\{g_i \mid i \in I\}$ in G, $g\varphi = \sup\{g_i\varphi \mid i \in I\}$ in H; or, equivalently, if the image $G\varphi$ is a complete subgroup of H.

COROLLARY 10. *Suppose* G *is a depressible* l*-subgroup of* $A(\Omega)$. *Then* G *is completely* l*-isomorphic to the* (*complete*) *depressible* l*-subgroup* $\tilde{G}$ *of* $A(\tilde{\Omega})$; *and sups are pointwise in* $\tilde{G}$.

Proof. By Theorem 3, the map $g \to \tilde{g}$ provides an l-isomorphism from G

onto $\tilde{G}$. By Proposition 5, $\tilde{G}$ is a depressible l-subgroup of $A(\tilde{\Omega})$. By Theorem 3 and Corollary 9, sups are pointwise in $\tilde{G}$, whence it follows that $\tilde{G}$ is a complete l-subgroup of $A(\tilde{\Omega})$.

COROLLARY 11. *Suppose that G is a depressible l-subgroup of $A(\Omega)$. Then a collection $\{s_i \mid i \in I\}$ of elements of G has a sup in G if and only if the function s^*, defined by*

$$\beta s^* = \sup\{\beta s_i \mid i \in I\},$$

is obtained by modifying as indicated below some $s \in G$ on a (possibly empty) collection $\mathfrak{M}$ of non-trivial maximal G-static o-blocks:

$$\beta s^* = \begin{cases} \inf \Gamma & \text{if} \quad \beta \in \Gamma \in \mathfrak{M}, \\ \beta s & \text{otherwise.} \end{cases}$$

Moreover, if this is the case, $s = \sup\{s_i \mid i \in I\}$. Thus if $\{s_i \mid i \in I\}$ has a sup in G which is not s, s^* cannot be one-to-one.

Proof. If $\{s_i \mid i \in I\}$ has a sup s, then the theorem guarantees that s^* is related to s in the prescribed way. Conversely, if s^* is related in this way to some $s \in G$, then certainly $s_i \leq s$ for all $i \in I$; and since each $\Gamma \in \mathfrak{M}$ is a G-static o-block, Lemma 1 guarantees that $s = \sup\{s_i\}$.

PROPOSITION 12. *Let G be an l-subgroup of $A(\Omega)$. Then sups are pointwise in G if and only if infs are pointwise in G.*

Proof. Suppose that sups are pointwise in G and that

$$s = \inf\{s_i \mid i \in I\}.$$

Suppose that $\beta s < \inf\{\beta s_i\}$ for some $\beta \in \Omega$, and pick $\gamma \in \Omega$ such that $\beta s < \gamma \leq \inf\{\beta s_i\}$. Then $\beta = (\beta s)s^{-1} < \gamma s^{-1} = \sup\{\gamma s_i^{-1}\}$ since $s^{-1} = \sup\{s_i^{-1}\}$ and sups are pointwise. Hence $\beta < \gamma s_i^{-1}$ for some i, and then $\beta s_i < \gamma \leq \inf\{\beta s_i\}$, a contradiction. Therefore infs are also pointwise in G. The proof of the converse is similar.

4. Complete distributivity

An l-group G is *completely distributive* if

$$\bigwedge_{i \in I} \bigvee_{k \in K} g_{ik} = \bigvee_{f \in K^I} \bigwedge_{i \in I} g_{if(i)}$$

for any collection $\{g_{ik} \mid i \in I,\ k \in K\}$ of elements of G for which the indicated sups and infs exist. If sups were pointwise in $A(\Omega)$, then complete distributivity in $A(\Omega)$ could be checked pointwise; and since totally ordered sets are completely distributive, so would $A(\Omega)$ be completely distributive. This was one of the reasons for wondering whether in $A(\Omega)$ sups were indeed pointwise. Although Lloyd [6, Theorem 1] has recently shown that $A(\Omega)$ is completely distributive, and although his proof also works for depressible l-subgroups of $A(\Omega)$, it is interesting to note that sups are close enough to being pointwise in $A(\Omega)$ that the original plan works.

THEOREM 13 (Lloyd). *Depressible l-subgroups of $A(\Omega)$ are completely distributive.*

Proof. If G is a depressible l-subgroup of $A(\Omega)$, then by Corollary 10, G is l-isomorphic to $\tilde{G}$. Sups are pointwise in $\tilde{G}$, so $\tilde{G}$ and hence G are completely distributive.

Of course, this proof depends upon Lloyd's proof that stabilizer subgroups are closed, and that proof is no easier than (and in fact is very similar to) his proof of complete distributivity. The following theorem is new.

THEOREM 14. *An l-group H is completely distributive if and only if it is completely l-isomorphic to a (complete) l-subgroup G of some $A(\Omega)$ such that sups are pointwise in G.*

Proof. Holland [5, Theorem 1] showed that every l-group H is l-isomorphic to an l-subgroup G of some $A(\Omega)$ such that Ω is the union of segments on each of which G (and hence $A(\Omega)$) is transitive. Hence $A(\Omega)$ has no non-trivial static o-blocks, and thus by Corollary 9, sups are pointwise in $A(\Omega)$. It is shown in [2, Theorem 3.10] that if H is completely distributive, it can be further arranged that the l-isomorphism is complete; and since then G is a complete l-subgroup of $A(\Omega)$ and sups are pointwise in $A(\Omega)$, sups are also pointwise in G. The converse is clear.

Finally, the characterization of complete distributivity in [2, Corollary 3.8], which states that an l-group is completely distributive if and only if its closed prime subgroups have trivial intersection, provides yet another proof that $A(\Omega)$ (and certain of its subgroups) are completely distributive.

COROLLARY 15. *Suppose that G is a (not necessarily depressible) l-subgroup of $A(\Omega)$ and that the stabilizer subgroups of G are closed. Then G is completely distributive.*

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