# BORDISM OF INVOLUTIONS ON MANIFOLDS

BY

H. L. ROSENZWEIG[1]

## I. Introduction and notation

In [7], Conner and Floyd computed the bordism groups of all involutions on closed manifolds. The purpose of this paper is to examine the bordism groups $\mathcal{O}_n(Z_2)$ of all orientation preserving involutions on closed oriented manifolds.

In section II we give a relation between certain bordism groups of an involution defined by Atiyah in [2] and the bordism groups of a space. In III we first examine the forgetful homomorphism $s : \Omega_n(Z_2) \to \mathcal{O}_n(Z_2)$, where $\Omega_n(Z_2)$ is the bordism group of fixed point free orientation preserving involutions. It is shown that the kernel of $s$ is exactly all of the torsion of $\Omega_n(Z_2)$. This result and that of section II enables us to show that all torsion of $\mathcal{O}_n(Z_2)$ has order 2 and that a free part occurs only in dimension $n = 4k$. From the computation of the kernel of $s$, it also follows that if $M^n$ bords orientably and $T$ is a fixed point free orientation preserving involution on $M^n$, then $M^n$ bounds some orientable $B^{n+1}$ to which $T$ can be extended, though $(T, B^{n+1})$ may not be fixed point free.

All manifolds will be smooth and compact. The bordism groups $\Omega_n$, $\mathfrak{N}_n$, $\Omega_n(X)$ and $\tilde{\Omega}_n(X)$ are defined in [7]. An element in $\mathcal{O}_n(Z_2)$ is represented by a pair $(T, M^n)$, where $M^n$ is a closed oriented $n$-manifold and $T$ is a smooth orientation preserving involution on $M^n$. Two such pairs $(T_1, M^n)$ and $(T_2, V^n)$ are bordant if there is an involution $T$ on a compact oriented $(n+1)$-manifold $B^{n+1}$ such that $\partial B^{n+1}$ is diffeomorphic to the disjoint union $M^n \cup -V^n$ and $T \mid \partial B^{n+1} = T_1 \cup T_2$. The bordism equivalence class of $(T, M^n)$ in $\mathcal{O}_n(Z_2)$ is denoted by $\{T, M^n\}$. The bordism group $\Omega_n(Z_2)$ differs from $\mathcal{O}_n(Z_2)$ only in that the involutions are required to be fixed point free. The bordism class of a fixed point free involution $(T, M^n)$ in $\Omega_n(Z_2)$ is denoted by $[T, M^n]$. An element $[T, M^n]$ in $\Omega_n(Z_2)$ is in the reduced group $\tilde{\Omega}_n(Z_2)$ if $[M^n/T] = 0$ in $\Omega_n$. Now suppose that $T$ is an involution on a space $X$. Consider triples $(M^n, \tau, f)$ where $\tau$ is a fixed point free orientation reversing involution on the closed oriented manifold $M^n$ and $f : (\tau, M^n) \to (T, X)$ is an equivariant map. Two such triples $(M^n, \tau_1, f_1)$ and $(V^n, \tau_2, f_2)$ are bordant if there is a triple $(B^{n+1}, \sigma, F)$ such that $\sigma$ is a fixed point free orientation reversing involution on $B^{n+1}$, $\partial B^{n+1}$ is the disjoint union $M^n \cup -V^n$, $F : (\sigma, B^{n+1}) \to (T, X)$ is equivariant, $\sigma \mid \partial B^{n+1} = \tau_1 \cup \tau_2$, and $F \mid \partial B^{n+1} = f_1 \cup f_2$. We denote the resulting bordism group by $\mathfrak{A}_n(T, X)$. These groups are essentially the groups $MSO_n(X, \alpha)$ defined in [2].

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[1] The results in this paper were contained in the author's doctoral dissertation at the University of Virginia.

If $\lambda \to X$ is a real vector bundle with group $O(k)$, the total space of the associated sphere, disk and projective space bundles will be denoted by $S(\lambda)$, $D(\lambda)$ and $\mathbf{R}P(\lambda)$ respectively.

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## II. An isomorphism on $\mathfrak{A}_k(T, X)$

Let $T$ be a fixed point free involution on a closed manifold $X$ or a fixed point free cellular involution on a finite CW-complex $X$. Let $\gamma \to X/T$ be the line bundle associated to the $Z_2$-bundle $X \to X/T$ and let $M(\gamma)$ be the Thom space of $\gamma$.

(2.1) THEOREM. *$\mathfrak{A}_k(T, X)$ is isomorphic to the reduced bordism group $\tilde{\Omega}_{k+1}(M(\gamma))$.*

Before proving (2.1) we mention two easily verified lemmas.

(2.2) LEMMA. *Let $M^{k+1}$ be an oriented $(k+1)$-manifold and let $K^k$ be a $k$-dimensional submanifold. Then we can identify the boundary of the normal tube to $K^k$ in $M^{k+1}$ with the orientation double covering of $K^k$.*

(2.3) LEMMA. *Let $M^n$ be a closed oriented manifold and let $T$ be a fixed point free orientation reversing involution on $M^n$. Then the covering $(T, M^n) \to M^n/T$ is the orientation double covering of $M^n/T$.*

Both of these lemmas follow from Lemma 2.2 in [3]. To obtain (2.3), consider the Gysin sequence of the line bundle associated to the $Z_2$-bundle $M^n \to M^n/T$.

*Proof of* (2.1). First consider the case when $X$ is a smooth compact $n$-manifold, $X = V^n$. To define $\varphi : \mathfrak{A}_k(T, V^n) \to \tilde{\Omega}_{k+1}(M(\gamma))$, consider an element $[M^k, \tau, f]$ in $\mathfrak{A}_k(T, V^n)$. If $\eta \to M^k/\tau$ is the line bundle associated to $M^k \to M^k/\tau$, then $f$ induces a map of Thom spaces, $F : M(\eta) \to M(\gamma)$. Define an involution $T_1$ on $M^k \times S^1$ by $T_1(x, z) = (\tau(x), \bar{z})$, where $\bar{z}$ denotes the complex conjugate of $z$. $T_1$ preserves orientation and $(M^k \times S^1)/T_1$ receives an orientation from the orientations of $M^k$ and $S^1$. Let

$$r : M^k \times S^1 \to (M^k \times S^1)/T_1$$

be the decomposition map. If $B$ is the subset of $M^k \times S^1$ consisting of all pairs $(x, a + bi)$ with $a > 0$, then collapsing $(M^k \times S^1)/T_1 - r(B)$ to a point in $(M^k \times S^1)/T_1$ yields the Thom space $M(\eta)$. This now defines a mapping

$$g : (M^k \times S^1)/T_1 \to M(\eta).$$

Letting $m : (M^k \times S^1)/T_1 \to M(\gamma)$ be the composition $m = F \cdot g$, we define

$$\varphi([M^k, \tau, f]) = [(M^k \times S^1)/T_1, m] \quad \text{in } \Omega_{k+1}(M(\gamma)).$$

$\varphi$ is a well defined homomorphism and since $T_1$ can be extended to a fixed point free involution on $M^k \times D^2$, the image of $\varphi$ lies in the reduced group $\tilde{\Omega}_{k+1}(M(\gamma))$.

To show $\varphi$ is an epimorphism, consider an element $[M^{k+1}, m]$ in $\tilde{\Omega}_{k+1}(M(\gamma))$. We may take the restricted map $m \mid (M^{k+1} - m^{-1}(\infty))$ to be smooth. Since $V^n/T$ is regularly embedded in $M(\gamma)$ as the zero section of $\gamma$, we may assume $m$ is transverse regular on $V^n/T$ [9, p. 22] and that $K^k = m^{-1}(V^n/T)$ is a non-empty regularly embedded $k$-dimensional submanifold of $M^{k+1}$. Let $(\tau, L^k) \to K^k$ be the orientation double covering of $K^k$. Since $m$ is transverse regular, the differential, $dm$, takes the normal bundle to $K^k$ in $M^{k+1}$ onto the normal bundle to $V^n/T$ in $M(\gamma)$, and we may assume $m$ is a bundle map of the normal tube to $K^k$ onto the normal tube to $V^n/T$. Thus we obtain an equivariant map $m_1 : (\tau, L^k) \to (T, V^n)$. Examine $\varphi([L^k, \tau, m_1])$. By (2.2) the line bundle associated to $(\tau, L^k) \to K^k$ is the normal tubular neighborhood of $K^k$ in $M^{k+1}$. The manifolds $L^k$ and $M^{k+1}$ are orientable, so we can identify the normal tube to $L^k$ in $M^{k+1}$ with $L^k \times [-1, 1]$. We choose this tube so that it does not intersect $K^k$ and so that $L^k \times \{-1\}$ always lies inside the normal tube to $K^k$. Define an involution $T_2$ on $L^k \times [-1, 1]$ by $T_2(x, t) = (\tau(x), t)$. Since $[M^{k+1}, m]$ is in $\tilde{\Omega}_{k+1}(M(\gamma))$, there is an oriented manifold $B^{k+2}$ with $\partial B^{k+2} = M^{k+1}$. In $B^{k+2}$ identify $q$ and $T_2(q)$ for all $q$ in $L^k \times [-1, 1]$ and let $U^{k+2}$ be the resulting manifold. $T_2$ reverses orientation, so $U^{k+2}$ is orientable. The boundary of $U^{k+2}$ is diffeomorphic to the disjoint union of $M^{k+1}$ and $(L^k \times S^1)/T_1$, where $T_1(x, z) = (\tau(x), \bar{z})$ as in the definition of $\varphi$. By taking the composition of $m$ followed by an appropriate deformation of $M(\gamma)$, $m$ is homotopic to a map, still denoted by $m$, which at each point $x$ of $K^k$ takes the fibre at $x$ of the normal tube $N$ of $K^k$ in $M^{k+1}$ "linearly" onto the fibre of $\gamma$ at $m(x)$ and which takes $M^{k+1} - N$ into the point at infinity. Similarly, if

$$\varphi([L^k, \tau, m_1]) = [(L^k \times S^1)/T_1, m_2],$$

then $(L^k \times S^1)/T_1 \to K^k$ is the $Z_2$-bundle with fibre $S^1$ associated to the $Z_2$-bundle $(\tau, L^k) \to K^k$ and above any point $x$ in $K^k$, $m_2$ takes half of the fibre $S^1$ "linearly" onto the fibre of $\gamma$ at $m(x)$ and takes the other half of the fibre into the point at infinity. Using the fact that a neighborhood of $M^{k+1}$ in $B^{k+2}$ has the form $M^{k+1} \times [0, 1)$, an examination of the formation of $U^{k+2}$ shows that the disjoint union map $m \cup m_2$ on

$$M^{k+1} \cup ((L^k \times S^1)/T_1) = \partial U^{k+2}$$

can be extended to all of $U^{k+2}$. Thus $\varphi([L^k, \tau, m_1]) = [M^{k+1}, m]$ and $\varphi$ is an epimorphism.

Now suppose $\varphi([M^k, \tau, f]) = 0$ in $\tilde{\Omega}_{k+1} M(\gamma)$, i.e., $((M^k \times S^1)/T_1, m)$ bounds some oriented pair $(B^{k+2}, \tilde{m})$. By the construction of $m$, it is transverse regular on $V^n/T$ and $m^{-1}(V^n/T) = M^k/\tau$, considering $M^k/\tau$ as the image of $M^k \times \{1\}$ under the decomposition

$$M^k \times S^1 \to (M^k \times S^1)/T_1.$$

Again, we may assume the restriction $\tilde{m} \mid (B^{k+2} - \tilde{m}^{-1}(\infty))$ is smooth and $\tilde{m}$ is transverse regular on $V^n/T$ without changing the values of $\tilde{m} = m$ on $M^k/\tau$.

Now let $K^{k+1} = \tilde{m}^{-1}(V^n/T)$ and let $(\sigma, L^{k+1}) \to K^{k+1}$ be the orientation double covering. Then $M^k/\tau = \partial K^{k+1}$, so by Lemma (2.3), $M^k = \partial L^{k+1}$ and $\sigma \mid M^k = \tau$. As before, there is an equivariant map $g : (\sigma, L^{k+1}) \to (T, V^n)$ with $g \mid M^k = f$. Thus $[M^k, \tau, f] = 0$ in $\mathfrak{A}_k(T, V^n)$ and $\varphi$ is a monomorphism.

I'd like to thank Robert Stong for showing me the following method of reducing the case when $X$ is a finite complex to the case where $X = V^n$ is a smooth manifold. Let $T$ be a fixed point free cellular involution on a finite complex $X$. Embed $X/T$ in some $\mathbf{R}^n$ and let $p : N \to X/T$ be a regular neighborhood of $X/T$. Then $N$ is a smooth manifold having the homotopy type of $X/T$. $p$ induces a principal $Z_2$-bundle $(T', X') \to N$ and $X'$ has the homotopy type of $X$ [8, Cor. 7.10]. Then we have a sequence of isomorphisms

$$\mathfrak{A}_k(T, X) \approx \mathfrak{A}_k(T', X') \approx \tilde{\Omega}_{k+1}(M(\gamma')) \approx \tilde{\Omega}_{k+1}(M(\gamma)).$$

This completes the proof of (2.1).

Now suppose that $T$ is any involution on a finite complex $X$. Set

$$\tau = A \times T : S^n \times X \to S^n \times X,$$

where $(A, S^n)$ denotes the antipodal map on the unit sphere in $\mathbf{R}^{n+1}$.

(2.4) THEOREM. *$\mathfrak{A}_k(\tau, S^N \times X)$ is isomorphic to $\mathfrak{A}_k(T, X)$ for $k < N$.*

*Proof.* Given $[M^k, \sigma, f]$ in $\mathfrak{A}_k(T, X)$, for $k < N$ there is an equivariant map $e : (\sigma, M^k) \to (A, S^N)$ and $e$ is unique up to equivariant homotopy. Define

$$\varphi([M^k, \sigma, f]) = [M^k, \sigma, e \times f].$$

If $[M^k, \sigma, g]$ is in $\mathfrak{A}_k(\tau, S^N \times X)$, express $g$ as $e \times f$ and define

$$\psi([M^k, \sigma, g]) = [M^k, \sigma, f].$$

It is clear that

$$\varphi : \mathfrak{A}_k(T, X) \to \mathfrak{A}_k(\tau, S^N \times X) \quad \text{and} \quad \psi : \mathfrak{A}_k(\tau, S^N \times X) \to \mathfrak{A}_k(T, X)$$

are well-defined inverse homomorphisms.

## III. The structure of $\mathcal{O}_*(Z_2)$

Let $s : \Omega_n(Z_2) \to \mathcal{O}_n(Z_2)$ be the homomorphism given by $s([T, M^n]) = \{T, M^n\}$.

(3.1) THEOREM. *The kernel of $s$ consists of all the torsion of $\Omega_n(Z_2)$.*

*Proof.* The sequence

$$0 \to \tilde{\Omega}_n(Z_2) \xrightarrow{\subset} \Omega_n(Z_2) \underset{\alpha}{\overset{\varepsilon}{\rightleftarrows}} \Omega_n \to 0$$

is a split short exact sequence, where

$$\varepsilon([T, M^n]) = [M^n/T] \quad \text{and} \quad \alpha([V^n]) = [A', V^n \times Z_2],$$

$A'$ being the map switching copies of $V^n$, $A'(x, i) = (x, 1 - i)$. Thus $\Omega_n(Z_2)$

is isomorphic to $\Omega_n \oplus \tilde{\Omega}_n(Z_2)$. Burdick [6, p. 51] showed that $\tilde{\Omega}_n(Z_2)$ is isomorphic to $\mathfrak{N}_{n-1}$ and thus consists entirely of 2-torsion. Now the torsion subgroup of $\Omega_n(Z_2)$ consists entirely of 2-torsion and can be written as

$$\text{Tor}\,(Z_2, \Omega_n) \oplus \tilde{\Omega}_n(Z_2),$$

where Tor $(Z_2, \Omega_n)$ is the 2-torsion of the group $\Omega_n$. To show that $s$ takes $\tilde{\Omega}_n(Z_2)$ into 0, we examine Burdick's isomorphism

$$\varphi : \mathfrak{N}_{n-1} \to \tilde{\Omega}_n(Z_2).$$

He defines $\varphi([V^{n-1}]_2) = [T, (E^{n-1} \times S^1)/T_1]$, where $(\tau, E^{n-1}) \to V^{n-1}$ is the orientation double covering, $T_1(x, z) = (\tau(x), \bar{z})$ and $T$ is induced on $(E^{n-1} \times S^1)/T_1$ by the involution $T(x, z) = (x, -z)$ on $E^{n-1} \times S^1$. By extending $T$ and $T_1$ to $E^{n-1} \times D^2$, we see that $(T, (E^{n-1} \times S^1)/T_1)$ bounds $(T, (E^{n-1} \times D^2)/T_1)$ so $s(\tilde{\Omega}_n(Z_2)) = 0$.

In [1], Anderson showed that every element of Tor $(Z_2, \Omega_n)$ may be represented as a sum of classes of manifolds of the form

$$V^n = \mathbf{R}P(\lambda \oplus \theta^{2k+1}(M)),$$

where $\lambda \to M$ is the line bundle with $w_1(\lambda) = w_1(M)$ and $\theta^{2k+1}(M) \to M$ is the trivial $(2k+1)$-bundle. On $S(\lambda \oplus \theta^{2k+1}(M))$ there is an orientation reversing involution $T = (-1) \oplus$ (identity). $T$ commutes with the bundle involution [7, p. 60] $A$ on $S(\lambda \oplus \theta^{2k+1}(M))$, so it induces an orientation reversing map, $T'$, on $V^n$, though $T'$ is not fixed point free. Now there is the fixed point free orientation reversing involution $\tilde{T}$ on $V^n \times Z_2$ given by $\tilde{T}(x, i) = (T'(x), 1 - i)$. $\tilde{T}$ commutes with $A'$, so $\{A', V^n \times Z_2\} = 0$ in $\mathcal{O}_n(Z_2)$.

Suppose $[T, M^n]$ is in the kernel of $s$. We have

$$[T, M^n] = [A', V^n \times Z_2] + [\tilde{T}, \tilde{M}^n]$$

where $[\tilde{T}, \tilde{M}^n]$ is in $\tilde{\Omega}_n(Z_2)$ and $[V^n]$ is in $\Omega_n$. Since

$$s([\tilde{T}, \tilde{M}^n]) = 0, \quad s([A', V^n \times Z_2]) = 0$$

and $V^n \times Z_2$ bounds some oriented $B^{n+1}$, so $2[V^n] = 0$. Thus

$$2[A', V^n \times Z_2] = \alpha(2[V^n]) = 0 \quad \text{and} \quad 2[T, M^n] = 0,$$

completing the proof of (3.1).

(3.2) Corollary. *If $[M^n] = 0$ in $\Omega_n$ and $T$ is a fixed point free orientation preserving involution on $M^n$, then $\{T, M^n\} = 0$ in $\mathcal{O}_n(Z_2)$.*

*Proof.* Under the isomorphism between $\Omega_n(Z_2)$ and $\Omega_n \oplus \tilde{\Omega}_n(Z_2)$, $[T, M^n]$ corresponds to $([M^n/T], [\tilde{T}, \tilde{M}^n])$ for some $[\tilde{T}, \tilde{M}^n]$ in $\tilde{\Omega}_n(Z_2)$. Since $2([M^n/T]) = [M^n] = 0$ in $\Omega_n$, then $s([A', Z_2 \times (M^n/T)]) = 0$ and thus $s([T, M^n]) = 0$, i.e., $\{T, M^n\} = 0$ in $\mathcal{O}_n(Z_2)$.

Now consider the orientation double covering

$$(\tau, BSO(n)) \to BO(n)$$

and let

$$\mathcal{A}_n = \sum_{k=0}^{[n/2]} \mathcal{A}_{n-2k}(\tau, BSO(2k)).$$

We define a homomorphism $\alpha : \mathcal{O}_n(Z_2) \to \mathcal{A}_n$ as follows. First define

$$\alpha(\{\text{identity}, M^n\}) = [\tilde{M}^n, \sigma, f]$$

where $(\sigma, \tilde{M}^n) \to M^n$ is the orientation double covering of $M^n$ and

$$f : (\sigma, \tilde{M}^n) \to (\tau, BSO(0))$$

is the obvious equivariant map, with $BSO(0) = \{\text{point}\} \times S^0$. Now look at an arbitrary $\{T, M^n\}$ in $\mathcal{O}_n(Z_2)$. If $F^m$ is the $m$-dimensional part of the fixed point set of $T$, then $n - m$ is even [5, p. 79] and $F^m$ is a regularly embedded submanifold of $M^n$. For $k > 0$, let $\eta_{2k} \to F^{n-2k}$ denote the normal bundle to $F^{n-2k}$ in $M^n$. The bundle $\eta_{2k}$ has a classifying map

$$\bar{f}_{2k} : F^{n-2k} \to BO(2k),$$

which induces a principal $Z_2$-bundle $(\tau_{2k}, V^{n-2k}) \to F^{n-2k}$ from the covering $(\tau, BSO(2k)) \to BO(2k)$ and hence an equivariant map

$$f_{2k} : (\tau_{2k}, V^{n-2k}) \to (\tau, BSO(2k)).$$

Since $\bar{f}_{2k}^*(w_1(BO(2k))) = w_1(\eta_{2k}) = w_1(F^{n-2k})$,

$$(\tau_{2k}, V^{n-2k}) \to F^{n-2k}$$

is the orientation double covering of $F^{n-2k}$ and $V^{n-2k}$ is canonically oriented. Define

$$\alpha(\{T, M^n\}) = \sum_{k=0}^{[n/2]} [V^{n-2k}, \tau_{2k}, f_{2k}],$$

where the case $k = 0$ is handled as in $\alpha(\{\text{identity}, M\})$.

Henceforth, $\xi \to BO(k)$ will denote the universal $k$-plane bundle and $A$ will denote the bundle involution on the indicated sphere or disk bundle. For $n \geq 1$, we define a homomorphism

$$\partial : \mathcal{A}_n \to \Omega_{n-1}(Z_2)$$

to be the sum of the homomorphisms

$$\partial : \mathcal{A}_{n-2k}(\tau, BSO(2k)) \to \Omega_{n-1}(Z_2)$$

given, for $k > 0$, by

$$\partial([V^{n-2k}, \tau_{2k}, f_{2k}]) = [A, S(\bar{f}_{2k}^* \xi)].$$

Here $A$ is the bundle involution on the sphere bundle associated to the induced bundle $\bar{f}_{2k}^* \xi$, where

$$\bar{f}_{2k} : V^{n-2k}/\tau_{2k} \to BO(2k)$$

is induced from the equivariant map

$$f_{2k} : (\tau_{2k}, V^{n-2k}) \to (\tau, BSO(2k)).$$

Because $w_1(\bar{f}_{2k}^* \xi) = w_1(F^{n-2k})$, $S(\bar{f}_{2k}^* \xi)$ and $D(\bar{f}_{2k}^* \xi)$ are orientable. Since

$(A, S(\bar{f}_{2k}^* \xi))$ bounds $(A, D(\bar{f}_{2k}^* \xi))$, $s([A, S(\bar{f}_{2k}^* \xi)]) = 0$ in $\mathcal{O}_{n-1}(Z_2)$. By (3.1), $2[A, S(\bar{f}_{2k}^* \xi)] = 0$ in $\Omega_{n-1}(Z_2)$, so it is not necessary to choose an orientation for $S(\bar{f}_{2k}^* \xi)$.

(3.3) THEOREM. *For $n \geq 0$, there is an exact sequence*

$$\cdots \longrightarrow \mathfrak{A}_{n+1} \xrightarrow{\partial} \Omega_n(Z_2) \xrightarrow{s} \mathcal{O}_n(Z_2) \xrightarrow{\alpha} \mathfrak{A}_n \xrightarrow{\partial} \cdots$$

*Proof.* It is clear that $\cdot\partial = \partial \cdot \alpha = \alpha \cdot s = 0$. Suppose that $s([T, M^n]) = 0$, i.e., $(T, M^n)$ bounds some $(T', B^{n+1})$. As usual, let $F^{n+1-2k}$ be the $(n+1-2k)$-dimensional part of the fixed point set of $T'$, $\eta_{2k} \rightarrow F^{n+1-2k}$ its normal bundle, and

$$\bar{f}_{2k} : F^{n+1-2k} \rightarrow BO(2k)$$

the classifying map. By removing the interiors of the normal tubular neighborhoods of the $F^{n+1-2k}$, we see that

$$[T, M^n] = \textstyle\sum_{k=0}^{[(n+1)/2]} [A, S(\eta_{2k})].$$

Each map $\bar{f}_{2k}$ induces an equivariant map

$$f_{2k} : (\tau_{2k}, V^{n+1-2k}) \rightarrow (\tau, BSO(2k)).$$

Then

$$\partial\left(\textstyle\sum_{k=0}^{[(n+1)/2]} [V^{n+1-2k}, \tau_{2k}, f_{2k}]\right) = \textstyle\sum_{k=0}^{[(n+1)/2]} [A, S(\eta_{2k})]$$

and hence kernel $(s)$ = image $(\partial)$.

If $\sum_{k=0}^{[n/2]} [V^{n-2k}, \tau_{2k}, f_{2k}]$ is in kernel $(\partial)$, then

$$\textstyle\bigcup_{k=0}^{[n/2)} (A, S(\bar{f}_{2k}^* \xi))$$

bounds some $(T, M^n)$ with $T$ fixed point free orientation preserving. Also, each $(A, S(\bar{f}_{2k}^* \xi))$ bounds $(A, D(\bar{f}_{2k}^* \xi))$, where the fixed point set of $A$ on the disk bundle is $V^{n-2k}/\tau_{2k}$. Let $B^n$ be the union of $M^n$ with the union $\bigcup_{k=0}^{[n/2]} D(\bar{f}_{2k}^* \xi)$ with their boundaries identified. There is an orientation preserving involution $T'$ on the closed manifold $B^n$ given by $T' = T$ on $M^n$ and $T' = A$ on the union of the $D(\bar{f}_{2k}^* \xi)$. The fixed point set of $T'$ is the union of the $V^{n-2k}/\tau_{2k}$ and the normal bundle to $V^{n-2k}/\tau_{2k}$ in $B^n$ is

$$\bar{f}_{2k}^* \xi \rightarrow V^{n-2k}/\tau_{2k}.$$

Thus

$$\alpha(\{T', B^n\}) = \textstyle\sum_{k=0}^{[n/2]} (V^{n-2k}, \tau_{2k}, f_{2k}].$$

Finally, suppose that $\alpha(\{T, M^n\}) = \sum_{k=0}^{[n/2]} (V^{n-2k}, \tau_{2k}, f_{2k}] = 0$ in $\mathfrak{A}_n$. Fix a Riemannian metric on $M^n$ for which $T$ is an isometry. For each $k$ for which $F^{n-2k}$ is non-empty there is a triple

$$(B^{n-2k+1}, \sigma_{2k}, g_{2k})$$

with boundary $(V^{n-2k}, \tau_{2k}, f_{2k})$, i.e., $\partial B^{n-2k+1} = V^{n-2k}$, $(\sigma_{2k}, B^{n-2k+1})$ is an orientation reversing involution,

$$g_{2k} : (\sigma_{2k}, B^{n-2k+1}) \rightarrow (\tau, BSO(2k))$$

is equivariant, and $\sigma_{2k}$ and $g_{2k}$ extend $\tau_{2k}$ and $f_{2k}$ respectively. We then have the map of quotient spaces

$$\bar{g}_{2k} : B^{n-2k+1}/\sigma_{2k} \longrightarrow BO(2k)$$

and the induced bundle

$$\bar{g}_{2k}^{*}\,\xi \longrightarrow B^{n-2k+1}/\sigma_{2k}\,.$$

If $\eta_{2k} \longrightarrow F^{n-2k}$ is the normal bundle to $F^{n-2k}$ in $M^n$, then

$$S(\bar{g}_{2k}^{*}\,\xi) \cup D(\eta_{2k}) = \partial(D(\bar{g}_{2k}^{*}\,\xi)) \quad \text{and} \quad S(\bar{g}_{2k}^{*}\,\xi) \cap D(\eta_{2k}) = S(\eta_{2k}).$$

Let $U^{n+1}$ be the union of $M^n \times [0, 1]$ with the union $\bigcup_{k=0}^{[n/2]} D(\bar{g}_{2k}^{*}\,\xi)$, identifying the two copies of $D(\eta_{2k})$ that lie in $M^n \times 1$ and in $D(\bar{g}_{2k}^{*}\,\xi)$. The boundary of $U^{n+1}$ is split into two parts: one consists of $M^n \times 0$, the other of $M^n \times 1$, but with $S(\bar{g}_{2k}^{*}\,\xi)$ replacing $D(\eta_{2k})$, the normal tubular neighborhood of the $(n - 2k)$-dimensional part of the fixed point set. Define an involution $T'$ on $U^{n+1}$ by $T'(x, t) = (T(x), t)$ on $M^n \times I$ and $T'$ is the bundle involution on each $D(\bar{g}_{2k}^{*}\,\xi)$. Since $T$ was an isometry $T'$ is well defined on $U^{n+1}$. $T' = T$ on $M^n = M^n \times 0$ and is fixed point free on the rest of the boundary of $U^{n+1}$. Thus $\{T, M^n\}$ is in the image of $s$ and the proof of (3.3) is completed.

(3.4) THEOREM. *All torsion of $\mathcal{O}_*(Z_2)$ has order* 2.

*Proof.* Let $\gamma_{2k} \longrightarrow BO(2\text{k})$ be the line bundle associated with the double covering $(\tau, BSO(2k)) \longrightarrow BO(2k)$. By (2.1),

$$\mathfrak{A}_{n-2k}(\tau, BSO(2k))$$

is isomorphic to

$$\tilde{\Omega}_{n-2k+1}(M(\gamma_{2k})),$$

which is in turn isomorphis to

$$\Omega_{n-2k+1}(D(\gamma_{2k}), S(\gamma_{2k})).$$

By Theorem (15.2) in [7] we know that if $(X, A)$ is a CW-pair such that each $H_m(X, A; Z)$ is finitely generated and has no odd torsion, then $\Omega_m(X, A)$ is isomorphic to $\sum_{p+q=m} H_p(X, A; \Omega_q)$. All homology and cohomology will now have coefficients in $Z$, the integers, unless indicated otherwise. The free parts of $H_m(D(\gamma_{2k}), S(\gamma_{2k}))$ and of $H^m(D(\gamma_{2k}), S(\gamma_{2k}))$ are isomorphic, as are the torsion subgroups of

$$H_m(D(\gamma_{2k}), S(\gamma_{2k})) \quad \text{and} \quad H^{m+1}(D(\gamma_{2k}), S(\gamma_{2k})).$$

Since $BO(2k)$ is a deformation retract of $D(\gamma_{2k})$ and $S(\gamma_{2k}) = BSO(2k)$, the exact cohomology triangle of the pair $(D(\gamma_{2k}), S(\gamma_{2k}))$ becomes

$$\begin{array}{ccc} H^*(BO(2k), BSO(2k)) & \xrightarrow{\;i^*\;} & H^*(BO(2k)) \\ {\scriptstyle\delta}\nwarrow & & \swarrow{\scriptstyle j^*} \\ & H^*(BSO(2k)). & \end{array}$$

By computations of the cohomology rings of $BO(k)$ and $BSO(k)$ given in [4] and [11], this exact triangle becomes

$$\begin{array}{ccc} H^*(BO(2k), BSO(2k)) & \xrightarrow{i^*} & Z[p_1, \cdots, p_k] + \text{2-torsion} \\ \nwarrow \delta & & \swarrow j^* \\ & Z[\tilde{p}_1, \cdots, \tilde{p}_{k-1}, X_{2k}] + \text{2-torsion} & \end{array}$$

where $p_m$ in $H^{4m}(BO(2k))$ is a universal Pontrjagin class, $\tilde{p}_m = j^*(p_m)$ for $1 \leq m \leq k-1$, $X_{2k}$ in $H^{2k}(BSO(2k))$ is the Euler class, $j^*(p_k) = X_{2k}^2$, and $j^*$ maps the 2-torsion of $H^*(BO(2k))$ onto the 2-torsion of $H^*(BSO(2k))$. Thus $H^*(BO(2k), BSO(2k))$ has no odd torsion or torsion of order 4 and hence neither do the $\mathfrak{A}_n$. The assertion then follows from Theorems (3.1) and (3.3).

We now consider the free part of $\mathcal{O}_*(Z_2)$. Since the image of

$$\partial : \mathfrak{A}_n \to \Omega_{n-1}(Z_2)$$

consists of torsion elements, for $Q$ the rationals,

$$\partial \otimes 1 : \mathfrak{A}_n \otimes Q \to \Omega_{n-1}(Z_2) \otimes Q$$

is the zero homomorphism. We then have a split short exact sequence

$$0 \to \Omega_n(Z_2) \otimes Q \xrightarrow{s \otimes 1} \mathcal{O}_n(Z_2) \otimes Q \xrightarrow{\alpha \otimes 1} \mathfrak{A}_n \otimes Q \to 0$$

and a sequence of isomorphisms

$$\mathcal{O}_n(Z_2) \otimes Q \approx (\Omega_n(Z_2) \otimes Q) \oplus (\mathfrak{A}_n \otimes Q) \approx (\Omega_n \otimes Q) \oplus (A_n \otimes Q).$$

Since $\Omega_* \otimes Q$ is known to be $Q[CP(2), CP(4), \cdots]$ [10], to determine the free part of $\mathcal{O}_n(X_2)$, we need only consider that of $\mathfrak{A}_n$. We have the isomorphism

$$(\#)\quad \mathfrak{A}_{n-2k}(\tau, BSO(2k)) \otimes Q \approx \textstyle\sum_{p+q=n-2k+1} H_p(BO(2k), BSO(2k); \Omega_q) \otimes Q.$$

The bordism group $\Omega_q$ is isomorphic to a sum $Z \oplus \cdots \oplus Z \oplus Z_2 \oplus \cdots \oplus Z_2$, so the free part of $\mathfrak{A}_n$ can be computed from the exact triangle

$$\begin{array}{ccc} H^*(BO(2k), BSO(2k); Q) & \xrightarrow{i^*} & Q[p_1, \cdots, p_k] \\ \nwarrow \delta & & \swarrow j^* \\ & Q[\tilde{p}_1, \cdots, \tilde{p}_{k-1}, X_{2k}]. & \end{array}$$

The free part of $\mathcal{O}_*(Z_2)$ is now easy to compute. In particular we have

(3.5) THEOREM. *If $n$ is not a multiple of 4, then $\mathcal{O}_n(Z_2) \otimes Q = 0$.*

*Proof.* We need only show the statement is true for $\mathfrak{A}_n \otimes Q$. First suppose $n$ is odd. A now-zero case can occur in (#) only if $q = 4m$, in which case

$p$ is even. Then $H^{p-1}(BSO(2k); Q) = 0$ and

$$j^* : H^p(BO(2k); Q) \to H^p(BSO(2k); Q)$$

is a monomorphism, so $H^p(BO(2k), BSO(2k); Q) = 0$. It then follows that $\mathcal{O}_n(Z_2) \otimes Q = 0$.

For $n$ even, in (♯) we must still have $q = 4m$, so now $p$ is odd. Thus $H^p(BO(2k); Q) = 0$ and we have the exact sequence

$$H^{p-1}(BO(2k); Q) \xrightarrow{j^*} H^{p-1}(BSO(2k); Q) \xrightarrow{\delta} H^p(BO(2k), BSO(2k); Q)) \to 0.$$

The homomorphism $j^*$ can fail to be onto only if $p - 1$ is an odd multiple of $2k$, the dimension of the Euler class, plus $4i$, so let

$$p - 1 = 4ak + 2k + 4i.$$

Then $n = p + q + 2k - 1 = 4(ak + m + k + i)$.

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Haverford College
Haverford, Pennsylvania
Western Maryland College
Westminister, Maryland