PROJECTIONS AND EXTENSION MAPS IN $C(T)$

BY
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1. Introduction

This paper is concerned principally with metric projections in $C(T)$ with special attention given to the subspace $R^{-1}0$ of functions that vanish on a closed set Q. The existence of a linear metric projection onto $R^{-1}0$ is shown to be equivalent to the existence of a bounded linear extension map of norm 1 from $C(Q)$ to $C(T)$ (Theorem 7). It is established that in a connected metric space $R^{-1}0$ has a linear metric projection of norm 2 (Corollary 9). Sufficient conditions are given in order for a certain subspace of codimension n to have a linear metric projection (Theorem 10).

2. Notation and definitions

A map P from a normed linear space X onto a subspace Y is called a *projection* if $Py = y$ for all $y \in Y$. The distance from a point x to a set Y is defined by

$$\text{dist}\ (x, Y) = \inf\{\| x - y \| : y \in Y\}.$$

If for each $x \in X$ there exists a $y \in Y$ such that $\| x - y \| = \text{dist}\ (x, Y)$ then Y is called an *E-space.* If the projection $P : X \to Y$ has the property that $\| x - Px \| = \text{dist}\ (x, Y)$ then we call P a *metric projection* or a promixity map. The *restriction* operator $R : C(T) \to C(Q)$ is defined by $(Rx)(q) = x(q)$ for all $x \in C(T)$ and all $q \in Q$. Thus, if Y is a subspace of $C(Q)$,

$$R^{-1}Y = \{x \in C(T) : Rx \in Y\}.$$

A function $E : C(Q) \to C(T)$ is called an *extension* map if $REx = x$ for all $x \in C(Q)$. The restriction of a function x to a set A is sometimes denoted by $x \mid A$. The difference of two sets is written $A - B = \{x : x \in A, x \notin B\}$. In topological nomenclature we follow J. L. Kelley's *General topology*,

3. E-spaces and linear metric projections

If T is a topological space then $C(T)$ will denote the Banach space of bounded continuous functions x defined on T with the supremum norm,

$$\| x \| = \sup\{ \mid x(t) \mid : t \in T\}.$$

Lemma 1. *Let Q be a closed set in a normal space T. If $x \in C(T)$ and $z \in C(Q)$ then z has an extension z' in $C(T)$ such that $\| Rx - z \| = \| x - z' \|$.*

Proof. Let $\alpha = \| Rx - z \|$. If $\alpha = 0$ then $Rx - z = 0$. Define $z' = x$.

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Thus, $\| Rx - z \| = \| x - z' \|$. Suppose $\alpha > 0$. By Tietze's Theorem, z has an extension $y \in C(T)$. Define the function z' by

$$\begin{aligned} z'(t) &= y(t) && \text{if } |x(t) - y(t)| \leq \alpha \\ &= x(t) - \alpha && \text{if } x(t) - y(t) > \alpha \\ &= x(t) + \alpha && \text{if } x(t) - y(t) < -\alpha. \end{aligned}$$

To verify that $z' \in C(T)$ it suffices to show that z' is continuous on the set

$$A = \{t \in T : |x(t) - y(t)| = \alpha\}.$$

Suppose $t \in A$ and $x(t) - y(t) = \alpha$. (The case $x(t) - y(t) = -\alpha$ is similar.) Let $\{t_i\}$ be a net in T converging to t. Since $x - y$ is continuous and $\alpha > 0$ we can assume $x(t_i) - y(t_i) > 0$. If $x(t_i) - y(t_i) \leq \alpha$ then

$$z'(t_i) = y(t_i) \to y(t).$$

If $x(t_i) - y(t_i) > \alpha$ then

$$z'(t_i) = x(t_i) - \alpha \to x(t) - \alpha = y(t).$$

Hence, in any case, $z'(t_i) \to y(t) = z'(t)$. Thus $z' \in C(T)$ and $\| x - z' \| = \alpha$.

LEMMA 2. *Let Q be a closed set in a normal space T. For all $x \in C(T)$ and for any subspace M in $C(Q)$*, dist $(x, R^{-1}M) =$ dist (Rx, M).

Proof. If $y \in R^{-1}M$, then $\| x - y \| \geq \| Rx - Ry \| \geq$ dist (Rx, M). Thus,

$$\text{dist } (x, R^{-1}M) \geq \text{dist } (Rx, M).$$

Assume there is an $x \in C(T)$ for which dist $(x, R^{-1}M) >$ dist (Rx, M). Then there is an $m \in M$ such that $\| Rx - m \| <$ dist $(x, R^{-1}M)$. By Lemma 1 there is an $m' \in C(T)$ such that $\| x - m' \| = \| Rx - m \|$, a contradiction.

THEOREM 3. *Let Q be a closed set in a normal space T. If Z is a subspace of $C(Q)$ then the following are equivalent:*

(1) *Z is an E-space in $C(Q)$*

(2) *$R^{-1}Z$ is an E-space in $C(T)$.*

Proof. Assume that (1) is true. Let $x \in C(T)$. Let z be a best approximation to Rx in Z. By Lemma 1, z has an extension $z' \in C(T)$ such that

$$\| Rx - z \| = \| x - z' \|.$$

If $y \in R^{-1}Z$ then $\| x - y \| \geq \| Rx - Ry \| = \| x - z' \|$. This shows that z' is a best approximation to x. Since $z' \in R^{-1}Z$, the latter is an E-space.

Next assume (2). Let $x \in C(Q)$. Let x' be a Tietze extension of x. Let y be a best approximation to x' in $R^{-1}Z$. If $z \in Z$, then by Lemma 1, z has an extension z' such that $\| x' - z' \| = \| x - z \|$. Thus,

$$\| x - Ry \| \leq \| x' - y \| \leq \| x' - z' \| = \| x - z \|.$$

So Ry is a best approximation to x, and Z is an E-space.

Since finite-dimensional spaces are E-spaces we have

COROLLARY 4. *Let Q be a closed set in a normal space T. If Z is a finite-dimensional subspace of $C(Q)$, then $R^{-1}Z$ is an E-space in $C(T)$.*

A subspace Y is said to be *complemented* if Y is the range of a bounded linear projection.

THEOREM 5. *Let Q be a closed set in a normal space T. If there exists a finite-dimensional subspace Z in $C(Q)$ such that $R^{-1}Z$ is complemented in $C(T)$, then there is a bounded linear extension operator from $C(Q)$ to $C(T)$.*

Proof. Let $z_1, z_2, \cdots, z_n$ be a basis for Z. By Tietze's Theorem, each z_i has an extension z_i' in $C(T)$ such that $\| z_i \| = \| z_i' \|$. If $z \in Z$, $z = \sum_{i=1}^{n} \alpha_i z_i$. Define E by the equation $Ez = \sum_{i=1}^{n} \alpha_i z_i'$. Then E is a bounded linear extension operator from Z to $C(T)$. If $x \in R^{-1}Z$ define L by $Lx = (I - ER)x$ and note that L is a bounded projection from $R^{-1}Z$ onto $R^{-1}0$. By hypothesis there is a bounded linear projection L' from $C(T)$ onto $R^{-1}Z$. Thus, LL' is a bounded linear projection from $C(T)$ onto $R^{-1}0$ and by a known result [4] there is a bounded linear extension operator from $C(Q)$ to $C(T)$.

The following elementary lemma will be needed.

LEMMA 6. *Let P be a linear projection from a normed linear space E onto a nontrivial subspace M. Then P is a metric projection if and only if $I - P$ is of norm 1.*

The next theorem is similar to a result of Dean [4].

THEOREM 7. *Let Q be a closed set in a normal space T. Then the following are equivalent:*

(1) *$R^{-1}O$ has a linear metric projection.*

(2) *There is a linear norm 1 extension operator from $C(Q)$ to $C(T)$.*

Proof. Assume (2) is true. Define $L = I - ER$. Since ER is a linear projection of norm 1, by Lemma 6, L is a metric projection. If $x \in C(T)$, then $RLx = 0$. Thus $Lx \in R^{-1}0$. Let $y \in R^{-1}0$. Then $Ly = y$ and L is a projection onto $R^{-1}0$.

If (1) is true, let P be a linear metric projection from $C(T)$ onto $R^{-1}0$. Let E be a Tietze extension map of norm 1 from $C(Q)$ to $C(T)$. We wish to obtain a linear norm 1 extension map. Define the map E' by $E' = (I - P)E$. Since $RE' = RE - RPE = I$, E' is an extension operator from $C(Q)$ to $C(T)$. By Lemma 6, $\| I - P \| = 1$ and therefore $\| E' \| = 1$.

To prove E' is linear it suffices to show that for arbitrary x and y in $C(Q)$, and scalars α and β,

$$(I - P)[E(\alpha x + \beta y) - (\alpha Ex + \beta Ey)] = 0.$$

Since P is a projection onto $R^{-1}0$, it follows that $(I - P)^{-1}(0) = R^{-1}0$.

Thus it suffices to show that $E(\alpha x + \beta y) - (\alpha Ex + \beta Ey) \in R^{-1}0$. Since

$$RE(\alpha x + \beta y) - \alpha REx - \beta REy = \alpha x + \beta y - \alpha x - \beta y = 0,$$

the conclusion follows.

THEOREM 8. *Let Q be a closed set in a T_4-space T. Then the following are equivalent:*

(1) *$R^{-1}0$ has a linear projection of norm < 2.*

(2) *Q is open.*

(3) *$R^{-1}0$ has a linear metric projection of norm 1.*

Proof. If (1) is true, let L be a linear projection from $C(T)$ onto $R^{-1}0$ of norm $2 - \varepsilon$, where $0 < \varepsilon \leq 1$. Let $y = 1 - L1$. If $t \in Q$ then $y(t) = 1$. Let $U = \{t : y(t) > 1 - \varepsilon$. Then U is an open set containing Q. If Q is not open, there exists a point $t_0 \in U - Q$. By Urysohn's Lemma there is a function $x \in C(T)$ such that $x \mid Q = 1$, $x(t_0) = -1$ and $\| x \| = 1$. Since $Ly = 0$, $Lx = L(x - y)$. Since $x - y \in R^{-1}0$, $L(x - y) = x - y$. Hence, $Lx = x - y$. However,

$$x(t_0) - y(t_0) = -1 - y(t_0) < -2 + \varepsilon,$$

so that $\| Lx \| > 2 - \varepsilon$. Since $\| x \| = 1$, $\| L \| > 2 - \varepsilon$. This contradiction implies Q is open.

Assume (2) is true and let v be the characteristic function of $T - Q$. For $y \in C(T)$ define $Py = vy$. Clearly, P is a linear projection of norm 1 onto $R^{-1}0$. That P is a metric projection follows by writing

$$\text{dist}\ (y, R^{-1}0) \leq \| y - Py \| = \| Ry - 0 \| = \text{dist}\ (Ry, 0) = \text{dist}\ (y, R^{-1}0).$$

If (3) is true, then (1) follows trivially.

COROLLARY 9. *Let Q be a closed set in a connected metric space T. Then $R^{-1}0$ has a linear metric projection of norm 2.*

Proof. If P is a metric projection note that $\| P \| \leq 2$, since

$$\| Px \| = \| Px - x + x \| \leq \| x - Px \| + \| x \| \leq 2 \| x \|.$$

By the Borsuk-Dugundji Theorem [5] there is a bounded linear norm 1 extension operator from $C(Q)$ to $C(T)$. By Theorem 7, $R^{-1}0$ has a linear metric projection and by Theorem 8 it is of norm 2.

If B is a set in T, define $R_B^{-1}0 = \{x \in C(T) : x \mid B = 0\}$. If $\phi \in C^*$ (continuous linear functionals on $C(T)$), we define the *support* of ϕ, denoted by $S(\phi)$, as the smallest closed set A such that $R_A^{-1}0 \subset \phi^{-1}(0)$.

THEOREM 10. *Let T be a normal space and let $\phi_1, \phi_2, \cdots, \phi_n$ be multiplicative linear functionals on $C(T)$ having disjoint supports. Then $\bigcap_{i=1}^n \phi_i^{-1}(0)$ has a linear metric projection.*

Proof. Each ϕ_i has the property that $\phi_i(1) = 1$ and $\| \phi_i \| = 1$. Since T

is normal, there exist disjoint open sets $U_1, U_2, \cdots, U_n$ such that $U_i \supset S(\phi_i)$. By Urysohn's Lemma there exist functions $y_1, y_2, \cdots, y_n$ such that $y_i \mid S(\phi_i) = 1$, $y_i \mid (T/U_i) = 0$, and $0 \leq y_i \leq 1$. Thus, $y_i \mid S(\phi_j) = \delta_{ij}$. Since $(1 - y_i) \mid S(\phi_i) = 0$, $\phi_i(1 - y_i) = \phi_i(1) - \phi_i(y_i) = 0$, which implies $\phi_i(y_i) = 1$. Thus $\phi_i(y_j) = \delta_{ij}$.

Let Y be the subspace generated by $y_1, y_2, \cdots, y_n$. Define the map P from $C(T)$ to Y by $Px = \sum_{i=1}^n \phi_i(x)y_i$. If $y \in Y$ then

$$y = \sum_{j=1}^n \alpha_j y_j \quad \text{and} \quad Py = \sum_{i=1}^n \phi_i(\sum_{j=1}^n \alpha_j y_j)y_i = \sum_{i=1}^n \alpha_i y_i = y.$$

It is clear that P is linear and therefore P is a linear projection from $C(T)$ onto Y.

Let $H = \bigcap_{i=1}^n \phi_i^{-1}(0)$. If $x \in C(T)$ then

$$\phi_j(I - P)x = \phi_j(x) - \sum_{i=1}^n \phi_i(x)\phi_j(y_i) = 0.$$

Thus $(I - P)x \in H$. If $h \in H$ then $\phi_i(h) = 0$ for each i and $(I - P)h = h$. Thus $I - P$ is a linear projection from $C(T)$ onto H. By the definition of the y_i and since $\| \phi_i \| = 1$ we have $\| P \| \leq 1$. Thus $\| P \| = 1$ and by Lemma 6, $I - P$ is a linear metric projection.

Since the point-evaluation functional $\hat{t}_i$ defined by $\hat{t}_i(x) = x(t_i)$ for each $x \in C(T)$ satisfy the hypotheses of Theorem 10 and $R^{-1}0 = \bigcap_{i=1}^n \hat{t}_i^{-1}(0)$ we obtain

COROLLARY 11. *Let T be a T_4-space and let $Q = \bigcup_{i=1}^n \{t_i\}$ where $t_i \in T$. Then $R^{-1}0$ has a linear metric projection.*

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