EMBEDDINGS OF BOUNDED TOPOLOGICAL MANIFOLDS

BY

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1. Introduction

Our main result gives sufficient conditions under which a map of a bounded topological manifold is homotopic to a locally flat embedding. We also establish the analogous unknotting theorem.

THEOREM 1. *Let M be a compact topological m-manifold with non-empty boundary and let Q be a topological q-manifold, with or without boundary, such that $q \geq m + 3$. Suppose that $(M, \partial M)$ is $(2m - q - 1)$-connected. Then any continuous map $f : M \to Q$ is homotopic to a locally flat embedding.*

THEOREM 2. *Let f and g be two locally flat embeddings of a compact topological m-manifold M with non-empty boundary into the interior of a topological q-manifold Q. Suppose that $(M, \partial M)$ is $(2m - q)$-connected and $q \geq m + 3$. If f and g are homotopic, then they are ambient isotopic.*

COROLLARY 1. *If M is a compact topological m-manifold, $m > 3$, such that each component of M has non-empty boundary, then there is a locally flat embedding of M in E^{2m-1} and any two locally flat embeddings of M in E^q, $q \geq 2m$ are ambient isotopic.*

COROLLARY 2. *If M is a compact topological m-manifold, then there is a locally flat embedding of M in E^{2m}, and if $m > 3$, then any two locally flat embeddings of M into a q-plane in E^{q+1}, $q \geq 2m$, are ambient isotopic in E^{q+1}.*

Proof. If $m \leq 3$, M is a combinatorial manifold, and the result is well known.

For $m > 3$, we remove the interior of a locally flat m-cell D_i from each component M_i of M. We can then embed each component of the resulting manifold with boundary in one of a set of parallel $(2m - 1)$-planes in E^{2m}, using Corollary 1. Then we embed the interiors of the cells we removed as cones over the ∂D_i's from points not in the $(2m - 1)$-planes. By Theorem 1.2 of [D2], if there is an embedding of M in E^{2m}, $m \geq 3$, then there is a locally flat embedding.

The proof of the isotopy part of the corollary is similar. One begins by moving the components of M into distinct parallel q-planes in E^{q+1}. Then proceed as in the embedding part using "relative" versions of Miller's taming

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theorem (Lemma 1 in Section 2) and Zeeman's codim 3 unknotting ball pair theorem [Z].

The proof of the next corollary is similar to the proof of Corollary 2.

COROLLARY 3. *Let M be a closed r-connected topological m-manifold $r \leq m - 4$. Then there is a locally flat embedding of M in E^{2m-r}.*

Lees [L] established Corollary 3 for M an orientable manifold. Other results on embedding compact topological manifolds have been obtained by Weller.

The proof of Theorem 1 is contained in Sections 2 and 3. The proof of Theorem 2 is in Section 4.

2. Preliminaries

In this section we list some definitions and lemmas which we will use in the proof of Theorem 1.

DEFINITION. Let X be a compact subset of a metric space Q. An *ε-push* P on (Q, X) is an ambient isotopy

$$\{H_t : Q \to Q, \quad t \in [0, 1] \quad \text{and} \quad H_0 = 1\}$$

such that each H_t is an ε-homeomorphism and

$$H_t(x) = x \quad \text{when} \quad d(x, X) \geq \varepsilon \quad \text{and} \quad t \in [0, 1].$$

Also we write $P(x) = H_1(x)$.

TAMING LEMMA 1 (Richard Miller [M]). *Suppose M and Q are PL combinatorial m- and q-manifolds, respectively, with M compact, $q \geq 5$, and $m \leq q - 3$. Let $f : M \to \text{Int } Q$ be a locally flat embedding, and let $\varepsilon > 0$ be given. Then there is an ε-push P of $(Q, f(M))$ such that Pf is piecewise linear.*

THEOREM 3 (Dancis [D1], Hudson, Tindell). *Let M be a compact PL m-manifold with non-empty boundary and let Q be a PL q-manifold. Suppose that $m \leq q - 2$ and $(M, \partial M)$ is $(2m - q - 1)$-connected. Then any map $f : M \to Q$ is homotopic to a PL embedding.*

Here we first prove a special version of this theorem (Lemma 2) which we will later use inside the coordinate neighborhoods of a topological manifold. A proof of Theorem 3 will follow the proof of Lemma 2. But first we need two definitions.

DEFINITION. Let $g : Y \to Z$ be a map where Y and Z are spaces. A subset X of Y is a *set of essential singularities of g* if g embeds $Y - X$ in Z.

DEFINITION. A complex R is *locally-tamely embedded* in a topological n-manifold M if for each point $x \in R$ there is a neighborhood $N(x)$ in M and an onto homeomorphism $h : N(x) \to I^n = [0, 1]^n$ such that $h \mid R \cap N(x)$ is PL (with respect to R and I^n).

LEMMA 2. *Let M be a compact topological m-manifold, $\partial M \neq \emptyset$, such that $(M, \partial M)$ is $(2m - q - 1)$-connected. Let R be a finite complex which is locally-tamely embedded in M. Let C be an open collar of ∂M in M. Suppose $g : R \to E^q$ is a PL general position map with $m \leq q - 2$. Then there is a compact set X of essential singularities for g and a push φ on M such that $\varphi(C) \supset X$.*

Furthermore if D is a compact subset of C then we may obtain $\varphi(C) \supset X \cup D$.

Note. g embeds $R - \varphi(C)$.

Proof. By removing from M an open collar containing D and properly contained in C, and then adding it back on at the end of the proof, we make it possible to ignore the set D during the proof.

Let K be a triangulation of R such that g is simplicial with respect to K and some subdivision of E^q. Let S be the set of singularities of g. Since g is in general position, $\dim S \leq 2m - q$. Let K_S be the triangulation S inherits from K and let K_S' be the $(2m - q - 1)$-skeleton of S. Then K_S' contains those points at which g is not locally a homeomorphism. Since K_S' is a locally tame subset of M, by Newman's Engulfing Theorem [N] there is a push φ_0 on M such that $\varphi_0(C) \supset K_S'$.

Now let us take care of $K_S - K_S'$. Let

$$\{\sigma_{ij} \mid i = 1, \cdots, n, \quad j = 1, \cdots, \quad s = s(i)\}$$

be the $(2m - q)$-simplices of K_S, where if σ_i is a $(2m - q)$-simplex in $g(K_S)$, then

$$g^{-1}(\sigma_i) = \sigma_{i1} \cup \cdots \cup \sigma_{is(i)}.$$

For each i, choose $s(i)$ open sets G_{ij} in σ_i whose closures are pairwise disjoint and disjoint from $\partial\sigma_i$. Let

$$F_{ij} = g^{-1}(\sigma_i - G_{ij}) \cap \sigma_{ij}.$$

Then there is a push φ_{ij} on each σ_{ij}, fixed on $\partial\sigma_{ij}$, such that $\varphi_{ij}\varphi_0(C) \supset F_{ij}$, and each φ_{ij} may be extended to all of M in such a way that each φ_{ij} is the identity outside $St(\hat{\sigma}_{ij}, \beta K)$.[2] Let φ be the composition of all the φ_{ij}'s with φ_0. If we let

$$X = K_S' \cup \bigcup_i \bigcup_{j=1}^{s(i)} F_{ij},$$

then $\varphi(C) \supset X$. Also g is an embedding on $R - X$, since if $g(\sigma_{ij}) = g(\sigma_{ip})$, $p \neq j$, then

$$g(\sigma_{ij} - F_{ij}) \cap g(\sigma_{ip} - F_{ip}) = G_{ij} \cap G_{ip} = \emptyset.$$

This completes the proof of Lemma 2.

Proof of Theorem 3. Let $R = M$ and $D = \emptyset$ in the above lemma. Let $\mathrm{Cl}(C)$ be a compact PL collar of ∂M. Let $i : M \to M - C$ be the PL

[2] $\hat{\sigma}$ is the barycenter of σ; βK is the first barycentric subdivision of K.

embedding obtained by pushing in the collar. (Thus i is homotopic to the identity.) Assume that f has been homotoped to a PL general position map f_1. Then the engulfing process described in the above proof may be carried out in such a way that φ is PL. Thus $f_1 \varphi i$ is the desired embedding of M into Q.

3. Proof of Theorem 1

Let $f : M \to Q$ be as in the hypothesis of the theorem, and let C be the interior of a locally flat closed collar of ∂M in M. The basic idea of the proof is to construct an approximation $h : M \to Q$ to f and engulf from C a set X of essential singularities for h (as defined in Section 2). If P is the engulfing push, then h will embed $M - P(C)$, which is a copy of M. On a neighborhood of $M - P(C)$, h will be locally PL, and hence locally flat by [Z].

In what follows, all neighborhood, interiors, and closures are understood to be neighborhoods, interiors, and closures in M. Let $\{I_j\}_{j=1}^{r}$ be a cover of M by m-cells such that each I_j is piecewise-linearly embedded in the interior of a triangulated coordinate patch on M whose image under f is contained in a triangulated coordinate patch in Q. We will assume during the course of the proof that all approximations to f retain this property. Let $N_j = \bigcup_{i=1}^{j} I_i$, for $j = 1, \cdots r$.

The map h, the set X, and the push P will be constructed inductively. This process will involve a double induction. In the main induction we will construct an approximation h_j to f and engulf from C a set of essential singularities for $h_j \mid N_j$. To do this it will be necessary to first engulf a set of essential singularities for $h_j \mid I_j$ and then, in a secondary induction, construct and engulf a set of essential singularities that arise from intersections of $h_j(I_j)$ with $h_j((N_j - I_j) \cap I_i)$ for each $i < j$.

The induction hypotheses are as follows.

Main Induction Hypothesis. For $j = 1, \cdots r$, there is:

(1) a map $h_j : M \to Q$ approximating f;

(2) an open set $K_j \subset M$ such that $h_j \mid K_j$ is a locally flat embedding;

(3) a push P_j on M such that $N_j \subset K_j \cup P_j(C)$.

Secondary Induction Hypothesis. With j as above and $i = 0, 1, \cdots j - 1$, there is:

(1) a map $g_{ji} : M \to Q$ approximating f;

(2) open sets L_{ji} and L_{j-1} in M such that g_{ji} is a locally flat embedding on each (L_{j-1} will be the same as K_{j-1}, except for some minor adjustments);

(3) a push ϕ_{ji} on M such that

(a) $N_i \cup I_j \subset L_{ji} \cup \phi_{ji}(C)$,

(b) $N_{j-1} \subset L_{j-1} \cup \phi_{ji}(C)$

The proof will be in three steps: the initial step of the main induction, construction of h_1; the initial step of the secondary induction, construction of

g_{j0} from h_{j-1}; and the general step of the secondary induction, construction of g_{ji} from $g_{j,i-1}$. The general step of the main induction is then completed by letting $h_j = g_{j,j-1}$, $K_j = L_{j,j-1}$, and $P_j = \phi_{j,j-1}$.

Construction of h_1. Let $h_1 : M \to Q$ be an approximation to f which is PL and in general position on some compact triangulated neighborhood W_1 of I_1. Then by Lemma 2 there is a compact set X_1 of essential singularities for $h_1 \mid W_1$ and a push P_1 on M such that $P_1(C) \supset X_1$. Let $K_r = \text{Int } W_1 - X_1$. This completes the initial step of the main induction.

Construction of g_{j0} from h_{j-1}, $j \geq 1$. Choose compact PL submanifolds R_j, W_j of the triangulated neighborhood of I_j such that

(a) $R_j \subset W_j \cap K_{j-1}$,
(b) $I_j \subset \text{Int } W_j$, and
(c) $\text{Cl } (W_j - R_j) \cap N_{j-1} \subset P_{j-1}(C)$.

By Lemma 1 there is a short push T of Q such that $Th_{j-1} \mid R_j$ is PL. By (c) we can find a closed neighborhood A_j of $\text{Cl } (W_j - R_j)$ in $\text{Cl } (M - R_j)$ such that

$$A_j \subset N_{j-1} \subset P_{j-1}(C).$$

Let g_{j0} be a PL general position approximation to Th_{j-1} which agrees with Th_{j-1} on R_j and off A_j. As in Lemma 2, there is a compact set X_j of essential singularities for $g_{j0} \mid W_j$ and a push ϕ_{j0} on M such that

$$\phi_{j0}(C) \supset (N_{j-1} - L_{j-1}) \cup X_j.$$

The initial step of the secondary induction is then completed by letting

$$L_{j0} = \text{Int } W_j - X_j \quad \text{and} \quad L_{j-1} = K_{j-1} - A_j.$$

Construction of g_{ji} from $g_{j,i-1}$, $1 \leq i \leq j-1$. Consider the compact set

$$N_i \cup I_j - (L_{j,i-1} \cap L_{j-1}) - \phi_{j,i-1}(C).$$

It is contained in $(N_i \cup I_j) - ((N_{i-1} \cup I_j) \cap N_{j-1})$ and hence in

$$(N_i - (I_j \cup N_{i-1})) \cup (I_j - N_{j-1}).$$

Therefore it is the union of two disjoint compact subsets, one in $I_i - (N_{i-1} \cup I_j)$ and the other in $I_j - N_{j-1}$. Let Z_i and R_i be disjoint, compact PL manifold neighborhoods of these sets, Z_i in the triangulated neighborhood of I_i and R_i in the triangulated neighborhood of I_j, such that $Z_i \subset L_{j-1}$ and $R_i \subset L_{j,j-1}$. By applying Lemma 1 to the manifolds Z_i and R_i and using general position we may alter $g_{j,i-1}$ (call the altered map g_{ji}) so that $g_{ji}(Z_i)$ and $g_{ji}(R_i)$ are polyhedra in general position; i.e., $g_{ji}(Z_i) \cap g_{ji}(R_i)$ is a polyhedron of dimension $2m - q$. If the taming and general position pushes are sufficiently short, no new singularities will be introduced in $L_{j,i-1} \cap L_{j-1}$ outside of $Z_i \cup R_j$. As in the proof of Lemma 2, there is a compact set $X_i \subset Z_i$

of essential singularities for $g_{ji} \mid Z_i \cup R_i$ and a push $\phi_{ji}(C)$ such that $\phi_{ji} \supset X_i$. Since $\phi_{j,i-1}(C)$ contains both

$$N_i \cup I_j - ((L_{j-1,i-1} \cap L_{j-1}) \cup \operatorname{Int} Z_i \cup \operatorname{Int} R_i) \quad \text{and} \quad N_{j-1} - L_{j-1},$$

we may assume that $\phi_{ji}(C)$ also contains these sets. We complete the induction step by letting $L_{ji} = (L_{j,i-1} \cap L_{j-1}) \cup \operatorname{Int} R_i \cup \operatorname{Int} Z_i - X_i$.

Finally, if we let $i : M \to M - P_r(C)$ be a homeomorphism homotopic to the identity, then $h_r i$ is the desired embedding of M into Q.

4. Proof of Theorem 2

A corollary to the method of proof of Theorem 1 is the following.

LEMMA 3. *Let f, g, M, and Q satisfy the hypotheses of Theorem 2. Then f and g are concordant; i.e., there is a locally flat embedding*

$$F : M \times I \to Q \times I$$

such that $F(x, 0) = (f(x), 0)$ *and* $F(x, 1) = (g(x), 0)$ *for all* $x \in M$ *and* $F^{-1}(Q \times i) = M \times i$, $i = 0, 1$.

Outline of Proof. Define

$$F \mid M \times [0, 1/10] = f \times 1 \quad \text{and} \quad F \mid M \times [9/10, 1] = g \times 1.$$

Then copy the proof of Theorem 1, replacing M by $M \times [1/10, 9/10]$ and C by $C \times [1/10, 9/10]$, and extending all pushes to $M \times I$ keeping $M \times \{0, 1\}$ fixed.

LEMMA 4 (Dancis and Richard T. Miller [D-M]) (Topological concordance implies ambient isotopy). *Let M be a compact topological m-manifold and let Q be a topological q-manifold, with $q \geq m + 3$. Let $F : M \times I \to (\operatorname{Int} Q) \times I$ be a locally flat embedding, with $F^{-1}(Q \times i) = M \times i$, $i = 0, 1$, and let f, $g : M \to Q$ be defined by $F(x, 0) = (f(x), 0)$ and $F(x, 1) = (g(x), 1)$ for all $x \in M$. Then there is an ambient isotopy*

$$\{H_t : Q \to Q, \quad t \in I, \quad H_0 = 1\}$$

such that $H_1 f = g$.

Theorem 2 is a direct consequence of Lemmas 3 and 4.

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