CONTINUOUS REPRESENTATIONS OF INFINITE SYMMETRIC GROUPS ON REFLEXIVE BANACH SPACES

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Abstract

Let S be an arbitrary infinite set and let G be the group of finitely supported permutations of S; give G the topology of pointwise convergence on S. Let B be a reflexive Banach space and let Γ be a continuous representation of G on B such that $\| \Gamma(g) \| \leq M$ for all $g \in G$ for some fixed positive number M. Through the use of a canonically defined dense subspace of cofinite vectors, it is shown that Γ is strongly continuous and contains an irreducible subrepresentation. An equivalence relation of cofinite equivalence of representations is defined; if Γ is irreducible, then Γ is cofinitely equivalent to an irreducible weakly continuous unitary representation of G on a Hilbert space.

1. Introduction and notation

The author [1] has shown that any weakly continuous unitary representation of an infinite symmetric group on a Hilbert space is the direct sum of irreducible representations; these irreducible representations were explicitly constructed.

Below, we consider a weakly continuous uniformly bounded representation Γ of an infinite symmetric group on a reflexive Banach space B. We show that B contains a canonically defined dense subspace which is invariant under Γ; a vector v is in this subspace iff it is invariant under the action of a certain type of subgroup. Γ is strongly continuous and contains an irreducible subrepresentation. If Γ is irreducible, then there is a unique (up to unitary equivalence) weakly continuous unitary representation Ω of G on a Hilbert space such that the restriction of Γ to its canonically defined dense subspace is algebraically equivalent to the restriction of Ω to its canonically defined dense subspace.

S will denote a fixed arbitrary infinite set and G will denote the group of those permutations Π of S such that $\{s \in S : \Pi(s) \neq s\}$ is finite. Give G the topology of pointwise convergence on S; G is a topological group but is not locally compact.

If B is a Banach space, then B' denotes the dual of B. If D is a subset of B, then $\mathrm{sp}(D)$ is the subspace of B that is spanned algebraically by D and $\mathrm{cl}\ \mathrm{sp}(D)$ is the closure of $\mathrm{sp}(D)$;

$$D^{\perp} = \{f \in B' : f(v) = 0 \text{ for all } v \in D\}.$$

A representation of G on the Banach space B is a homomorphism of G

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into the bounded linear operators on B. The representation Γ is weakly (strongly) continuous if $\Gamma(g)v$ is a weakly (strongly) continuous function of g for every $v \in B$; Γ is uniformly bounded if there is a real number M such that $\| \Gamma(g) \| \leq M$ for every $g \in G$.

If $T \subseteq S$, let

$$G_T = \{g \in G : g(s) = s \text{ if } s \notin T\}$$

and let

$$G^T = \{g \in G : g(s) = s \text{ if } s \in T\}.$$

Note that $G_{S-T} = G^T$.

DEFINITION. Let Γ be a representation of G on the Banach space B.

(1) Let $v \in B$. Then v is a cofinite vector for Γ if there is a finite subset T of S such that $\Gamma(g)v = v$ for all $g \in G.^T$ The vector v is cofinite of type n if T may be chosen to have cardinal number n and if it is impossible to choose T with cardinal number less than n.

(2) Let v be a non-zero cofinite vector for Γ of type n. Then v is a minimally cofinite vector for Γ if no non-zero vector in cl sp$(\Gamma(G)v)$ is cofinite of type less than n.

(3) Γ is uniformly of type n if no non-zero vector in B is cofinite of type m with $m < n$ and if cl sp$(\{\text{cofinite vectors of type } n\}) = B$.

(4) If T is a finite subset of S, then

$$B_T = \{v \in B : \Gamma(g)v = v \text{ for all } g \in G^T\}.$$

(5) B_Γ is the subspace of all cofinite vectors for Γ.

(6) Let D be a Banach space and let Ω be a representation of G on D. Then Γ is cofinitely equivalent to Ω iff there is a 1-1 linear mapping U from B_Γ onto D_Ω such that $U\Gamma(g)v = \Omega(g)Uv$ for every $v \in B$ and $g \in G$.

(7) A Banach space B is an $\mathfrak{N}_1$ space [2, p. 60] if there is an arbitrary index set Λ and a family $\{N_\lambda : \lambda \in \Lambda\}$ of finite-dimensional subspaces, directed by inclusion, whose union is dense in B and such that each N_λ is the range of a projection P_λ of norm one of B.

(8) Γ is completely decomposable [4, p. 230] if $\{v \in B : T(G) \text{ acts irreducibly on cl sp}(\Gamma(G)v)\}$ is dense in B.

Remarks. (1) Clearly cofinite equivalence is an equivalence relation on the set of representations of G on Banach spaces.

(2) It follows from [1] that two weakly continuous unitary representations of G are cofinitely equivalent iff they are unitarily equivalent.

(3) Complete decomposability as defined in definition 8 agrees with the usual definition in the case when Γ is a unitary representation on a Hilbert space.

Example. Let M_0 be a positive number and let m be a measure on S such that $M_0 \geq m(\{s\}) \geq 1/M_0$ for each $s \in S$. Let $1 < p < \infty$, let n be a positive integer, and let $B = L^p(S^n, m^n)$. Then B is a reflexive Banach space and the canonical representation Γ of G on B, defined by

$$\Gamma(g)v(s_1, s_2, \cdots, s_n) = v(g^{-1}(s_1), g^{-1}(s_2), \cdots g^{-1}(s_n))$$

is weakly continuous and is uniformly bounded by M_0^{2n}.

If $v \in B$, then v is a cofinite vector for Γ iff there is a finite subset T of S such that support$(v) \subseteq T^n$. The non-zero vector v is minimally cofinite iff v is the characteristic function of $\{(s, s, \cdots, s)\}$ for some $s \in S$.

2. Statement of the results

In the next section, the following theorems are proved.

THEOREM 1. *Let Γ be a uniformly bounded weakly continuous representation of G on the reflexive Banach space B. Then*

(1) *B_Γ is dense in B,*

(2) *Γ is strongly continuous.*

THEOREM 2. *Let Γ be a uniformly bounded weakly continuous representation of G on the reflexive Banach space B. Assume there is a non-negative integer n such that Γ is uniformly of type n. Then Γ is completely decomposable.*

If, in addition, $\Gamma(g)$ is an isometry for each $g \in G$ and there is a minimally cofinite cyclic vector for Γ, then B is an $\mathfrak{M}_1$ space.

THEOREM 3. *Let Γ be a uniformly bounded weakly continuous representation of G on the reflexive Banach space B. Then Γ contains an irreducible subrepresentation.*

THEOREM 4. *Let Γ be an irreducible uniformly bounded weakly continuous representation of G on the reflexive Banach space B. Then there is a Hilbert space H and an irreducible weakly continuous unitary representation Ω of G on H such that Γ is cofinitely equivalent to Ω. Ω is unique up to unitary equivalence.*

Remarks. (4) The theorems and their proofs hold, with trivial modifications, if Γ is an antirepresentation of G.

(5) If β is an infinite cardinal number, let G_β be the set of those permutations Π of S such that $\{s \in S : \Pi(s) \neq s\}$ has cardinal number less than β; note that G_β is the group of all permutations of S if β is sufficiently large. Give G_β the topology of pointwise convergence on S. Theorems 1, 2, 3, and 4 remain valid if G is replaced by G_β. The proofs require trivial modifications.

3. Proof of the theorems

For the remainder of this paper, $\mathcal{Z}$ will denote the set of all finite subsets of S. If $Q \in \mathcal{Z}$, $|Q|$ will denote the cardinal number of Q and $\mathcal{Z}_Q$ will denote the set of all finite subsets of $S - Q$. $\mathcal{Z}$ and $\mathcal{Z}_Q$ are directed sets with respect to set inclusion as the partial order relation. If m is a non-negative integer, $\mathcal{S}_m = \{Q \in \mathcal{Z} : |Q| = m\}$.

M will always denote a fixed real number such that $\| \Gamma(g) \| \leq M$ for all $g \in G$.

If Λ is an index set and $x_\lambda \in B$ for each $\lambda \in \Lambda$, then $x_\lambda = 0$ a.a. (almost always) if $\{\lambda \in \Lambda : x_\lambda \neq 0\}$ is finite.

Proof of Theorem 1. (1) Let $f \in B'$, $f \neq 0$. Pick $w \in B$ so that $f(w) = 1$. By continuity, there is a neighborhood U of e such that real $f(\Gamma(g)w) > \frac{1}{2}$ if $g \in U$. Therefore, there is a finite subset T of S such that $f(\Gamma(g)w) > \frac{1}{2}$ if $g \in G^T$.

If $Q \in \mathcal{Z}_T$, let $R_Q = |Q|!^{-1} \sum_{g \in G_Q} \Gamma(g)$. Note that $\Gamma(h)R_Q = R_Q$ if $h \in G_Q$. $\| R_Q \| \leq M$. Therefore $\| R_Q w \| \leq \| R_Q \| \| w \| \leq M \| w \|$. Since B is reflexive, the ball of radius M in B is weakly compact. Consequently, there is a subnet $\{R_Q : Q \in \mathcal{Y}_T\}$ of $\{R_Q : Q \in \mathcal{Z}_T\}$, where $\mathcal{Y}_T \subseteq \mathcal{Z}_T$, such that $\{R_Q w : Q \in \mathcal{Y}_T\}$ converges weakly to a limit point v.

Let $g \in G^T$ be given. Let $Y = \{s \in S : g(s) \neq s\}$; then $Y \in \mathcal{Y}_T$.

$$\begin{aligned} \Gamma(g)v &= \Gamma(g) \text{ weak limit}_{Q \in \mathcal{Y}_T} R_Q w \\ &= \text{weak limit}_{Q \in \mathcal{Y}_T} \Gamma(g) R_Q w \\ &= \text{weak limit}_{Q \in \mathcal{Y}_T, Q \supseteq Y} \Gamma(g) R_Q w \\ &= \text{weak limit}_{Q \in \mathcal{Y}_T, Q \supseteq Y} R_Q w = v. \end{aligned}$$

Consequently, v is a cofinite vector for Γ.

Let $Q \in \mathcal{Y}_T$. Then

$$\text{real } f(R_Q w) = \text{real } |Q|!^{-1} \sum_{g \in G_Q} f(\Gamma(g)w) \geq |Q|!^{-1} \sum_{g \in G_Q} \tfrac{1}{2} = \tfrac{1}{2}.$$

Therefore real $f(v) \geq \frac{1}{2}$. Consequently, $(B_\Gamma)^\perp = 0$ and B_Γ is dense in B.

(2) Let $x \in B_\Gamma$. Assume $x \in B_T$, where T is some finite subset of S. Then G^T is a neighborhood of e, and $\Gamma(g)x = x$ if $g \in G^T$. The strong continuity of Γ follows immediately from the density of B_Γ and the uniform boundedness of Γ.

Proof of Theorem 2. Let v be a non-zero cofinite vector of type n; assume $T \in \mathcal{S}_n$ and $\Gamma(g)v = v$ for all $g \in G^T$. Let $D = \text{cl sp}(\Gamma(G)v)$. If $g \in G$, let $\Omega(g) = \Gamma(g) \mid D$.

Since G_T is a finite group, $\text{sp}(\Gamma(G_T)v)$ is a finite-dimensional subspace and consequently there is a positive integer m and subspaces D_{jT}, $1 \leq j \leq m$ of $\text{sp}(\Gamma(G_T)v)$ such that $\text{sp}(\Gamma(G_T)v) = \oplus_{1 \leq j \leq m} D_{jT}$ and $\Gamma(G_T)$ acts irreducibly on D_{jT} for $1 \leq j \leq m$. There are vectors v_j, $1 \leq j \leq m$, such that $v = \sum_{1 \leq j \leq m} v_j$ and $v_j \in D_{jT}$. Let $D_j = \text{cl sp}(\Gamma(G)v_j) = \text{cl sp}(\Gamma(G)D_{jT})$. To prove that Γ is completely decomposable, it suffices to prove that $\Gamma(G)$ acts irreducibly on D_j for $1 \leq j \leq m$.

This will be proved by a sequence of lemmas; the hypotheses of Theorem 2 are assumed in these lemmas. The second conclusion of Theorem 2 is an immediate consequence of Lemma 3 and Lemma 5. Several of these lemmas are used in the proof of Theorem 4.

LEMMA 1. *Let* $Q \in \mathcal{S}_n$ *and* $x \in B_Q$. *Let* Q_0 *be a non-empty subset of* Q. *Assume* $Q_0 = \{t_1, t_2, \cdots t_m\}$, *where* $t_j \neq t_k$ if $j \neq k$. *For* $1 \leq k \leq m$, *let* s_{kj} *be a sequence of elements of* $S - Q$. *Assume* $s_{kj} \neq s_{k'j'}$ *unless* $k = k'$ *and* $j = j'$. *Let* $g_j = h_j \prod_{k=1}^{m} (t_k, s_{kj})$, *where* $h_j \in G^Q$, $h_j(s_{kq}) = s_{kq}$ *for all* j, k, *and* q, *and* $h_j^2 = e$. ((t_k, s_{kj}) *is the 2-cycle which interchanges* t_k *with* s_{kj}.)

Then there is a subsequence g_{j_i} *of* g_j *such that*

$$\text{weak limit}_{i\to\infty} \Gamma(g_{j_i})x = 0.$$

Proof. $\| \Gamma(g_j)x \| \leq M \| x \|$ for all j. Since B is reflexive, there is a vector $w \in B$ and a subsequence g_{j_i} of g_j such that

$$\text{weak limit}_{i\to\infty} \Gamma(g_{j_i})x = w.$$

Pick $h \in G^{Q-Q_0}$ arbitrarily. Pick a positive integer p so that $h(s_{kj_i}) = s_{kj_i}$ if $1 \leq k \leq m$ and $i \geq p$.

$$\begin{aligned}\Gamma(h)w &= \Gamma(h)\ \text{weak limit}_{i\to\infty} \Gamma(g_{j_i})x \\ &= \text{weak limit}_{i\to\infty} \Gamma(h)\Gamma(g_{j_i})x \\ &= \text{weak limit}_{i\to\infty} \Gamma(g_{j_i})\Gamma(g_{j_i} h g_{j_i})x \\ &= \text{weak limit}_{i\to\infty} \Gamma(g_{j_i})x \\ &= w\end{aligned}$$

since $g_{j_i} h g_{j_i} \in G^Q$ if $i \geq p$ and since $g_j^2 = e$ for each j.

Therefore $\Gamma(h)w = w$ if $h \in G^{Q-Q_0}$. Since $Q - Q_0$ has cardinal number less than n, this implies that $w = 0$.

LEMMA 2. *Let* W *be a finite subset of* S. *Let*

$$x = \sum_{Q\in\mathcal{S}_n} x_Q,$$

where $x_Q \in B_Q$ *for each* $Q \in \mathcal{S}_n$ *and* $x_Q = 0$ a.a. *Then*

$$\| x \| \geq M^{-1} \| \sum_{Q\in\mathcal{S}_n, Q\subseteq W} x_Q \|.$$

The subspaces B_Q, $Q \in \mathcal{S}_n$, *are linearly independent.*

Proof. Let X be a finite subset of S such that $x_Q = 0$ if $Q \in \mathcal{S}_n$ and $Q \not\subseteq X$. Let $X_1 = X - W$. Assume $X_1 = \{t_1, t_2, \cdots t_q\}$, for some integer q, where $t_j \neq t_k$ if $j \neq k$. Let s_{kj} be a sequence in $S - (X \cup W)$ for each k, $1 \leq k \leq q$; assume $s_{kj} \neq s_{k'j'}$ unless $k = k'$ and $j = j'$.

Let $g_j = \prod_{k=1}^{q} (t_j, s_{kj})$. By repeatedly applying Lemma 1, obtain a subsequence g_{j_i} such that weak limit $_{i\to\infty} \Gamma(g_{j_i})x_Q = 0$ if $Q \in \mathcal{S}_n$ and $Q \not\subseteq W$.

Then weak $\text{limit}_{i\to\infty} \Gamma(g_{j_i})x = \sum_{Q\in\mathcal{S}_n, Q\subseteq W} x_Q$. Therefore

$$\| \sum_{Q\in\mathcal{S}_n, Q\subseteq W} x_Q \| \leq \sup_{1\leq i<\infty} \| \Gamma(g_{j_i})x \| \leq M \| x \|.$$

If $W \in \mathcal{Z}$ and $x = \sum_{Q\in\mathcal{S}_n} x_Q$, with $x_Q = 0$ a.a., let

$$P_W\, x = \sum_{Q\in\mathcal{S}_n, Q\subseteq W} x_Q.$$

$P_W^2 x = x$; $\| P_W x \| \leq M \| x \|$ by Lemma 2. P_W has been defined on the subspace spanned by the minimally cofinite vectors; consequently, P_W can be extended in a unique way to a linear operator on B; this operator is a projection and has norm $\leq M$. Note that

$$P_W x = \sum_{Q\in\mathcal{S}_n, Q\subseteq W} P_Q x \quad \text{if } x \in B.$$

The range of $P_W = \text{sp}(\{B_Q : Q \in \mathcal{S}_n, Q \subseteq W\}) = \text{cl sp}(\{B_Q : Q \in \mathcal{S}_n, Q \subseteq W\})$.

LEMMA 3. *Let $x \in B$. Then* strong $\text{limit}_{W\in\mathcal{Z}} P_W x = x$. *If $P_Q x = 0$ for all $Q \in \mathcal{S}_n$, then $x = 0$.*

Proof. Assume $x = \sum_{Q\in\mathcal{S}_n} x_Q$, where $x_Q \in B_Q$ for each $Q \in \mathcal{S}_n$ and $x_Q = 0$ a.a. Then $P_W x = x$ if $W \supseteq \bigcup \{Q \in \mathcal{S}_n : x_Q \neq 0\}$. The result for a general vector x follows from the density of the subspace spanned by the minimally cofinite vectors and the uniform boundedness of the P_W.

LEMMA 4. *Let $x \in B$ and $W \in \mathcal{Z}$. Then $P_W x \in \text{cl sp}(\Gamma(G)x)$. If in addition $\Gamma(g)x = x$ for all $g \in G^W$, then $P_W x = x$.*

Proof. Let $\varepsilon > 0$ be given. By Lemma 3, there exists $Y \in \mathcal{Z}$ such that $Y \supseteq W$ and $\| x - P_Y x \| < \varepsilon$. By the proof of Lemma 2 and a weak compactness argument, there is a sequence $g_j \in G^W$ and an element $y \in B$ such that

$$y = \text{weak limit}_{j\to\infty} \Gamma(g_j)x, \quad \text{and} \quad P_W P_Y x = \text{weak limit}_{j\to\infty} \Gamma(g_j)P_Y x.$$

However, $P_W P_Y x = P_W x$ and

$$\| y - P_W x \| = \| y - P_W P_Y x \| \leq M \| x - P_Y x \| < M\varepsilon.$$

Since $y \in \text{cl sp}(\Gamma(G)x)$, it follows that $P_W x \in \text{cl sp}(\Gamma(G)x)$.

Assume now that $\Gamma(g)x = x$ for all $g \in G^W$. Then $\Gamma(g_j)x = x$ for all j so that $y = x$. Then $\| x - P_W x \| < M\varepsilon$ for arbitrary $\varepsilon < 0$, so that $x = P_W x$.

LEMMA 5. *Assume now that x is a minimally cofinite vector for Γ, $X \in \mathcal{S}_n$, $x \in B_X$, and $\text{cl sp}(\Gamma(G)x) = B$. Then $B_X = \text{sp}(\Gamma(G_X)x)$.*

Proof. Let $y \in B_X$. Since x is a cyclic vector for Γ, if $\varepsilon > 0$ is given we can find a finite subset $\{g_i : 1 \leq i \leq k\}$ of G such that

$$\| y - \sum_{1\leq i\leq k} \Gamma(g_i)x \| < \varepsilon.$$

If $Q \in S_n$, let

$$k(Q) = \{i : 1 \leq i \leq k \text{ and } g_i(X) = Q\} \quad \text{and} \quad x_Q = \sum_{i\in k(Q)} \Gamma(g_i)x.$$

Note that $x_Q \in B_Q$, $x_Q = 0$ a.a., and $\sum_{Q\in S_n} x_Q = \sum_{1\leq i\leq k} \Gamma(g_i)x$. Apply

Lemma 2 with $W = X$ to obtain

$$M^{-1} \| y - x_X \| \leq \| y - \sum_{Q \in S_n} x_Q \| < \varepsilon.$$

Since $x_X \in \text{sp}(\Gamma(G_X)x)$, $y \in \text{cl sp}(\Gamma(G_X)x)$. Since G_X is a finite group, $\text{sp}(\Gamma(G_X)x)$ is a finite-dimensional subspace of B and is therefore closed, so that $y \in \text{sp}(\Gamma(G_X)x) = \text{cl sp}(\Gamma(G_X)x)$.

LEMMA 6. *Assume the hypotheses of Lemma* 5. *Assume further that* $\Gamma(G_X)$ *acts irreducibly on* B_X. *Then* Γ *is an irreducible representation of* G.

Proof. Let $w \in B$ be any non-zero vector. By Lemma 3, there exists $Q \in \mathcal{S}_n$ such that $P_Q w \neq 0$. By Lemma 4, $P_Q w \in \text{cl sp}(\Gamma(G)w)$. Pick $g \in G$ such that $g(Q) = X$. Then

$$\Gamma(g)P_Q w \in B_X \quad \text{and} \quad \Gamma(g)P_Q w \in \text{cl sp}(\Gamma(G)w).$$

Then

$$\text{cl sp}(\Gamma(G)w) \supseteq \text{cl sp}(\Gamma(G)\Gamma(g)P_Q w) = \text{cl sp}(\Gamma(G)\Gamma(G_X)\Gamma(g)P_Q w)$$
$$= \text{cl sp}(\Gamma(G)B_X) = B.$$

Since any non-zero vector in B is a cyclic vector for the representation Γ, Γ is irreducible.

Proof of Theorem 3. By Theorem 1 and the well-ordering of the non-negative integers, there is a non-zero $v \in B$ such that v is minimally cofinite for Γ. The subspace $\text{cl sp}(\Gamma(G)v)$ is invariant under $\Gamma(G)$ and by Theorem 2, the subrepresentation of Γ on $\text{cl sp}(\Gamma(G)v)$ contains an irreducible subrepresentation.

Proof of Theorem 4. Assume $x \in B$, $x \neq 0$, and x is a minimally cofinite vector for Γ. Assume x is cofinite of type n, $T \in \mathcal{S}_n$, and $x \in B_T$.

It follows from Lemma 5 that $\Gamma(G_T)$ acts irreducibly on B_T. Since G_T is a finite group, there is a finite-dimensional Hilbert space H_T, an irreducible unitary representation Λ of G_T on H_T, and a 1-1 linear operator U_T from B_T onto H_T such that $U_T\,\Gamma(g)v = \Lambda(g)U_T\,v$ if $g \in G_T$ and $v \in B_T$.

By part Ib of Theorem 2 of [1], there is a Hilbert space H and an irreducible weakly continuous unitary representation Ω of G on H such that

(1) $H_T \subseteq H$ and $H_T = \{v \in H : \Omega(g)v = v \text{ for all } g \in G^T\}$.
(2) x is a minimally cofinite vector for Ω.
(3) $\Lambda(g) = \Omega(g) \mid H_T$ if $g \in G_T$.

By Lemma 4 and Lemma 5, $B_\Gamma = \text{sp}(\Gamma(G)B_T)$ and $H_\Omega = \text{sp}(\Omega(G)H_T)$. Let $y \in B_\Gamma$. Then there is a finite subset $\mathcal{Y}$ of $\mathcal{S}_n$ such that

$$y = \sum_{Q \in \mathcal{Y}} a_Q \Gamma(g_Q)x_Q,$$

where a_Q is a scalar, $g_Q \in G$ and $g(T) = Q$, and $x_Q \in B_T$, for each $Q \in \mathcal{Y}$. Let $Uy = \sum_{Q \in \mathcal{Y}} a_Q \Omega(g_Q)U_T x_Q$. The function U is well defined and satisfies the conclusions of the theorem; this may be shown by direct computation.

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