# GROUP EXTENSIONS AND TWISTED COHOMOLOGY THEORIES

BY

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## Introduction

In this paper we continue the study of group extensions initiated in [7]. The specific problem discussed there was the computation of extensions in the exact sequence of groups obtained by mapping a space into a principal fibration sequence. Here we consider the same problem, but in a different category—the category of spaces "over and under" a fixed space (see [9], [1]). This means in particular that the solution to the extension problem is given in terms of "twisted" cohomology operations [9], whereas in [7] only ordinary cohomology operations were needed.

In §1 we discuss the category we will use. In §2 we state our extension problem, and in §§3–4 we give a general solution. Finally, in §§5–6 we give applications of our theory—in §5 we compute the (affine) group of immersions of an $n$ manifold in $R^{2n-1}$, while in §6 we compute the (affine) group of vector 1-fields on a manifold.

## 1. The Category $\mathfrak{X}_B$

Let $B$ be a fixed topological space. We define a category $\mathfrak{X}_B$ as follows: an object of $\mathfrak{X}_B$ is an ordered triple $(E, \check{e}, \hat{e})$ such that $E$ is a topological space, $\hat{e} : E \to B$ is a continuous function, and $\check{e} : B \to E$ is a section of $\hat{e}$, i.e., $\hat{e} \circ \check{e} = 1_B$. If $e = (E, \check{e}, \hat{e})$ and $y = (Y, \check{y}, \hat{y})$ are objects, we say that $g : e \to y$ is a map if $g : E \to Y$ is a topological map and if $\hat{y} \circ g = \hat{e}$ and $g \circ \check{e} = \check{y}$; see McClendon and Becker [9], [1]. We say that two maps in $\mathfrak{X}_B$ are homotopic if there exists a homotopy of $\mathfrak{X}_B$-maps connecting them. Thus, we have the concept of homotopy equivalence in $\mathfrak{X}_B$.

Let $X$ be any space and $f : X \to B$ a map. If $e = (E, \check{e}, \hat{e})$ and $g : X \to E$ is a map such that $\hat{e} \circ g = f$, we say that $g$ is an $f$-map. Two $f$-maps are $f$-homotopic if they are connected by a homotopy of $f$-maps.

Let $[X, f; e]$ be the set of $f$-homotopy classes of $f$-maps from $X$ to $E$. If $A \subset X$ is a subspace, let $[X, A, f; e]$ be the set of rel $A$ $f$-homotopy classes of $f$-maps $X \to E$ which send $A$ to $\check{e}(B)$.

Let $(K, k_0)$ be a pointed CW complex, and let $e = (F, \check{e}, \hat{e})$ be an object in $\mathfrak{X}_B$. We define $e^K = (E^K_B, \check{e}^K, \hat{e}^K)$ as follows: $E^K_B$ is the space of all maps (with the compact-open topology) $g : K \to E$ such that $g(k_0) \in \check{e}(B)$ and $\hat{e} \circ g$ is constant. For all $b \in B$ and $k \in K$, $\check{e}^K(b)(k) = \check{e}(b)$; for all $g \in E^K_B$, $\hat{e}^K(g) = \hat{e} \circ g(k_0)$. Let $\Omega e = e^S$ and $Pe = e^I$, where $S = S^1$ and $I = [0, 1]$ with basepoint 0.

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If $e = (E, \check{e}, \hat{e})$ and $y = (Y, \check{y}, \hat{y})$ are $\mathfrak{X}_B$-objects, we let $e \times y = (Z, \check{z}, \hat{z})$, where Z is the pullback:

$$\begin{array}{ccc} Z & \longrightarrow & E \\ \downarrow & & \downarrow \hat{e} \\ Y & \xrightarrow{\check{y}} & B \end{array}$$

and $\check{z}$ and $\hat{z}$ are defined in the obvious way.

If $\theta : e \to y$ is a map in the category $\mathfrak{X}_B$, we define the *fiber* of $\theta$ to be the object $z = (Z, \check{z}, \hat{z})$, where $Z$ is the pullback:

$$\begin{array}{ccc} Z & \longrightarrow & Y_B^I \\ \downarrow p & & \downarrow \varepsilon \\ E & \xrightarrow{\theta} & Y \end{array}$$

where $\varepsilon$ is evaluation at 1. That is, $Z = \{(e, g) \in E \times Y_B^I \mid \theta(e) = g(1)\}$. We leave to the reader the definitions of $\check{z}$ and $\hat{z}$. If $x$ is any $\mathfrak{X}_B$-object and $h : x \to e$ any map, we say that $x \xrightarrow{h} e \xrightarrow{\theta} y$ is a fibration sequence if there a homotopy equivalence $\phi : x \to z$ (in $\mathfrak{X}_B$, of course) such that the following triangle is homotopy commutative:

$$\begin{array}{ccc} x & & \\ \downarrow \phi & \searrow h & \\ z & \xrightarrow{p} & e \end{array}$$

where, in both of the above diagrams, $p(e, g) = e$ for all $(e, g) \in Z$. We leave to the reader the verification of the following long fibration sequence:

$$\Omega^2 y \xrightarrow{\Omega\lambda} \Omega z \xrightarrow{\Omega p} \Omega e \xrightarrow{\Omega\theta} \Omega y \xrightarrow{\lambda} z \xrightarrow{p} e \xrightarrow{\theta} y.$$

If $f : X \to B$ is any map, where $X$ is a topological space, then $[X, f; e]$ is a set with a distinguished element, $[X, f; \Omega e]$ is a group, and $[X, f; \Omega^2 e]$ is an Abelian group. If we apply the functor $[X, f; \;]$ to a fibration sequence, we obtain an exact sequence; as the reader can easily verify. (See [6] for a similar theorem in a slightly more restricted case.)

Let $e = (E, \check{e}, \hat{e})$ be any object of the category $\mathfrak{X}_B$, and let $(Y, y_0)$ be a pointed space. We define $e \wedge Y$ to be the $\mathfrak{X}_B$-object

$$(E \wedge_B Y, \check{e} \wedge Y, \hat{e} \wedge Y),$$

where $E \wedge_B Y$ is the quotient space of $E \times Y$ under the equivalence relation, $(x, y_0) \sim (x', y_0)$ whenever $\hat{e}x = \hat{e}x'$, and $(\check{e}b, y) \sim (\check{e}b, y')$ for all $b \in B$ and $y, y' \in Y$. We let $[x, y]$ denote the equivalence class of $(x, y) \in E \times Y$, and let $(\check{e} \wedge Y)b = [\check{e}b, y_0]$ for all $b \in B$, and $(\hat{e} \wedge Y)[x, y] = \hat{e}x$ for all $[x, y] \in E \wedge_B Y$.

If $e = (E, \check{e}, \hat{e})$ is any $\mathfrak{X}_B$-object, we write

$$H^*(e; G) = H^*(E, \check{e}(B); G)$$

for any group $G$. We leave it to the reader to verify the following simple remarks, where $F$ is a field: (Henceforth, we assume that all spaces have singular homology of finite type.)

*Remark* 1.1. If $e$ and $e'$ are $\mathfrak{X}_B$-objects,

$$H^*(e \times e'; F) = H^*(e; F) \otimes_F H^*(e'; F).$$

*Remark* 1.2. If $e$ is an $\mathfrak{X}_B$-object and $Y$ is a pointed space,

$$H^*(e \wedge Y; F) = H^*(e; F) \otimes_F H^*(Y; F).$$

## 2. The problem

Suppose $\theta : e \to y$ is any twice deloopable $\mathfrak{X}_B$-map, i.e., there exist $\mathfrak{X}_B$-objects $e''$ and $y''$ and a map $\theta''$ such that $\Omega^2 e'' = e$, $\Omega^2 y'' = y$, and $\Omega^2 \theta'' = \theta$. Let $X$ be a CW-complex and $f : X \to E$ a map. According to §1, we then have an exact sequence of Abelian groups:

$$0 \longrightarrow A = \operatorname{Coker} \Omega\theta_\# \xrightarrow{\lambda_\#} [X, f; z] \xrightarrow{p_\#} C = \operatorname{Ker} \theta_\# \longrightarrow 0$$

where $p : z \to e$ is the fiber of $\theta$. Our problem is to evaluate $[X, f; z]$ as an extension of $C$ by $A$.

DEFINITION 2.1. For each positive integer $n$, let

$$C_n = \{x \in C \mid nx = 0\}.$$

Let $\Phi_n : C_n \to A/nA$ be the homomorphism given by

$$\Phi_n(x) = \lambda_\#^{-1} n p_\#^{-1} x \quad \text{for all } x \in C_n .$$

According to Theorem 5.1 of [6], it is only necessary to know $\Phi_n$ for all $n$ which are powers of primes in order to solve our problem. If $2A = 0$ or $2C = 0$, it suffices to know $\Phi_2$.

## 3. The general theory

Suppose that $\psi : v \to w$ is any map in the category $\mathfrak{X}_B$; we then have a long fibration sequence

$$\Omega v \xrightarrow{\Omega\psi} \Omega w \xrightarrow{\chi} u \xrightarrow{\gamma} v \xrightarrow{\psi} w$$

where $u$ is the fiber of $\psi$. Let $X$ be a CW complex, and let $f : X \to B$ be a map. Let

$$L \xrightarrow{i} K \xrightarrow{q} K/L \xrightarrow{r} SL \xrightarrow{Si} SK$$

be any cofibration of pointed CW complexes. We shall assume that $L$ and $K$ are suspensions, though $i$ need not be a suspension. We leave it to the reader to verify that the following diagram of $\mathfrak{X}_B$ -objects and maps is commutative, and that all rows and columns are fibration sequences (this diagram is analogous to diagram (2.5) of [7]):

(3.1)
$$\begin{array}{ccccccccc}
 & & & & & & & & \Omega v^L \\
 & & & & & & & & \downarrow{\scriptstyle \Omega\theta^L} \\
 & & & & & & \Omega w^K & \xrightarrow{\Omega w^i} & \Omega w^L \\
 & & & & & & \downarrow{\scriptstyle \chi^K} & & \downarrow{\scriptstyle \chi^L} \\
 & & & & & & u^K & \xrightarrow{u^i} & u^L \\
 & & & & & & \downarrow{\scriptstyle \gamma^K} & & \downarrow{\scriptstyle \gamma^L} \\
 & & v^{SL} & \xrightarrow{v^r} & v^{K/L} & \xrightarrow{v^q} & v^K & \xrightarrow{v^i} & v^L \\
 & & \downarrow{\scriptstyle \theta^{SL}} & & \downarrow{\scriptstyle \theta^{K/L}} & & \downarrow{\scriptstyle \theta^K} & & \\
w^{SK} & \xrightarrow{w^{Si}} & w^{SL} & \xrightarrow{w^r} & w^{K/L} & \xrightarrow{w^q} & w^K & &
\end{array}$$

If we apply the functor $[X, f;\ \ ]$ to Diagram (3.1), we obtain a commutative diagram of groups with exact rows and columns:

(3.2)
$$\begin{array}{ccccccc}
 & & [X, f; v^{SL}] & \xrightarrow{v^r_\#} & [X, f; v^{K/L}] & \xrightarrow{v^q_\#} & \\
 & & \downarrow{\scriptstyle \theta^{SL}} & & \downarrow{\scriptstyle \theta^{K/L}} & & \\
[X, f; w^{SK}] & \xrightarrow{w^{Si}_\#} & [X, f; w^{SL}] & \xrightarrow{w^r_\#} & [X, f; w^{K/L}] & \xrightarrow{w^q_\#} &
\end{array}$$

$$\begin{array}{ccccc}
 & & & & [X, f; \Omega v^L] \\
 & & & & \downarrow{\scriptstyle \Omega\theta^L_\#} \\
 & & [X, f; \Omega w^K] & \xrightarrow{\Omega w^i_\#} & [X, f; \Omega w^L] \\
 & & \downarrow{\scriptstyle \chi^K_\#} & & \downarrow{\scriptstyle \chi^L_\#} \\
 & & [X, f; u^K] & \xrightarrow{u^i_\#} & [X, f; u^L] \\
 & & \downarrow{\scriptstyle \gamma^K_\#} & & \downarrow{\scriptstyle \gamma^L_\#} \\
 & \xrightarrow{v^q_\#} & [X, f; v^K] & \xrightarrow{v^i_\#} & [X, f; v^L] \\
 & & \downarrow{\scriptstyle \theta^K_\#} & & \\
 & \xrightarrow{w^q_\#} & [X, f; w^K] & &
\end{array}$$

We define maps

$$\Phi : \operatorname{Ker} v^i_\# \cap \operatorname{Ker} \theta^K_\# \to [X, f; \Omega w^L]/\Omega w^i_\#[X, f; \Omega w^K] + \Omega\theta^L_\#[X, f; \Omega v^L],$$

$$\tilde{\Phi} : \operatorname{Ker} v^i_\# \cap \operatorname{Ker} \theta^K_\# \to [X, f; w^{SL}]/w^{Si}_\#[X, f; w^{SK}] + \theta^{SL}_\#[X, f; v^{SL}]$$

as follows: for any $x \in [X, f; v^K]$ such that $v^i_\# x = 0$ and $\theta^K_\# x = 0$, let

$$\Phi(x) = (\chi^L_\#)^{-1}x \quad \text{and} \quad \tilde{\Phi}(x) = (w^r_\#)^{-1}\theta^{K/L}_\#(v^q_\#)^{-1}x.$$

Now $\Omega w^L$ and $w^{SL}$ are of the same homotopy type in the category $\mathfrak{X}_B$ ; we can identify them in such a manner that the following theorem, analogous to Theorem 2.5 of [6], holds:

THEOREM 3.2. $\Phi = \tilde{\Phi}$.

We leave the proof to the reader. Note that if the map $\theta$ is deloopable, i.e., $\theta = \Omega\psi$ for some $\psi$, $\Phi = \tilde{\Phi}$ is a homomorphism.

*Remark* (added in proof). We take this opportunity to correct an error (of sign) that occurs in [7]. Namely, Theorem 2.5 should read $-\Phi_1 = \Phi_2$, while in Corollary 3.7, a minus sign should be appended to the left hand side of each equation. The error occurs at the top of page 232 where the fifth line should read

$$-\phi_1 = \tilde{\Phi}_1(\mathrm{u}), \quad \phi_2 = \tilde{\Phi}_2(\mathrm{u}).$$

For any integer $n \geq 1$ and any group $\pi$ (where $\pi$ is Abelian if $n > 1$), and for any $a \in H^1(B; \text{Aut}\, \pi)$, we say that the $\mathfrak{X}_B$-object $k_B(\pi, n, a) = (K, \check{k}, \hat{k})$ is an Eilenberg MacLane object of type $(\pi, n, a)$ if $k_B(\pi, n, a)$ is of the homotopy type of an object $(K, \check{k}, \hat{k})$ where $\hat{k} : K \to B$ is a fibration with fiber an Eilenberg-MacLane space of type $(\pi, n)$, and if $a$ classifies the action of the fundamental group of $B$ on $\pi_n$ of that fiber. See Gitler [2] and Siegel [11] for construction of $k_B(\pi, n, a)$, which we briefly describe as follows. Let $K'$ be an Eilenberg-MacLane space of type $(\pi, n)$, and let $\Gamma = \text{Aut}\, \pi$, the automorphism group of $\pi$, which acts on $K'$ in the obvious way. Let $W$ be a $\Gamma$-free acyclic complex. Now projection onto the second factor, $p : K' \times W \to W$ induces a fibration

$$q : K' \times W/\Gamma \to W/\Gamma = K(\Gamma, 1).$$

Let $f_a : B \to K(\Gamma, 1)$ be a map which classifies $a$, and define $K$ and $\hat{k}$ by the pullback diagram

$$\begin{array}{ccc} K & \longrightarrow & K' \times W/\Gamma \\ \downarrow{\scriptstyle \hat{k}} & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{f_a} & K(\Gamma, 1) \end{array}$$

Inclusion along the second factor $W \to K' \times W$ induces a lifting

$$K(\Gamma, 1) \to K' \times W/\Gamma$$

which induces the lifting $\check{k} : B \to K$.

Now if $(X, A)$ is a CW pair and $f : X \to B$ is a map, then

$$[X, A, f; k_B(\pi, n, a)] = H^n(X, A; \pi[f^*a]),$$

where $\pi[f^*a]$ is the local system of groups over $X$, isomorphic to $\pi$, classified by $f^*a \in H^1(X, \Gamma)$. Let $\iota_n \in H^n(K, k(B); \pi[a])$ be the fundamental class of $k_B(\pi, n, a)$, classified by the identity map.

In the special case that $\pi = Z_2$, then $\Gamma = 0$. We write $k_B(Z_2, n)$ for $k_B(Z_2, n, 0)$.

Let $\mathcal{A}$ be the mod 2 Steenrod algebra; take cohomology with $Z_2$ coefficients. We define an algebra over $Z_2$, $H^*(B)\cdot\mathcal{A}$, the semi-tensor product (see [8]), as follows. As a module over $Z_2$, $H^*(B)\cdot\mathcal{A} = H^*(B) \otimes \mathcal{A}$. Its multiplication is the composition

$$H^*(B) \otimes \mathcal{A} \otimes H^*(B) \otimes \mathcal{A} \xrightarrow{1 \otimes A \otimes 1} H^*(B) \otimes H^*(B) \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow{\cup \otimes M} H^*(B) \otimes \mathcal{A}$$

where $A$ is the composition

$$\mathcal{A} \otimes H^*(B) \xrightarrow{\mu \otimes 1} \mathcal{A} \otimes \mathcal{A} \otimes H^*(B) \xrightarrow{1 \otimes T} \mathcal{A} \otimes H^*(B) \otimes \mathcal{A} \xrightarrow{\alpha \otimes 1} H^*(B) \otimes \mathcal{A}$$

where $\mu$ is the comultiplication of $\mathcal{A}$, $T$ exchanges coordinates, and $\alpha$ is the action of $\mathcal{A}$ on $H^*(B)$.

Now if $(X, A)$ is a CW pair and $f : X \to B$ is a map, $H^*(X, A)$ is a module over $H^*(B)\cdot\mathcal{A}$ in an obvious way; if

$$x \in H^*(X, A) \quad \text{and} \quad b \otimes \theta \in H^*(B)\cdot\mathcal{A},$$

for $b \in H^*(B)$ and $\theta \in \mathcal{A}$, then let $(b \otimes \theta)x = (f^*b)\theta x$. We leave it to the reader to verify this action.

Let $n \geq 1$ and $m \geq 1$ be integers. Let $\pi$ and $\tau$ be groups; $\pi$ Abelian if $n > 1$, $\tau$ Abelian if $m > 1$. Let $a \in H^1(B; \text{Aut } \pi)$ and $b \in H^1(B; \text{Aut } \tau)$. Let

$$[k_B(\pi, n, a); k_B(\tau, m, b)]$$

denote $[K, \check{k}(B), k; k_B(\tau, m, b)]$, where $k_B(\pi, n, a) = (K, \check{k}, \hat{k})$. The elements of

$$[k_B(\pi, n, a); k_B(\tau, m, b)]$$

we call *cohomology operations* of type $(\pi, n, a; \tau, m, b)$.

Applying the functor $\Omega$ to any map, we obtain a "suspension"

$$\sigma : [k_B(\pi, b, a); k_B(\tau, m, b)] \to [k_B(\pi, n-1, a); k_B(\tau, m-1, b)]$$

if $n, m > 1$. Let ${}^1\psi$ denote $\sigma\psi$ for any $\psi$.

If $\pi$ and $\tau$ are Abelian and if $k$ is any integer, let $\mathcal{S}^k[\pi, a, \tau, b]$ denote the set of stable operations of type $(\pi, a, \tau, b)$ and degree $k$, defined as the inverse limit as $n$ approaches $\infty$ (via $\sigma$) of $[k_B(\pi, n, a); k_B(\tau, n + k, b)]$.

Finally, we remark that $H^*(B)\cdot\mathcal{A}$ can be identified with $\mathcal{S}^*[Z_2, Z_2]$, the algebra of stable operations of type $(Z_2, 0, Z_2, 0)$ as follows: if $n$ and $m$ are integers and $b \otimes \theta \in H^*(B) \cdot \mathfrak{A}$, let $b \otimes \theta$ correspond to

$$(b \otimes \theta)\iota_n = (k^*b)\theta\iota_n \in H^m(K, \check{k}(B); Z_2) = [k_B(Z_2, n); k_B(Z_2, m)]$$

where $k_B(Z_2, n) = (K, \check{k}, \hat{k})$.

We wrote $\mathfrak{A}_B = H^*(B) \cdot \mathfrak{A}$. Let $\varepsilon : \mathfrak{A}_B \to \mathfrak{A}_B$ be the homomorphism, of degree $-1$, given by $\varepsilon(b \otimes \theta) = b \otimes \varepsilon\theta$, where $\varepsilon : \mathfrak{A} \to \mathfrak{A}$ is the Kristensen map, dual to multiplication by $\xi_1$ in the dual algebra [4]. For any $\psi \in \mathfrak{A}_B$, we let $\tilde{\psi} = \varepsilon\psi$. [5], [7].

## 4. The functor $P$

Henceforth in this paper, all coefficients will be in $Z_2$, unless otherwise specified.

Consider diagram (3.2) with the cofibration

$$S \xrightarrow{\gamma} S \xrightarrow{q} P \xrightarrow{r} S^2 \xrightarrow{S\gamma} S^2,$$

where $S = S^1$ and $\gamma$ is multiplication by 2. Then $-\tilde{\Phi} = \Phi$, which equals the homomorphism $\Phi_2$ defined in Section 2.

We consider only cases in which $v$ and $w$ are both Eilenberg-MacLane objects of type $(Z_2, n)$ or $(Z, n, a)$, and where $\theta$ is a stable cohomology operation. $P$ is the real projective plane; for $i = 1, 2$, let $e^i \in H^i(P)$ be the generator mod 2.

We wish to compute the operation

$$\theta^P : k_B(\pi, n, a)^P \to k_B(\tau, m, b)^P,$$

where $\theta : k_B(\pi, n, a) \to k_B(\tau, m, b)$. We first note the following facts: If $\theta$ and $\theta'$ are two cohomology operations where $\theta + \theta'$ is meaningful, then $(\theta + \theta')^P = \theta^P + \theta'^P$. If $\theta$ and $\theta'$ are operations where $\theta \circ \theta'$ is meaningful, then $(\theta \circ \theta')^P = \theta^P \circ \theta'^P$.

Henceforth, let $k_n$ denote $k_B(Z_2, n)$ and let $k_n^*(a)$ denote $k_B(Z, n, a)$ for any integer $n \geq 1$ and any $a \in H^1(B)$. (Since Aut $Z \cong Z_2$.) If $n \geq 3$, we have (as in the untwisted case) $k_n^*(a)^P = k_{n-2}$, and $k_n^P = k_{n-2} \times k_{n-1}$, where product is taken in the category $\mathfrak{X}_B$, i.e., over $B$. The proofs are essentially identical to those of corresponding theorems in the untwisted case [7]; we leave them to the reader. The following is the analogue of Theorem 3.6 of [7]:

THEOREM 4.1. *Let $\theta$ be a stable cohomology operation.*

*Case* I. $\theta : k_n \to k_m$. *Then*

$$(\theta^P)^*(\iota_{m-2} \otimes 1) = \theta\iota_{n-2} \otimes 1 + 1 \otimes \tilde{\theta}\iota_{n-1}, \quad \textit{and}$$

$$(\theta^P)^*(1 \otimes \iota_{m-1}) = 1 \otimes \theta\iota_{n-1}.$$

*Case* II. $\theta : k_n \to k_{n+1}^*(a)$ *is the Bokstein homomorphism of the exact se-*

*quence of sheaves*

$$0 \to Z[a] \xrightarrow{\times 2} Z[a] \to Z_2 \to 0.$$

*Then* $(\theta^P)^* \iota_{n-1} = (Sq^1 + a)\iota_{n-2} \otimes 1 + 1 \otimes \iota_{n-1}$.

*Case* III. $\theta : k_n^*(a) \to k_n$ *is reduction mod* 2. *Then*

$$(\theta^P)^*(\iota_{n-2} \otimes 1) = \iota_{n-2} \quad \textit{and} \quad (\theta^P)^*(1 \otimes \iota_{n-1}) = (Sq^1 + a)\iota_{n-2}.$$

*Proof.* We do the details of the proof only for Case I. The homotopy equivalence $k_{n-2} \times k_{n-1} \to k_n^P$ can be chosen to be adjoint to an $\mathfrak{X}_B$-map $f_n : (k_{n-2} \times k_{n-1}) \wedge P \to k_n$ such that

$$f_n^* \iota_n = \iota_{n-2} \otimes 1 \otimes e^2 + 1 \otimes \iota_{n-1} \otimes e^1.$$

We have a commutative diagram of $\mathfrak{X}_B$-objects and maps

$$\begin{array}{ccc} (k_{n-2} \times k_{n-1}) \wedge P & \xrightarrow{f_n} & k_n \\ \big\downarrow{\scriptstyle \theta^P \wedge P} & & \big\downarrow{\scriptstyle \theta} \\ (k_{m-2} \times k_{m-1}) \wedge P & \xrightarrow{f_m} & k_m \end{array}$$

which induces a commutative diagram in mod 2 cohomology:

$$\begin{array}{ccc} H^*((k_{n-2} \times k_{n-1}) \wedge P) & \xleftarrow{f_n^*} & H^*(k_n) \\ \big\uparrow{\scriptstyle (\theta^P)^* \otimes 1} & & \big\uparrow{\scriptstyle \theta^*} \\ H^*((k_{m-2} \times k_{m-1}) \wedge P) & \xleftarrow{f_m^*} & H^*(k_m). \end{array}$$

We have that $\theta = \sum_{i=1}^N (b_i \otimes \psi_i)$ for some integer $N$ and some choices of $\psi_i \in \mathfrak{A}$ and $b_i \in H^*(B)$ (the index $i$ does not denote degree). Note that $\tilde{\theta} = \sum_{i=1}^N (b_i \otimes \tilde{\psi}_i)$. Now

$$\begin{aligned} f_n^* \theta^* \iota_m &= f_n^* \textstyle\sum_{i=1}^N (b_i \otimes \psi_i)\iota_n \\ &= \textstyle\sum_{i=1}^N (b_i \otimes \psi_i(\iota_{n-2} \otimes 1 \otimes e^2 + 1 \otimes \iota_{n-1} \otimes e^1)) \\ &= \textstyle\sum_{i=1}^N (b_i \otimes (\psi_i \iota_{n-2} \otimes 1 \otimes e^2 + 1 \otimes \psi_i \iota_{n-1} \otimes e^1 + 1 \otimes \tilde{\psi}_i \iota_{n-1} \otimes e^2). \end{aligned}$$

On the other hand, $f_m^* \iota_m = \iota_{m-2} \otimes 1 \otimes e^2 + 1 \otimes \iota_{m-1} \otimes e^1$. Comparing coefficients, we see that

$$\begin{aligned} (\theta^P)^*(\iota_{m-2} \otimes 1) &= \textstyle\sum_{i=1}^N (b_i \otimes (\psi_i \iota_{n-2} \otimes 1 + 1 \otimes \tilde{\psi}_i \iota_{n-1})) \\ &= \theta(\iota_{n-2} \otimes 1) + \tilde{\theta}(1 \otimes \iota_{n-1}), \end{aligned}$$

while $(\theta^P)^*(1 \otimes \iota_{m-1}) = \sum_{i=1}^N b_i \otimes (1 \otimes \psi_i \iota_{n-1}) = \theta(1 \otimes \iota_{n-1})$, as claimed.

## 5. Applications to immersions

Let $\pi : B \to B'$ be a fibration, where the fiber is $(r-1)$-connected for some $r$. If $X$ is a CW complex of dimension $n$, and if $f : X \to B'$ is a map, let $L(X, B, f)$ be the set of rel $f$ homotopy classes of liftings of $f$ to $B$. If $n \leq 2r - 2$, then $L(X, B, f)$ naturally has the structure of an affine group, according to Becker [1].

Let us assume that $f$ has a lifting, $g$. Then $L(X, B, f)$ is naturally an Abelian group with identity $[g]$, isomorphic to $[X, g, e]$, where $e = (E, \check{e}, \hat{e})$ is the $\mathfrak{X}_B$ -object, where $E$ is the pullback

$$\begin{array}{ccc} E & \longrightarrow & B \\ \Big\downarrow{\scriptstyle \hat{e}} & & \Big\downarrow{\scriptstyle \pi} \\ B & \xrightarrow{\ \pi\ } & B' \end{array}$$

and $\check{e}(b) = (b, b) \in E$ for all $b \in B$. To compute the group structure on $L(X, B, f)$, we then use McClendon's techniques [9] to obtaiin a Postnikov tower for $e$ (in the category $\mathfrak{X}_B$, of course), and hence a spectral sequence for $[X, g, e]$. (See also [6], Section 5.)

Consider the fibration $\pi : B \to B'$, where $B = BO_r$ and $B' = BO$. Let $e$ be the $\mathfrak{X}_B$ -object defined above. Let $M$ be a connected, smooth, $n$-dimensional manifold, and let $\nu : M \to B'$ classify the stable normal bundle of $M$. The set of regular homotopy classes of immersions of $M$ into $R^{n+r}$ is in one-to-one correspondence with $[M, g; e]$, if $g : M \to B$ classifies the normal bundle of any immersion $M \subseteq R^{n+r}$. If $n \leq 2r - 2$, $[M, g; e]$ is an Abelian group, which we call the immersion group, $Im_{n+r}(M)$, and which does not depend on the choice of $g$ (up to isomorphism) [1], [3]. In this section we shall compute $\mathrm{Im}_{2n-k}(M)$ for sufficiently small $k$. Toward that end, we construct a Postnikov tower for $e$. In the range we are considering, four cases are necessary, corresponding to the equivalence class of $r$ modulo 4.

*Case* I. $r \equiv 1$ (4), $r \geq 5$. The following diagram is the first two stages of the Postnikov tower for $e$ (recall that all objects and maps in this diagram are in the category $\mathfrak{X}_B$):

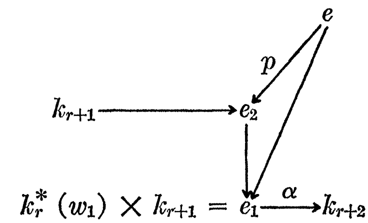
$$\begin{array}{ccccc} & & & & e \\ & & & \swarrow{\scriptstyle p} & \\ k_{r+1} & \longrightarrow & e_2 & & \\ & & \Big\downarrow{\scriptstyle \pi} & \swarrow & \\ k_r = & & e_1 & \xrightarrow{\ \alpha\ } & k_{r+2} \end{array}$$

where $\alpha^{*}\iota_{r+2}$ W $(1 \otimes Sq^2 + w_2 \otimes 1)\iota_r$. Also, $k_{r+1}$ is the fiber of $\pi$, $e_2$ is the fiber of $\alpha$, and $p$ is an equivalence through dimension $r + 1$; i.e.,

$$p_{\#} : [X, h; e] \to [X, h; e_2]$$

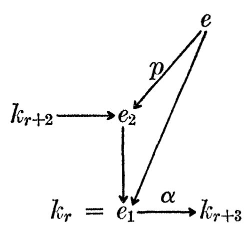
is an isomorphism for any complex $X$ of dimension $\leq r + 1$.

*Case* II. $r \equiv 2\ (4)$, $r \geq 6$. The Postnikov tower of $e$, where $p$ is an equivalence through dimension $r + 2$, begins



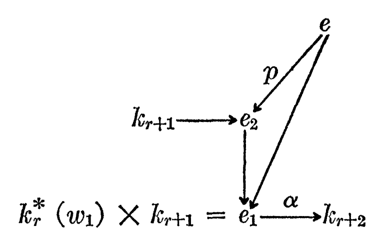

where $\alpha^* \iota_{r+2} = (1 \otimes Sq^2 + w_2 \otimes 1)\iota_r \otimes 1 + 1 \otimes (1 \otimes Sq^1)\iota_{r+1}$.

*Case* III. $r \equiv 3\ (4)$, $r \geq 3$. Then




where $\alpha^* \iota_{r+3} = (1 \otimes Sq^2 Sq^1 + (w_2 + w_1^2) \otimes Sq^1)\iota_r$; $p$ is an equivalence through dimension $r + 1$ if $r = 3$, $r + 2$ if $r > 3$.

*Case* IV. $r \equiv 0\ (4)$, $r \geq 4$. Then




where $\alpha^* \iota_{r+2} = (1 \otimes Sq^2 + w_2 \otimes 1)\iota_r \otimes 1 + 1 \otimes (w_1 \otimes 1)\iota_{r+1}$; $p$ is an equivalence through dimension $r + 1$.

We then obtain, via Theorem 4.1, the following (in each case, $w_i$ and $\bar{w}_i$ are the $i^{\text{th}}$ Stiefel-Whitney classes of the tangent bundle and the normal bundle of $M$, respectively):

THEOREM 5.1. *Assume $n \geq 4$. As above, $M$ is a connected, smooth $n$-manifold. Then $\mathrm{Im}_{2n-1}(M)$ is as follows.*

*Case* I. $n \equiv 1\ (4)$, *$M$ orientable. Then*

$$\mathrm{Im}_{2n-1}(M) \cong H^{n-1}(M; Z) \oplus Z_2 \oplus Z_2 .$$

*Case* II. $n \equiv 1\ (4)$, *$M$ non-orientable. Then*

$$\mathrm{Im}_{2n-1}(M) \cong H^{n-1}(M; Z[w_1]) \oplus Z_2 .$$

*Case* III. $n \equiv 2$ (4), *M orientable. Then*

$$\mathrm{Im}_{2n-1}(M) \cong H^{n-1}(M) \oplus Z_2 .$$

*Case* IV. $n \equiv 2$ (4), *M non-orientable. Then*

$$\mathrm{Im}_{2n-1}(M) \cong K \oplus Z_4 ,$$

*where K is the kernel of* $Sq^1 : H^{n-1}(M) \to H^n(M)$.

*Case* V. $n \equiv 3$ (4), *M orientable. Then*

$$\mathrm{Im}_{2n-1}(M) \cong H^{n-1}(M; Z) \oplus Z_4 .$$

*Case* VI. $n \equiv 3$ (4), *M non-orientable. Then*

$$\mathrm{Im}_{2n-1}(M) \cong H^{n-1}(M; Z[w_1]) \oplus Z_2 .$$

*Case* VII. $n \equiv 0$ (4), *and M immerses in* $R^{2n-1}$. *Then*

$$\mathrm{Im}_{2n-1}(M) \cong H^{n-1}(M).$$

*Proof. Cases* I *and* II. We have an exact sequence

$$\cdots \to H^{n-2}(M; Z[w_1]) \oplus H^{n-1}(M) \xrightarrow{\theta} H^n(M) \to \mathrm{Im}_{2n-1}(M)$$
$$\to H^{n-1}(M; Z[w_1]) \oplus H^n(M) \to 0$$

where $\theta(u, v) = (Sq^2 + \bar{w}_2)u + w_1 v$. In the oriented case, $\theta = 0$, and

$$\Phi_2(u, v) = (Sq^2 + \bar{w}_2)\, \delta^{-1} u,$$

according to Theorem 4.1. But $Sq^2 + \bar{w}_2 = 0 : H^{n-2}(M) \to H^n(M)$, so $\Phi_2 = 0$, and the extension is trivial. In the unoriented case, Coker $\theta = 0$ since $w_1 v$ is the top class for some $v \in H^{n-1}(M)$; there is no extension problem.

*Cases* III *and* IV. We have an exact sequence

$$\cdots \to H^{n-2}(M) \xrightarrow{\theta} H^n(M) \to \mathrm{Im}_{2n-1}(M) \to H^{n-1}(M) \to 0$$

where $\theta u = (Sq^2 + \bar{w}_2)u = 0$. Now for $v \in H^{n-1}(M)$, $\Phi_2(v) = Sq^1 v$, and we are done.

*Cases* V *and* VI. We have an exact sequence

$$\cdots \to H^{n-2}(M; Z[w_1]) \oplus H^{n-1}(M) \xrightarrow{\theta} H^n(M) \to \mathrm{Im}_{2n-1}(M)$$
$$\to H^{n-1}(M; Z[w_1]) \oplus H^n(M) \to 0$$

where $\theta(u, v) = (Sq^2 + \bar{w}_2)u + Sq^1 v$. In the orientable case, $\theta = 0$, and $\Phi_2(u, v) = Sq^1 u + v = v$ for all $u \in H^{n-1}(M; Z)$, $v \in H^n(M)$, and we are done. In the unoriented case, Coker $\theta = 0$ and there is no extension problem.

Case VII has a trivial proof, which we leave to the reader.

*Remark.* For calculation, it is useful to recall that by duality,

$$H^{n-i}(M; Z[w_1]) \cong H_i(M; Z), \quad 0 \le i \le n.$$

THEOREM 5.2. *Let $M$ be a connected smooth $n$-manifold, for $n \equiv 1$ (4), $n \geq 9$. Let $\gamma : H^{n-2}(M) \to H^n(M)$ be multiplication by $w_1^2$. Then* $\mathrm{Im}_{2n-2}(M)$ *is as follows (provided $M \subseteq R^{2n-2}$).*

*Case* I. $\gamma = 0$. *Then* $\mathrm{Im}_{2n-2}(M) \cong H^{n-2}(M) \oplus Z_2$.
*Case* II. $\gamma \neq 0$, *but* $\gamma Sq^1 = 0 : H^{n-3}(M) \to H^n(M)$. *Then*

$$\mathrm{Im}_{2n-2}(M) \cong \mathrm{Ker}\,\gamma \oplus Z_4 .$$

*Case* III. $\gamma\, Sq^1 \neq 0$. *Then* $\mathrm{Im}_{2n-2}(M) \cong H^{n-2}(M)$.

*Proof.* We have an exact sequence

$$\cdots H^{n-3}(M) \xrightarrow{\theta} H^n(M) \to \mathrm{Im}_{2n-2}(M) \to H^{n-2}(M) \to 0$$

where $\theta u = Sq^2 Sq^1 u + (\bar{w}_2 + w_1^2) Sq^1 u = w_1^2 Sq^1 u$. If $x \in H^{n-2}(M)$,

$$\Phi_2(x) = (Sq^2 + \bar{w}_2 + w_1^2)x = w_1^2 x;$$

we are done.

Examples of manifolds satisfying the three conditions are $P_{13}$, $P_{12} \times S^1$, and $P_2 \times S^{11}$, respectively (where $P_k$ = real projective $k$-space).

Finally, suppose that $M$ is a smooth connected $n$-manifold, where $n \equiv 0$ (4), $n \geq 8$, and $M$ immerses in $R^{2n-2}$. There is no particularly neat way of expressing the group $\mathrm{Im}_{2n-2}$ in general, but the information below is sufficient to determine it. First of all we have an exact sequence:

$$\cdots \to H^{n-3}(M; Z[w_1]) \oplus H^{n-2}(M) \xrightarrow{\theta} H^{n-1}(M) \to \mathrm{Im}_{2n-2}(M)$$
$$\to H^{n-2}(M; Z[w_1]) \oplus H^{n-1}(M) \to 0$$

where

$$\theta(u, v) = (Sq^2 + \bar{w}_2)u + w_1 v$$

for all $u \in H^{n-3}(M; Z[w_1])$ and $v \in H^{n-2}(M)$: secondly,

$$\Phi_2(x, y) = (Sq^2 + \bar{w}_2)\delta^{-1}x + Sq^1 \rho x$$

for all $x \in H^{n-2}(M; Z[w_1])$ such that $2x = 0$ and all $y \in H^{n-1}(M)$. ($\rho$ = reduction modulo 2.)

For completeness' sake, we mention that if $n \geq 2$,

$$\mathrm{Im}_{2n}(M) \cong Z \quad \text{if } n \text{ is even}$$
$$\cong Z_2 \quad \text{if } n \text{ is odd}$$

and that if $n \geq 1$, $\mathrm{Im}_{2n+k}(M) = 0$ for all $k \geq 1$. We leave the proofs to the reader; cf. [3].

These results extend those in [4] and [10]. In [4], only the cardinality of the immersion group was computed, while in [10] an exact sequence for the group was constructed, but no extensions were computed.

## 6. Applications to vector fiields

Throughout this section, let $M$ be a smooth connected $n$-dimensional manifold. Let $V^k(M)$ be the set of homotopy classes of $k$-fields on $M$. Now $V^k(M)$ is in one-to-one correspondence with the set of rel $\tau$ homotopy classes of liftings of $\tau$ to $BO_{n-k}$ :

$$\begin{array}{ccc} & & BO_{n-k} \\ & & \downarrow \\ M & \xrightarrow{\tau} & BO_n \end{array}$$

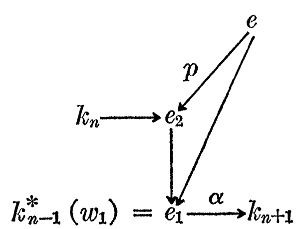
where $\tau$ classifies the tangent bundle. If $n \geq 2k + 2$, $V^k(M)$ is an affine group [1].

Let $B = BO_{n-k}$, and $B' = BO_n$ ; then $V^k(M) \cong [M, g; e]$, where $e$ is the $\mathfrak{X}_B$ -object defined in the previous section and $g : M \to B$ is any given lifting of $\tau$. The techniques of the previous sections can then be applied to compute $[M, g; e]$. We have a complete answer only in the case $k = 1$.

THEOREM 6.1. *If $n \geq 4$ and $M$ admits a vector field, then*

$$V^1(M) = H^{n-1}(M; Z[w_1]) \oplus H^n(M) = H_1(M; Z) \oplus Z_2 .$$

*Proof.* We have a Postnikov tower $e$, where $p$ is an equivalence through dimension $n$:




where $\alpha^* \iota_{n+1} = (Sq^2 + w_2)\iota_{n-1}$. We have an exact sequence

$$\cdots \to H^{n-2}(M; X[w_1]) \xrightarrow{\theta} H^n(M) \to V^1(M) \to H^{n-1}(M; Z[w_1]) \to 0$$

where

$$\theta x = (Sq^2 + w_2)x = w_1^2 x = w_1 Sq^1 x = Sq^1 Sq^1 x = 0.$$

If $y \in H^{n-1}(M; X[w_1])$ and $2y = 0$, choose $u \in H^{n-2}(M)$ such that $\delta u = y$. Then

$$\Phi_2 y = (Sq^2 + w_2)u + Sq^1 \rho y = w_1(w_1 u + \rho y)$$
$$= w_1(w_1 u + (Sq^1 + w_1)\rho y) = Sq^1 Sq^1 \rho y = 0;$$

we are done.

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