GROUPS WHOSE SYLOW SUBGROUPS ARE THE DIRECT PRODUCT OF TWO SEMI-DIHEDRAL GROUPS

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1. Introduction

The purpose of this paper is to classify all finite fusion-simple groups whose Sylow 2-subgroups are the direct product of two semi-dihedral groups. We prove the following:

THEOREM. *If G is a finite fusion-simple group* ($O^2(G) = G$ *and* $O(G) = Z(G) = 1$) *whose Sylow 2-subgroups are a direct product of two semi-dihedral groups, then G possesses a normal subgroup of odd index of the form $F_1 \times F_2$ where $F_i \cong M_{11}, L_3(q_i), q_i \equiv -1 \pmod 4$, or $U_3(q_i), q_i \equiv 1 \pmod 4$, $i = 1, 2$.*

In an unpublished paper John Thompson proved the following:

If T is a Sylow 2-subgroup of a nonabelian simple group G, T contains no normal elementary subgroups of order greater than 4, $N_G(T) = TC_G(T)$, and $Z(T)$ is noncyclic, then $T = T_1 \times T_2$ where T_i is a dihedral or a semi-dihedral group, $i = 1, 2$.

Our theorem, combined with the main results of [5] and [6], shows that there is no simple group whose Sylow 2-subgroups satisfy the conditions of Thompson's result.

Since many of the arguments of this paper are quite similar to corresponding ones in [6], we have omitted the proofs of some lemmas. It is thus necessary that the reader is familiar with [6] and the notation and definitions in that paper.

2. Centralizers of involutions

Henceforth, G denotes a minimal counter-example to our theorem and S is a Sylow 2-subgroup of G.

As in Section 3 of [6], we can find semi-dihedral subgroups S_1 and S_2 in S such that $S = S_1 \times S_2$ and all the involutions and elements of order 4 in S_i are conjugate in G. Let $\langle x_1 \rangle = Z(S_1)$ and $\langle y_1 \rangle = Z(S_2)$. Then the involutions x_1, y_1, and $x_1 y_1$ are mutually nonconjugate in G. It is easy to see that all involutions and elements of order 4 in S_2 are conjugate in $C = C_G(x_1)$. An analogous statement holds for $C_G(y_1)$. Let $\langle x_1, x_2 \rangle$ be a four-group in S_1. Then x_2 is not conjugate in C to any involution in $T = T_1 \times S_2$ where T_1 is a generalized quaternion group of index 2 in S_1. It follows that C has a nor-

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mal subgroup E of index 2 and that T is a Sylow 2-subgroup of E. We now apply the main result of [6] to $E/\langle x_1 \rangle O(E)$, a fusion-simple group whose Sylow 2-subgroups are the direct product of a dihedral and a semi-dihedral group. We conclude that $\bar{C} = C/O(C)$ has a normal subgroup of odd index of the form $\bar{C}_1 \times \bar{C}_2$ where $\bar{C}_1 \cong SL^{\pm}(2, r_1)$, $r_1 \equiv -1 \pmod 4$ or $SU^{\pm}(2, r_1)$, $r_1 \equiv 1 \pmod 4$ (using the results in Chapter 2 of [1], and $C_2 \cong M_{11}$, $L_3(q_2)$, or $U_3(q_2)$. A similar structure is possessed by $C_G(y_1)$. We thus obtain

LEMMA 2.1. *If $C = C_G(x_1)$ and $\bar{C} = C/O(C)$, then $\bar{C}$ has a normal subgroup $\bar{C}_0 = \bar{C}_1 \times \bar{C}_2$ where $\bar{C}_1$ and $\bar{C}_2$ have the following structures:*

(i) *$\bar{S}_1 \subseteq \bar{C}_1$ and $\bar{C}_1 \cong SL^{\pm}(2, q_1)$ or $SU^{\pm}(2, q_1)$.*

(ii) *$\bar{S}_2 \subseteq \bar{C}_2$ and $\bar{C}_2 \cong M_{11}$ and $q_2 = 3$, $L_3(q_2)$, $q_2 \equiv -1 \pmod 4$, or $U_3(q_2)$, $q_2 \equiv 1 \pmod 4$.*

If $D = C_G(y_1)$ and $\bar{D} = D/O(D)$, then $\bar{D}$ has a normal subgroup $\bar{D}_0 = \bar{D}_1 \times \bar{D}_2$ where $\bar{D}_1$ and $\bar{D}_2$ have the following structures:

(i) *$\bar{S}_1 \subseteq \bar{D}_1$ and $\bar{D}_1 \cong M_{11}$ and $q_1 = 3$, $L_3(q_1)$, $q_1 \equiv -1 \pmod 4$, or $U_3(q_1)$, $q_1 \equiv 1 \pmod 4$.*

(ii) *$\bar{S}_2 \subseteq \bar{D}_2$ and $\bar{D}_2 \cong SL^{\pm}(2, q_2)$ or $SU^{\pm}(2, q_2)$.*

If $B = C_G(x_1 y_1)$, then $B = (B \cap C \cap D)O(B)$.

Let C_i be the preimage in C of $\bar{C}_i$ and define D_i similarly, $i = 0, 1, 2$. We use this notation for the remainder of the paper.

As a consequence of Lemma 2.1, we have

LEMMA 2.2. *If X and Y are four-groups in S_1 and S_2 respectively and $M = C_G(X)$, $N = C_G(Y)$, $\bar{M} = M/O(M)$, and $\bar{N} = N/O(M)$, then*

(i) *$\bar{M} = \bar{X} \times \bar{M}_1$ where $\bar{S}_2 \subseteq \bar{M}_1$ and $\bar{M}_1$ has a normal subgroup $\bar{M}_0$ of odd index isomorphic to $C_2/O(C)$ and*

(ii) *$\bar{N} = \bar{Y} \times \bar{N}_1$ where $\bar{S}_1 \subseteq \bar{N}_1$ and $\bar{N}_1$ has a normal subgroup $\bar{N}_0$ of odd index isomorphic to $D_1/O(D)$.*

In the remainder of the paper M_i denotes the preimage in M of $\bar{M}_i$ and N_i is defined similarly in N, $i = 0, 1$.

3. Subgroup structure of G

In this section H always denotes a proper subgroup of G, $S \cap H$ is a Sylow 2-subgroup of H, and $S \cap H$ contains an elementary abelian subgroup A of order 16. We set $X = A \cap S_1$, $Y = A \cap S_2$, and denote the involutions in X and Y by x_i and y_i respectively, $i = 1, 2, 3$. As above $x_1 \in Z(S_1)$ and $y_1 \in Z(S_2)$.

LEMMA 3.1. *If H has an isolated involution, then $C_H(z)$ covers $H/O(H)$ for $z = x_1$ or y_1.*

Proof. As Lemma 4.1 in 6.

LEMMA 3.2. *If H has no isolated involutions and J is a subgroup of H containing $O^2(H)A$ such that $J \cap S_1 \times J \cap S_2$ is a Sylow 2-subgroup of J, then $\bar{J} = J/O(J)$ has a normal subgroup of odd index of the form $\bar{J}_1 \times \bar{J}_2$ where $(S_i \cap J)^-$ is a Sylow 2-subgroup of $\bar{J}_i$ and $\bar{J}_i \cong M_{11}, L_3(r_i), U_3(r_i), A_7, L_2(r_i), PGL(2, r_i), PGL^*(2, r_i)$ (described in chapter 2 of [1]), r_i odd, or a four-group, $i = 1, 2$.*

Proof. As in Lemmas 4.2 and 4.3.

Next, we require results on the transitivity of maximal A-invariant p-subgroups of G, p an odd prime, under conjugation by $N_G(A)$. If L is a simple SD-group and Z is a four-group in L, then $N_L(Z)$ does not act transitively on the set of maximal Z-invariant p-subgroups of L when p divides $| C_L(Z) |$, however if D is a Sylow 2-subgroup of $N_L(Z)$, then $N_L(D)$ does act transitively on the set of maximal D-invariant p-subgroups of L for such a prime p. Furthermore, every maximal Z-invariant p-subgroup is a maximal D-invariant p-subgroup for some Sylow 2-subgroup D in $N_L(Z)$ when p divides $| C_L(Z) |$. As a consequence, it is necessary to make the following subdivisions.

Let π be the set of all odd primes dividing the order of G. Let ρ_1 and ρ_2 be the set of odd primes dividing the orders of $C_{N_0/O(N)}(X)$ and $C_{M_0/O(M)}(Y)$ respectively. Now set

$$\pi_1 = \pi - (\rho_1 \cup \rho_2), \quad \pi_2 = \rho_2 - \rho_1, \quad \pi_3 = \rho_1 - \rho_2, \quad \pi_4 = \rho_1 \cap \rho_2.$$

If $T = T \cap S_1 \times T \cap S_2$ is a 2-group in $N_G(A)$ containing A and $T_i = T \cap S_i$, $i = 1, 2$, then

$$T \in \tau_1 \text{ if } T = A, \qquad T \in \tau_2 \text{ if } T_1 = X \text{ and } T_2 \supset Y,$$
$$T \in \tau_3 \text{ if } T_1 \supset X \text{ and } T_2 = Y, \qquad T \in \tau_4 \text{ if } T_1 \supset X \text{ and } T_2 \supset Y.$$

LEMMA 3.3. *Suppose that $T \in \tau_i$ and $p \in \pi_i$ for $i = 1, 2, 3$, or 4. If $T \subseteq H$, then $N_H(T)$ acts transitively on the maximal T-invariant p-subgroups of H.*

Proof. As in Lemma 4.4 of 6.

LEMMA 3.4. *If $p \in \pi_i$ and $T \in \tau_i$, $1 \leq i \leq 4$ and if P_1 and P_2 are maximal T-invariant p-subgroups of G, then one of the following holds:*

(i) $P_1 \sim P_2$ *in* $N_G(T)$,
(ii) $P_1 \cap P_2 = 1$.

Proof. This lemma follows from the preceding and a standard argument.

LEMMA 3.5. *Suppose that $p \in \pi_i$ and $T \in \tau_i$, $1 \leq i \leq 4$. If P_j is a maximal T-invariant p-subgroup of G such that*

$$C_{P_j}(\langle x, y \rangle) \neq 1 \quad \text{for some} \quad x \in X^\# \cap Z(T), y \in Y^\# \cap Z(T), \quad j = 1, 2,$$

then $P_1 \sim P_2$ in $N_G(T)$.

Proof. Let V_j be a maximal T-invariant p-subgroup of G containing a maximal T-invariant p-subgroup of $H = C_G(\langle x, y\rangle)$ as well as $P_j \cap H, j = 1, 2$. Then $V_1^h \supseteq P_2 \cap H$ for some $h \in N_H(T)$ and we have $P_1 \sim V_1 \sim V_1^h \sim V_2 \sim P_2$ in $N_G(T)$.

LEMMA 3.6. *Suppose $Q \neq 1$ is an A-invariant p-subgroup, p an odd prime, such that Q covers a maximal X-invariant p-subgroup of $N_0/O(N)$ and a maximal Y-invariant p-subgroup of $M_0/O(M)$. If P_i is a maximal A-invariant p-subgroup of G containing Q, $i = 1, 2$, then $P_1 \sim P_2$ in $N_G(A)$.*

Proof. Assume that the lemma is false and choose P_1 and P_2 so that $R = P_1 \cap P_2$ has maximal order. If $J = O^2(N_G(R))A$, we can assume that

$$J \cap S_1 \times J \cap S_2$$

is a Sylow 2-subgroup of J. Let U_i be a maximal A-invariant p-subgroup of J containing $P_i \cap J$, $i = 1, 2$. Then $U_1 \not\sim U_2$ in $N_J(A)$ by our choice of R. If J has an isolated involution z, the structure of $C_G(z)$ gives a contradiction. Otherwise $\bar{J} = J/O(J)$ has normal subgroups $\bar{J}_1$ and $\bar{J}_2$ as given in Lemma 3.2. Since $\bar{J}_1 = (J_1 \cap N_0)^-$, $\bar{J}_2 = (J_2 \cap M_0)^-$, and $Q \subseteq R \subseteq O(J)$, we conclude that $\bar{U}_i \cap \bar{J}_1 \bar{J}_2 = 1, i = 1, 2$. It follows that $\bar{U}_i \subseteq (C_J(A))^-$ and thus, that $U_1 \sim U_2$ in $N_J(A)$, a contradiction.

4. An A-signalizer functor

Our main goal in this section is to show that if for $a \in A^\#$, we set

$$\theta(C_G(a)) = \langle C_G(a) \cap O(C_G(x)) \cap O(C_G(y)) \mid x \in X^\#, y \in Y^\#\rangle,$$

then θ is an A-signalizer functor on G.

If K is an A-invariant subgroup of odd order in G and

$$K_{x,y} = K \cap O(C_G(x)) \cap O(C_G(y)), \quad x \in X^\#, \quad y \in Y^\#,$$

then we say that K is *XY-generated* if

$$K = \langle K_{x,y} \mid x \in X^\#, y \in Y^\#\rangle.$$

As an immediate consequence of this definition and the structures of involutions we have

LEMMA 4.1. *If R is an XY-generated p-subgroups of G and $R \subseteq C_G(a)$ for some $a \in A$, then $R \subseteq O(C_G(a))$.*

LEMMA 4.2. *If $R \neq 1$ is an XY-generated p-subgroup of G, $p \in \pi_i$, $1 \leq i \leq 4$, and $R \subseteq C_G(a)$ where $a = x_1, y_1$, or $x_1 y_1$, then for some $T \in \tau_i$ we can find a T-invariant p-subgroup R_1 of $C_G(a)$ such that*

$$R \subseteq R_1, \quad \langle x_1, y_1\rangle \subseteq Z(T), \quad \text{and} \quad C_{R_1}(\langle x_1, y_1\rangle) \neq 1$$

for some $x \in X^\# \cap Z(T)$, $y \in Y^\# \cap Z(T)$.

Proof. By the preceding lemma, $R \subseteq O(C_G(a))$. Set

$$E = O(C_G(a))O(C_G(\langle x_1, y_1\rangle))O(N \cap M_0)O(M \cap N_0).$$

Let R_1 be an A-invariant Sylow p-subgroup of E which contains R. Let Q be a T_1-invariant Sylow p-subgroup of E where T_1 is chosen in τ_i such that $Z(T_1) \supseteq \langle x_1, y_1\rangle$. Since $A \subseteq T_1$, we have $Q^e = R_1$ for some $e \in C_E(A)$. Then

$$T = T_1^e \in \tau_i \quad \text{and} \quad Z(T) \supseteq \langle x_1, y_1\rangle.$$

If $p \in \rho_1$, then $R_1 \cap O(M \cap N_0) \neq 1$ and if $p \in \rho_2$, then $R_1 \cap O(N \cap M_0) \neq 1$. It follows that

$$C_{R_1}(\langle x, y\rangle) \neq 1 \quad \text{for some} \quad x \in Z(T) \cap X^{\#}, \quad y \in Z(T) \cap Y^{\#}.$$

We find it convenient to single out the following two primes. If $N_0/O(N) \cong L_3(q_1)$, let p_1 be the prime divisor of q_1, if $M_0/O(M) \cong L_3(q_2)$, let p_2 be the prime divisor of q_2.

LEMMA 4.3. *Let $p \in \pi_i$ and $T \in \tau_i$, $1 \leq i \leq 4$ and assume that p divides the orders of both $O(C)$ and $O(D)$. If R is a T-invariant p-subgroup such that*

$$C_R(\langle x, y\rangle) \neq 1 \quad \textit{for some} \quad x \in X^{\#} \cap Z(T), \quad y \in Y^{\#} \cap Z(T),$$

then one of the following holds:

(i) *X and Y centralize Sylow p-subgroups of $O(D)$ and $O(C)$ respectively.*

(ii) *There exist p-local subgroups H and K of G which cover $N_0/O(N)$ and $M_0/O(M)$ respectively such that $H \cap K \supseteq PA$ where P is a maximal A-invariant p-subgroup of G containing R.*

Proof. We assume that (i) is false and that X does not centralize a Sylow p-subgroup of $O(D)$ (the argument being symmetrical). For definiteness set $x = x_1$ and $y = y_1$. Set $T_1 = T \cap N_0$ and $T_2 = T \cap M_0$ so that $X \subseteq T_1$, $Y \subseteq T_2$, and $T = T_1 \times T_2$.

Let R_0 be a T-invariant p-subgroup of D containing both $C_R(\langle x_1, y_1\rangle)$ and a Sylow p-subgroup of $O(D)$. Then D contains a p-local subgroup which covers $N_0/O(N)$ and contains R_0T. Among all such p-local subgroups in G choose H such that a T-invariant p-subgroup P_0 of H containing R_0 has maximal order. Without loss we can assume that $H = FP_0T$, $T_1O(H) \subseteq F$, and in $\bar{H} = H/O(H)$ we have $\bar{F} = (F \cap N_0)^- \cong N_0(/O(N)$ and $\bar{H} = \bar{F}\bar{P}_0 \times \bar{T}_2$. If $Q = O(H) \cap P_0$, then X does not centralize $Q \supseteq R_0 \cap O(D)$, and we can assume $Q \lhd H$.

We consider first the case that P_0 is not a maximal T-invariant p-subgroup of G and let P_1 denote a T-invariant p-subgroup of G properly containing and normalizing P_0. If $p \neq p_1$, we apply Lemma 2.6 of [6] and contradict our choice of H and P_0. Thus $p = p_1 \notin \rho_1$.

Suppose that $p \notin \rho_2$ and so $T_2 = Y$. Now for some $y \in Y^{\#}$, $U = P_0C_{P_1}(y)$

$\supset P_0$. Since $[U, y] = [Q, y]$ is normal in $F \cap N_0$ and is UT-invariant, we must have $[U, y] = 1$ by our choice of H. But then we can find a p-local subgroup in $C_G(y)$ containing UT and covering $N_0/O(N)$, a contradiction. Thus $p \in \rho_2$, $Z(T_2) = \langle y_1 \rangle$, and $T = X \times T_2$.

Let R_1 be a T-invariant p-subgroup of C containing a Sylow p-subgroup of $O(C)$ as well as $C_R(\langle x_1, y_1 \rangle)$. In C we can find a p-local subgroup which covers $M_0/O(M)$ and contains R_1T. Among all such subgroups in G choose K such that a T-invariant p-subgroup of K containing R_1, P_2, has maximal order. Since $T_1 = X$, we can argue as above to show that P_2 is a maximal T-invariant p-subgroup of G.

Since $P_0 \cap P_2 \neq 1$ and $P_2 \cap R \neq 1$, we can assume that $R \subseteq P_2$ and $P_0 \subseteq P_2$ by Lemma 3.4. The argument above also shows that $C_{P_2}(y_1) \subseteq P_0$. Since $p \in \rho_2$, $C_{P_2}(y_1)$ covers a maximal Y-invariant p-subgroup of $M_0/O(M)$. Since H covers $N_0/O(N)$, P_0 covers a maximal X-invariant p-subgroup of $N_0/O(N)$. If U and U_1 are maximal A-invariant p-subgroups of G such that $P_0 \subseteq U \cap U_1$, then $U \sim U_1$ in $N_G(A)$ by Lemma 3.6.

Among all p-local subgroups of G containing $P_0 A$ and covering $N_0/O(N)$ choose one with an A-invariant p-subgroup U containing P_0 of maximal order. The arguments above show that U is a maximal A-invariant p-subgroup of G. Among all p-local subgroups of G containing P_2A and covering $M_0/O(M)$ choose one with an A-invariant p-subgroup U_1 of maximal order containing P_2. Again U_1 must be a maximal A-invariant p-subgroup of G. Since $U \cap U_1 \supseteq P_0$, $U^g = U_1$ for some $g \in N_G(A)$. This proves the lemma in the case that P_0 is not a maximal T-invariant p-subgroup.

We consider the case that P_0 is maximal. Since $P_0 \cap R \neq 1$, we can assume that $R \subseteq P_0$. Let P_2 and K be as above. We can assume that $K = LP_2 T$, $T_2 O(K) \subseteq L$, and in $\bar{K} = K/O(K)$, $\bar{L} = (L \cap M_0)^- \cong M_0/O(M)$, and $\bar{K} = \bar{L}\bar{P}_2 \times \bar{T}_1$. Since $P_2 \cap P_0 \neq 1$, we can assume that $P_2 \subseteq P_0$. We claim that P_2 covers a maximal X-invariant p-subgroup of $N_0/O(N)$ and a maximal Y-invariant p-subgroup of $M_0/O(M)$. This is clear if $P_2 = P_0$. If $P_2 \subset P_0$, then $p \in \rho_1$ and $Z(T_1) = \langle x_1 \rangle$. However in this case as above $C_{P_0}(x_1) \subseteq P_2$ and $C_{P_0}(x_1)$ covers a maximal X-invariant p-subgroup of $N_0/O(N)$. The argument in the preceding two paragraphs completes the proof of the lemma.

LEMMA 4.4 *Let E be an A-invariant subgroup of odd order and assume that $AE \subseteq H \cap K$ where H and K are proper subgroups of G covering $N_0/O(N)$ and $M_0/O(M)$ respectively. Then E is XY-generated if and only if $E \subseteq O(H) \cap O(K)$.*

Proof. As Lemma 5.2 in [6].

Set $K = \theta(C_G(a))$ for some $a \in A^{\#}$. Assume $R \neq 1$ is an XY-generated p-subgroup of K and for definiteness assume $a \in \langle x_1, y_1 \rangle$. If X and Y centralize Sylow p-subgroups of $O(D)$ and $O(C)$ respectively, then $R \subseteq O(M) \cap$

$O(N)$ and every-subgroup of R is XY-generated by the preceding lemma. In the contrary case, every A-invariant subgroup of R is XY-generated by Lemmas 4.2–4.4. Thus K satisfies condition (d) of Proposition 2.1 of [5]. Conditions (a)–(c) follow as in Lemmas 5.3–5.5 in [6]. The arguments in Proposition 5.7 and lemma 6.2 of [6] provide the following result:

LEMMA 4.5. *The functor θ is an A-signalizer functor on G and the group*

$$W = \langle \theta(C_G(a)) \mid a \in A^{\#} \rangle$$

is of odd order. Moreover, every A-invariant subgroup of W is XY-generated.

5. Proof of the main theorem

Set $I = N_G(W)$. We shall show that I is a strongly imbedded subgroup of G. Since G has no proper normal subgroup of odd index, being a minimal counterexample, we need only show that $C_G(z) \subseteq I$ for all involutions z in I.

LEMMA 5.1. *The group I contains $N_G(A)$, $O(M)$, $O(N)$, $O(C_G(x))$, a $\in X^{\#}$, and $O(C_G(y))$, $y \in Y^{\#}$.*

Proof. As in Lemma 6.1 of [6].

LEMMA 5.2. *If R is an A-invariant Sylow p-subgroup of W, then $N_G(R)$ covers both $M_0/O(M)$ and $N_0/O(N)$.*

Proof. We can assume $R \neq 1$. By Lemma 4.5, R is XY-generated. Let $p \in \pi_i$, $1 \leq i \leq 4$. If X and Y centralize Sylow p-subgroups of $O(D)$ and $O(C)$ respectively, then $R \subseteq O(C) \cap O(D)$, if Q is an A-invariant Sylow p-subgroup of $O(D)$ containing R, then $C_D(Q)$ covers $N_0/O(N)$. Similarly, $N_C(R)$ covers $M_0/O(M)$. Thus we can assume this is not the case.

Since $R \neq 1$, we have $C_R(\langle x, y\rangle) \neq 1$ for some $x \in X^{\#}$, $y \in Y^{\#}$. Since $N_G(A) \subseteq I$, we can find $T \in \tau_i$ such that $Z(T) \supseteq \langle x, y\rangle$ and such that R is T-invariant.

By Lemma 4.3 we can find p-local subgroups H and K of G such that H covers $N_0/O(N)$ and K covers $M_0/O(M)$ and such that $PA \subseteq H \cap K$ where P is a maximal A-invariant p-subgroup of G containing R. We can assume that $H = FPA$, $XO(H) \subseteq F$, in $\bar{H} = H/O(H)$, $\bar{F} = (F \cap N_0)^{-} \cong N_0/O(N)$, and $\bar{H} = (FP) \times \bar{Y}$ and that $K = LPA$, $YO(K) \subseteq L_i$ in $\bar{K} = K/O(K)$, $\bar{L} = (L \cap M_0)^{-} \cong M_0/O(M)$, and $\bar{K} = (LP)^{-} \times \bar{X}$. We can also assume that $Q = P \cap O(H) \lhd H$ and that $V = P \cap O(K) \lhd K$. By Lemma 4.4, $R = Q \cap V \lhd P$.

Assume, by way of contradiction, that $N_G(R)$ does not cover $M_0/O(M)$. The following results are proved under this assumption.

(a) *We have $p \neq p_2$ (where p_2 is defined immediately preceding Lemma* 4.3.

Proof. If $p = p_2$, then $U = P \cap M_0$ covers a maximal Y-invariant p-subgroup of $\bar{M}_0 = M_0/O(M)$. Since R is a Sylow p-subgroup of W, $N_{M_0}(R) \cap N_G(Y)$

contains a subgroup Z such that $\bar{Z} \cong S_4$. By Lemma 2.3 of [6], $\bar{M}_0 = \langle \bar{Z}, \bar{U} \rangle$ and $(N_{M_0}(R))^- = \bar{M}_0$, contrary to our assumption.

(b) *We have Y does not centralize V.*

Proof. Else $C_L(V)$ covers $M_0/O(M)$ and since $R \supseteq V$, this is a contradiction.

(c) *Either X centralizes Q or $p = p_1$ (defined before Lemma 4.3). In both cases $N_G(R)$ covers $N_0/O(N)$.*

Proof. If $p = p_1$, we argue as in (a) to see that $N_G(R)$ covers $N_0/O(N)$. Suppose that $p \neq p_1$ and X does not centralize Q. By Lemma 2.6 of [6], we conclude that $J = N_G(Z(J(P)))$ covers both $M_0/O(M)$ and $N_0/O(N)$. Since R is a Sylow p-subgroup of $O(J)$ by Lemma 4.4, $N_J(R)$ covers $M_0/O(M)$ a contradiction. This proves (c).

Set

$$R_i = R \cap O(C_G(x_i)), \quad V_i = V \cap O(C_G(x_i)) \quad \text{and } V_0 = \langle V_i \mid i = 1, 2, 3 \rangle.$$

Since R is XY-generated, $R \subseteq V_0$.

(d) *We have $[R, L \cap M_0] \subseteq V_0$.*

Proof. Since V_i is normalized by $L \cap M_0$, $i = 1, 2, 3$, this is clear.

(e) *We have $V_0 \subseteq RC_V(A)$.*

Proof. If $\bar{H} = H/O(C)$, we see that $\bar{V}_i \subseteq O(C_{\bar{H}}(\bar{x}_i))$, $i = 1, 2, 3$. Thus we have

$$\bar{V}_0 = C_{\bar{V}_0}(\bar{A}) = (C_{V_0}(A))^- ,$$

since $O(C_{\bar{H}})(\bar{x}) \subseteq O(C_{\bar{H}}(\bar{A}))$, $x \in X^{\#}$. It follows that

$$V_0 \subseteq C_{V_0}(A)Q \cap V \subseteq (Q \cap V)C_V(A) = RC_V(A).$$

This proves (e).

We now set $C^{(i)} = C_G(x_i)$ and we let $C_j^{(i)}$ in $C^{(i)}$ correspond to C_j in C, $i = 1, 2, 3; j = 0, 1, 2$.

(f) *We have $[C_0^{(i)}, R] \subseteq WC_V(A)O(C^{(i)})$, $[C_0^{(i)}, R]$ is of odd order, and*

$$[C_0^{(i)}, R] \lhd [C_0^{(i)}, R]R \quad \textit{for } i = 1, 2, 3.$$

Proof. Since $N_G(R)$ covers $N_0/O(N)$ by (c), we have that if $c \in C_0^{(i)}$, then $c = c_1 c_2 c_3$ where $c_1 \in C_1^{(i)} \cap N_{N_0}(R)$, $c_2 \in C_2^{(i)} \cap L \cap M_0$, and $c_3 \in O(C^{(i)})$. We then have

$$R^c = R^{c_2 c_3} \subseteq (RC_V(A))^{c_3} \subseteq WC_V(A)O(C^{(i)}).$$

This is sufficient to prove (f).

We are now in a position to contradict our assumption that $N_G(R)$ does not cover $M_0/O(M)$. If $M^1 = (L \cap M_0 \cap C_2^{(1)})[C_0^{(1)}, R]R$, then M^1 is A-invariant and covers $M_0/O(M)$. If $E_1 = M^1 \cap L$ and $U_1 = E_1 \cap ([C_0^{(1)}, R]R)$, then $R \subseteq U_1$ which is normal and of odd order in E_1 and E_1 covers $M_0/O(M)$. Let R_1 be an A-invariant Sylow p-subgroup of U_1 containing R and set $G_1 = N_{E_1}(R_1)$. Then G_1 also covers $M_0/O(M)$. Next, set

$$M^2 = (G_1 \cap M_0 \cap C_2^{(2)})[C_0^{(2)}, R]R, \quad E_2 = M^2 \cap G_1 \quad \text{and} \quad U_2 = E_2 \cap ([C_0^{(2)}, R]R).$$

Then $R \subseteq U_2$ which is of odd order and normal in E_2 and since $E_2 \supseteq G_1 \cap M_0 \cap C_2^{(2)}$, E_2 covers $M_0/O(M)$. Let R_2 be an A-invariant Sylow p-subgroup of U_2 containing R and set $G_2 = N_{E_2}(R_2)$. We then have G_2 covers $M_0/O(M)$ and $G_2 \subseteq E_2 \subseteq G_1 \subseteq E_1$. Next, we set

$$M^3 = (G_2 \cap M_0 \cap C_2^{(3)})[C_0^{(3)}, R]R, \quad E_3 = M^3 \cap G_2 \quad \text{and} \quad U_3 = E_3 \cap ([C_0^{(3)}, R]R).$$

Then $R \subseteq U_3$ which is of odd order and normal in E_3 and since $E_3 \supseteq G_2 \cap M_0 \cap C_2^{(3)}$, E_3 covers $M_0/O(M)$. Let R_3 be an A-invariant Sylow p-subgroup of U_3 containing R and set $G_3 = N_{E_3}(R_3)$. Then G_3 covers $M_0/O(M)$ and we have $G_3 \subseteq E_3 \subseteq G_2 \subseteq E_2 \subseteq G_1 \subseteq E_1 \subseteq L$. It follows that

$$[G_3, R] \subseteq R_1 \cap R_2 \cap R_3 \cap V.$$

Since $R_i \subseteq WC_V(A)O(C^{(i)})$ and $R = R_i \cap W$, we have $R_i = RC_{R_i}(x_i)$, $i = 1, 2, 3$. Since

$$[C_0^{(i)}, C_{R_i}(x_i)] \subseteq C_0^{(i)} \cap ([C_0^{(i)}, R]R)$$

which is of odd order, we conclude that $C_{R_i}(x_i) \subseteq O(C^{(i)})$ and so

$$R_i = R(R_i \cap O(C^{(i)})), \quad i = 1, 2, 3.$$

Set $R_0 = R_1 \cap R_2 \cap R_3 \cap V$. We claim that $R_0 \subseteq O(H)$. If $\bar{H} = H/O(H)$, then

$$\bar{R}_i \subseteq O(C_{\bar{H}}(\bar{x}_i))$$

and so $\bar{R}_0 \subseteq \bar{C}(C_{\bar{H}}(\bar{x}))$, $x \in X^{\#}$. By Lemma 2.4 of [6], we conclude that $\bar{R}_0 = 1$, as asserted. Since $R \subseteq R_0 \subseteq V \cap O(H) = V \cap Q = R$, we see that $[G_3, R] \subseteq R$ and since G_3 covers $M_0/O(M)$, we have a contradiction. It follows that $N_G(R)$ covers $M_0/O(M)$ and a symmetrical argument then shows that $N_G(R)$ covers $N_0/O(N)$. This proves our lemma.

LEMMA 5.3. *If $W \neq 1$, then I is a strongly imbedded subgroup of G.*

Proof. Since I has even order and is a proper subgroup of G if $W \neq 1$, we need only show that $C_G(i) \subseteq I$ for every involution $i \in I$. By the preceding lemma we conclude that I covers both $M_0/O(M)$ and $N_0/O(N)$. This and Lemma 5.1 imply that $C_G(a) \subseteq I$ for all $a \in A^{\#}$. Since every involution in S is conjugate in I to an involution in A, our lemma is proved.

PROPOSITION 5.4. *We have $O(C_G(x)) \cap O(C_G(y)) = 1$ for all $x \in X^{\#}$, $y \in Y^{\#}$.*

Proof. Clearly, it is sufficient to show that $W = 1$. Since G has three conjugacy classes of involutions, G does not possess a strongly imbedded subgroup. By the preceding lemma, we conclude that $W = 1$.

The proof of our theorem now follows exactly as in Section 7 of [6].

References

1. J. Alperin, R. Brauer and D. Gorenstein, *Finite groups with quasidihedral and wreathed Sylow 2-subgroups*, Trans. Amer. Math. Soc., vol. 151 (1970), pp. 1–261.
2. G. Glauberman, *Central elements in core-free groups*, J. Algebra, vol. 4(1966), pp. 403–420
3. D. Goldschmidt, *Solvable signalizer functors on finite groups*, J. Algebra, to appear.
4. D. Gorenstein, *Finite groups*, Harper and Row, New York, 1968.
5. D. Gorenstein and K. Harada, *Finite groups whose Sylow 2-subgroups are the direct product of two dihedral groups*, to appear.
6. F. Smith, *Finite groups whose Sylow 2-subgroups are the direct product of a dihedral and a semi-dihedral group*, Illinois J. Math., vol. 17 (1973), pp. 352–386 (this issue).
7. M. Suzuki, *A characterization of the 3-dimensional projective unitary group over a finite field of odd characteristic*, J. Algebra, vol. 2(1965), pp. 1–14.
8. J. Walter, *The characterization of finite groups with abelian Sylow 2-subgroups*, Ann. of Math., vol. 89(1969), pp. 405–514.

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