VECTOR VALUED KÖTHE FUNCTION SPACES I

BY
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1. Introduction

The purpose of the present series of papers is to study spaces of vector valued functions defined on a measure space. We call these spaces vector valued Köthe function spaces (v.f.s.). This study encompasses Dieudonné's theory of Köthe spaces [4] and Gregory's work [5] on spaces whose elements are sequences of vectors.

The present paper defines and establishes the basic properties of the universal spaces $\Omega(E)$ and $\bar{\Omega}(E')$ of which v.f.s.'s are subspaces. In particular, a Radon-Nikodym theorem for vector valued measures which would seem to have independent interest is proved.

In the next two papers we shall investigate properties of v.f.s.'s. A completeness criterion and various compactness theorems will be proved. Also, results concerning the topological duals of v.f.s.'s will be established. Most of the results about duals are, as far as I know, new even for Köthe spaces.

In the third paper we shall also investigate a special type of v.f.s. $\Lambda(E)$ formed in a natural way from a Köthe space Λ and a locally convex topological vector space E. Special spaces of this type have been investigated by Các [3].

These papers form the major part of the author's doctoral dissertation at the University of Michigan. I wish to thank my advisor, Professor M. S. Ramanujan, for his interest and help. I wish to also thank Professor M. M. Day for a suggestion which has shortened several proofs.

2. Terminology and notation

Let Z be a locally compact Hausdorff topological space which is countable at infinity. Let π be a positive Radon measure on Z. Recall [1, p. 169] that a function f from Z into a topological space is *measurable* if given a compact set $K \subseteq Z$ and $\varepsilon > 0$ there is a compact set $K' \subseteq K$ with $\pi(K - K') < \varepsilon$ such that $f|_{K'}$ is continuous.

Let E be a locally convex Hausdorff topological vector space over the real field with topological dual E' and completion $\hat{E}$. Let P be the set of continuous seminorms on E. If $p \in P$, E_p will denote the completion of the normed space $E/p^{-1}(0)$ and $\theta_p : E \to E_p$ will denote the canonical map. If $p \in P$, p^0 will denote the gauge of the polar (in E') of the closed unit ball U of p. Note that $p^0(y) = \operatorname{Sup}_{x \in U} |\langle x, y\rangle|$ and that if $p^0(y) < \infty$, then $|\langle x, y\rangle| \leq p(x)p^0(y)$. If $R \subseteq Z$, $c(R)$ will denote the characteristic function of R.

A function $f : Z \to E$ is *p-measurable* if $\theta_p \circ f$ is measurable for every

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$p \in P$. The function f is *weakly measurable* if it is measurable when E is given the weak topology $\sigma(E, E')$ and is *scalarly measurable* if $\langle f(\cdot), x'\rangle$ is measurable for every $x' \in E'$. It is not difficult to show, using the fact that a weakly measurable function into a Banach space is measurable [2, p. 96, Ex. 25] that a function which is weakly measurable is also p-measurable.

3. The spaces $\Omega(E)$ and $\bar{\Omega}(F)$

Consider the space of functions $f: Z \to E$ which are p-measurable and such that $\int_K p \circ f\, d\pi < \infty$ for every compact K and $p \in P$. Define $\Omega_0(E)$ to be the separated space associated with this space when equipped with the seminorms $\int_K p \circ f\, d\pi$ and $\Omega(E)$ to be its completion. If we wish to emphasise the space Z we write $\Omega_Z(E)$. If E is the real field, $\Omega_0(E) = \Omega(E) = \Omega$, the space of all measurable, locally integrable real valued functions. The space $\Omega(E)$ was introduced in [6, pp. 71–73]. It is shown there that $\Omega(E) = \Omega \hat{\otimes}_\pi E$ and if E is a Fréchet space, $\Omega_0(E) = \Omega(E)$.

Now suppose that F is a separating subset of E'. We define $\bar{\Omega}(F)$ to be the set of $\sigma(F, E)$ scalarly measurable functions $g: Z \to F$ satisfying the following condition: for every compact set $K \subseteq Z$, $g|_K = bg_0$ where b is real valued and integrable and g_0 is a $\sigma(F, E)$ scalarly measurable function satisfying $p^0 \circ g \leq 1$ (everywhere) for some $p \in P$. We identify g_1 and g_2 and write $g_1 \equiv g_2$ if $g_1 = g_2$ scalarly a.e. (i.e., if $\langle x, g_1(\cdot)\rangle = \langle x, g_2(\cdot)\rangle$ a.e. for all $x \in E$). If E and F are the real field, $\bar{\Omega}(F) = \Omega$.

When E is separable, $\bar{\Omega}(E')$ has several nice properties.

PROPOSITION 3.1. (1) *If E is separable then any $g \in \bar{\Omega}(E')$ is weakly measurable.*

(2) *If $g_1 \equiv g_2$ in $\bar{\Omega}(E')$ and both are weakly measurable, then $g_1 = g_2$* a.e.

(3) *If $g \in \bar{\Omega}(E')$ is weakly measurable and $p \in P$, then $p^0 \circ g$ is measurable.*

Proof. (1) Let G be the linear span of the dense set $\{x_n\} \subseteq E$. Let $g \in \bar{\Omega}(E')$, a compact set $K \subseteq Z$, and $\varepsilon > 0$ be given. Write $g|_K = bg_0$ where $b \in \Omega$ and $p^0 \circ g_0 \leq 1$. Since for every n, $\langle x_n, g_0\rangle$ is measurable, there is a compact set $K' \subseteq K$ such that $\pi(K - K') \leq \varepsilon$ and $\langle x_n, g_0(z)\rangle|_{K'}$ is continuous for all n [1, p. 170]. Thus $\langle x, g_0(z)\rangle|_{K'}$ is continuous for any $x \in G$ and so $g_0|_{K'}$ is continuous when E' is given the topology $\sigma(E', G)$. Since the topologies $\sigma(E', E)$ and $\sigma(E', G)$ agree on the $\sigma(E', E)$ relatively compact range of g_0 on K', $g_0|_{K'}$ is continuous when E' is given the topology $\sigma(E', E)$, whence g_0 and so g are weakly measurable.

(2) This follows from [6, p. 21] which states that two measurable scalarly a.e. equal functions are a.e. equal.

(3) Since p^0 is lower semicontinuous when E' is given the weak topology, $(p^0)^{-1}([0, a])$ is weakly closed for any $a \geq 0$. Since g is weakly measurable, $g^{-1} \circ (p^0)^{-1}([0, a])$ is measurable [1], p. 179]. Thus $p^0 \circ g$ is measurable [1, p. 180]. ∎

We define $\Gamma(E)$ as the set of functions $f: Z \to E$ of the form $\sum_{j=1}^{n} c(R_j)x_j$ where $\{R_j\}$ is a set of disjoint relatively compact measurable sets in Z. The space $\Delta(E)$ will be the set of functions $f: Z \to E$ of the form $\sum_{j=1}^{\infty} c(R_j)x_j$ where $\{R_j\}$ is as above. The spaces $\Gamma(F)$ and $\Delta(F)$ are defined similarly. We note that a function in $\Delta(E)$ or $\Delta(F)$ is measurable (no matter what topology E or F is given; see [1, p. 169].

In spite of the fact that $f_1 \equiv f_2$ in $\Omega_0(E)$ does not imply $f_1 = f_2$ a.e. (it only implies that $p \circ (f_1 - f_2) = 0$ a.e. for all $p \in P$) and the fact that $g_1 \equiv g_2$ in $\bar{\Omega}(F)$ does not imply $g_1 = g_2$ a.e. we have the following result.

THEOREM 3.2. (1) *If* $f_1 \equiv f_2$ *in* $\Omega_0(E)$ *and* $g_1 \equiv g_2$ *in* $\Omega(F)$, *then* $\langle f_1, g_1\rangle = \langle f_2, g_2\rangle$ *a.e.*

(2) *If* $f \in \Omega_0(E)$ *and* $g \in \Omega(F)$, *then* $\langle f, g\rangle$ *is measurable.*

Proof. (1) First, let $f \equiv 0$ in $\Omega_0(E)$, $g \in \bar{\Omega}(F)$, and a compact set $K \subseteq Z$ be given. Set $g|_K = bg_0$ with $b \in \Omega$ and $p^0 \circ g_0 \leq 1$. Then

$$|\langle f, g\rangle| \,|_K \leq (p \circ f)|b| = 0 \quad \text{a.e.}$$

since $f \equiv 0$. It follows that $\langle f, g\rangle = 0$ a.e.

Next, let $g \equiv 0$ in $\bar{\Omega}(F)$, $f \in \Omega_0(E)$, a compact set $K \subseteq Z$ and $\varepsilon > 0$ be given. Set $g|_K = bg_0$ with $b \in \Omega$ and $p^0 \circ g_0 \leq 1$. Since $\theta_p \circ f$ is measurable, there is a compact set $K' \subseteq K$ such that $\pi(K - K') < \varepsilon$ and $\theta_p \circ f|_{K'}$ is continuous. Now let $\delta > 0$ be arbitrary. For any $w_0 \in K'$ pick an open (in K') neighborhood W_0 of w_0 such that $p(f(z) - f(w_0)) < \delta$ on W_0. Find such a neighborhood for each $w_0 \in K'$ and choose a finite subcovering of K' giving points $w_1, w_2, \cdots, w_m$ with neighborhoods $W_1, W_2, \cdots, W_m$. Set $V_1 = W_1$ and $V_j = W_j - \bigcup_{k=1}^{j-1} W_k$ for $j = 2, 3, \cdots, m$. Then

$$\begin{aligned}
|\langle f, g\rangle| \,|_{K'} &= \textstyle\sum_{j=1}^{m} |\langle f, g\rangle| \,|_{V_j} \\
&\leq \textstyle\sum_{j=1}^{m} \{|\langle f(z) - f(w_j), g(z)\rangle| + |\langle f(w_j), g(z)\rangle|\} \,|_{V_j} \\
&\leq \textstyle\sum_{j=1}^{m} \{p(f(z) - f(w_j))p^0(g(z)) + 0\} \,|_{V_j} \\
&\leq \delta|b|
\end{aligned}$$

where the zero was inserted above since $g \equiv 0$. Since δ is arbitrary, $\langle f, g\rangle|_{K'} = 0$ a.e. and so $\langle f, g\rangle = 0$ a.e.

Combining the last sentences of the last two paragraphs, we have proved (1).

(2) Let $f \in \Omega_0(E)$, $g \in \bar{\Omega}(F)$, and a compact set $K \subseteq Z$ be given. Set $g|_K = bg_0$ with $b \in \Omega$ and $p^0 \circ g_0 \leq 1$. Using [1, p. 178], choose $(f_n) \subseteq \Gamma(E)$ so that $p \circ (f|_K - f_n) \to 0$ a.e. Now $\langle f_n, g\rangle$ is measurable since g is scalarly measurable. Also,

$$|\langle f|_K, g\rangle - \langle f_n, g\rangle| \leq p \circ (f|_K - f_n)|b| \to 0 \quad \text{a.e.}$$

showing that $\langle f, g\rangle$ is measurable. ▌

Example. This example shows that various conjectures that one might make concerning $\Omega_0(E)$ and $\bar{\Omega}(F)$ are not true. No proofs are given; none is difficult. Let E be a Hilbert space with orthonormal basis e_z, $z \in [0, 1]$. Let $Z = [0, 1]$ with Lebesgue measure. Now $E' = E$ but we shall write E' for the dual of E. Let f_1 and f_2 be functions from Z into E defined by $f_1(z) = e_z$ and $f_2(z) = 0$. Let g_1, g_2, and g_3 be functions from Z into E' defined by $g_1(z) = e_z$, $g_2(z) = 0$, and $g_3(z) = c(A)(z)e_z$ where A is a non-measurable subset of Z (cf. [2, p. 81]). The functions f_1, g_1, and g_3 are scalarly a.e. equal to zero and so scalarly measurable. But none is norm or weakly measurable. The g_i are all in $\bar{\Omega}(E')$ and are equivalent in spite of the fact that they are not a.e. equal. Also, g_3 gives an example of a function in $\bar{\Omega}(E')$ such that $\| g_3(z) \| = c(A)(z)$ is not measurable. Thus Proposition 3.1 (1) is not true for a general E. The function $f_1 \notin \Omega_0(E)$ even though $\| f_1(z) \| \equiv 1$ is measurable and $\int \| f_1 \| \, d\pi < \infty$. If we give E the topology $\sigma(E, E')$, then $f_1 \in \Omega_0(E)$ and $f_1 \equiv f_2$. But it is not the case that $f_1 = f_2$ a.e.

Example. Even though the function g_1 above is not $\sigma(E', E)$ measurable, the function g_2, which is equivalent to it, is $\sigma(E', E)$ measurable. This always happens when E is a reflexive Banach space [2, p. 95, Ex. 25]. The following example, adapted from Thomas [8, p. 83], shows that it is possible for a class of functions in $\bar{\Omega}(E')$ to contain no $\sigma(E', E)$ measurable function. Let $Z = [0, 1]$. Let $E = l_I^1$ where I is the unit ball of $L^\infty[0, 1]$. Then $E' = l_I^\infty$. Define $g : Z \to E'$ by $g(t) = (b_i(t))_i$ where b_i is some function in the ith class of functions in the unit ball of $L^\infty[0, 1]$ satisfying $| b_i(t) | \leq 1$ everywhere. Then $g \in \bar{\Omega}(E')$. If $g' \equiv g$ where $g'(t) = (c_i(t))_i$ then $c_i(t) = b_i(t)$ a.e. for every i. Now suppose g' is $\sigma(E', E)$ measurable. Then there is a compact set $K \subseteq Z$ with $\pi(K) > \frac{1}{2}$ such that $g' |_K$ is $\sigma(E', E)$ continuous. But this implies that $c_i |_K$ is continuous for every i, which is impossible.

4. The dual of $\Omega(E)$

We recall ([4], p. 97) that the space $\Phi \subseteq \Omega$ is defined as the set of all measurable bounded (a.e.) functions of compact support (a.e.). An element of $\bar{\Omega}(F)$, i.e., a class of functions in $\bar{\Omega}(F)$, will belong to $\bar{\Phi}(F)$ if there is a function g in the class and a $p \in P$ such that $p^0 \circ g$ has compact support and is bounded. $\bar{\Phi}_Z(F)$ will be used to emphasize the space Z. If E is a Banach space, we set $\Phi(E) = \{f \in \Omega(E) : \| f \| \in \Phi\}$.

The dual of Ω is Φ [4, p. 96].

THEOREM 4.1. $\Omega(E)' = \bar{\Phi}(E')$.

Proof. $\Omega_0(E)$ and its completion $\Omega(E)$ have the same dual and so we shall prove that $\Omega_0(E)' = \bar{\Phi}(E')$. It is easy to show that for $g \in \bar{\Phi}(E')$, $f \to \int < f, g > d\pi$ is a continuous linear functional on $\Omega_0(E)$. It remains to show that any $\phi \in \Omega_0(E)'$ is so represented. As in [4, p. 96], there is a compact set K such that $\phi(f) = \phi(f |_K)$. Thus we may consider ϕ as a member of the dual of $\Omega_0(E)$ for the set K. Now $\Omega_K(E) = \Omega_K \hat{\otimes}_\pi E$ [6, p. 71] and so $\phi \in (\Omega_K \otimes_\pi E)'$. Thus ϕ is a continuous bilinear functional on $\Omega_K \times E$. By the Dunford-Pettis Theorem [2, p. 45] and [7, p. 544], there is a unique $g \in \bar{\Phi}_K(E')$ such that $\phi(f) = \int \langle f, g\rangle \, d\pi$ for any $f \in \Omega_K \otimes E$. Since $\Omega_K \otimes E$ is dense in $\Omega_K(E)$ and since we have shown that $f \to \int \langle f, g\rangle \, d\pi$ is a continuous linear functional on $\Omega_0(E)$, we must have $\phi(f) = \int \langle f, g\rangle \, d\pi$ for all $f \in \Omega_0(E)$. ▮

5. Extension of operations on $\Omega_0(E)$

In spite of the fact that the elements of $\Omega(E)$ are not all functions most of the operations performed on the functions of $\Omega_0(E)$ can be extended to all of $\Omega(E)$.

DEFINITION. (1) Let $g \in \bar{\Omega}(E')$ be fixed. For a given compact set K set $g|_K = bg_0$ with $b \in \Omega$ and $p^0 \circ g_0 \leq 1$. The map $f \to \langle f, g_0\rangle|_K$ is, by Theorem 3.2, a well defined linear continuous map of $\Omega_0(E)$ into Ω. It thus has a continuous extension from $\Omega(E)$ into Ω which we denote $\langle f, g_0\rangle_K$. Set $\langle f, g\rangle_K = \langle f, g_0\rangle_K \cdot b|_K$. Now $\langle f, g\rangle_{K_1}|_{K_1 \cap K_2} = \langle f, g\rangle_{K_2}|_{K_1 \cap K_2}$ a.e. since these functions agree when $f \in \Omega_0(E)$. Thus there is a measurable function $\langle f, g\rangle$ such that for any compact set $K \subseteq Z$, $\langle f, g\rangle|_K = \langle f, g\rangle_K$ a.e. [4, p. 83, footnote].

(2) For $p \in P$, the map $f \to \theta_p \circ f$ of $\Omega_0(E)$ into $\Omega_0(E_p) = \Omega(E_p)$ is continuous. We denote the continuous extension of this map by $\theta_p \circ f$. For $f \in \Omega(E)$, $p \circ f$ can now be defined as $p \circ \theta_p \circ f$.

(3) If $a \in L^\infty$, then the map $f \to af$ on $\Omega_0(E)$ is continuous. We denote the continuous extension of this map by af. If $R \subseteq Z$ is measurable and $f \in \Omega(E)$, $f|_R$ can now be defined as $c(R)f$.

Not only do the operations on the functions in $\Omega_0(E)$ extend to $\Omega(E)$ but most of the properties of these operations continue to hold.

PROPOSITION 5.1. (1) *Let $f \in \Omega(E)$, $a \in \Omega$, $p \in P$, and a $\sigma(F, E)$ measurable function $g \in \bar{\Omega}(F)$ be given. Suppose that $p \circ f \leq a$ and that $ap^0 \circ g$ is defined (i.e., $p^0(g(z)) \neq \infty$ when $a(z) = 0$). Then $|\langle f, g\rangle| \leq ap^0 \circ g$.*

(2) *If $f \in \Omega(E)$ and $g = bg_0 \in \bar{\Omega}(E')$ where $b \in \Omega$ and $p^0 \circ g_0 \leq 1$, then $|\langle f, g\rangle| \leq (p \circ f)|b|$.*

(3) *The form $\langle f, g\rangle$ is bilinear.*

(4) *For $a \in L^\infty$, $f \in \Omega(E)$, and $g \in \bar{\Omega}(F)$,*

$$a\langle f, g\rangle = \langle af, g\rangle = \langle f, ag\rangle.$$

(5) *The extension of the seminorm $\int_K p \circ f \, d\pi$ on $\Omega_0(E)$ to $\Omega(E)$ is $\int_K p \circ f \, d\pi$.*

(6) *The element of $\Omega(E)'$ represented by $g \in \bar{\Phi}(E')$ is given by $f \to \int \langle f, g\rangle \, d\pi$ for any $f \in \Omega(E)$.*

(7) *For $f_1, f_2 \in \Omega(E)$ and $p \in P$,*

$$p \circ (f_1 + f_2) \leq p \circ f_1 + p \circ f_2.$$

(8) *If $p \in P$, $f \in \Omega(E)$, and $a \in L^\infty$, then $ap \circ f = p \circ (af)$.*

Proof. We prove only (1) since the proofs of the other parts are simpler. Set $R = \{z : p^0(g(z)) = \infty\}$. Let a compact set K and $\varepsilon > 0$ be given. By Proposition 3.1(3), there is a compact set $K' \subseteq K - R$ with $\pi((K - R) - K') < \varepsilon$ and $p^0 \circ g|_{K'}$ continuous and so bounded by, say, M. Now for $f \in \Omega_0(E)$,

$$(*) \qquad |\langle f, g\rangle|\,|_{K'} \leq (p \circ f)(p^0 \circ g)\,|_{K'} \leq Mp \circ f|_{K'}.$$

Thus the maps $f \to |\langle f, g\rangle||_{K'}$ and $f \to (p \circ f)(p^0 \circ g)|_{K'}$ are continuous as maps from $\Omega_0(E)$ into Ω. Since the positive cone of Ω is closed (it is the intersection of the weakly closed sets

$$\{a \in \Omega : \textstyle\int_K a\, d\pi \geq 0\}$$

as K runs through the set of compact sets K in Z), the continuous extension of these maps satisfies $(*)$ for any $f \in \Omega(E)$. The result follows immediately. ▌

LEMMA 5.2. *Let $g : Z \to F$ be weakly measurable. Let $a \in \Omega$ and $p \in P$ be given and suppose $a(z) \neq 0$ when $p^0(g(z)) = \infty$ (so $ap^0 \circ g$ is defined). Then:*

$$(1) \int |a|\, p^0 \circ g\, d\pi = \operatorname{Sup}\left\{ \left| \int \langle f, g\rangle\, d\pi \right| : f \in \Gamma(E) \text{ and } p \circ f \leq |a| \right\}.$$

(2) *If $p^0 \circ g$ is finite a.e. and if for every $f \in \Delta(E)$ with $p \circ f \leq |a|$ we have $\int |\langle f, g\rangle|\, d\pi < \infty$, then $\int |a| p^0 \circ g\, d\pi < \infty$.*

Proof. Consider any compact set K of positive measure such that $g|_K$ is weakly continuous and $p^0 \circ g|_K$ is continuous and finite. We claim that (1) is valid if we only integrate over K. Suppose first that $a = c(K')$ where $K' \subseteq K$ is compact and let $\varepsilon > 0$ be given. Fix $z_0 \in K'$ and choose $x_0 \in U$ such that

$$p^0(g(z_0)) \leq \langle x_0, g(z_0)\rangle + \varepsilon(2\pi(K'))^{-1}.$$

Find a neighborhood (in K') N_0 of z_0 such that

$$p^0(g(z)) \leq \langle x_0, g(z)\rangle + \varepsilon(\pi(K'))^{-1}$$

for $z \in N_0$. If we do this for each $z_0 \in K'$ we get an open covering of K'. Let $N_1, \cdots, N_m$ be a finite subcovering with $x_1, \cdots, x_m$ the associated elements of U. Set $R_n = N_n - \bigcup_{j=1}^{n-1} N_j$ and $f = \sum_{j=1}^m c(R_j)x_j$. Then $p \circ f \leq c(K') = a$ and $\int_{K'} p^0 \circ g\, d\pi \leq \int \langle f, g\rangle\, d\pi + \varepsilon$. This establishes the claim for $a = c(K')$. It now follows when a is a simple function easily. The claim is now justified by observing that both sides of (1) (integrating over K) are continuous as functions of $a \in \Omega$ and that the simple functions are dense in Ω. Set

$$N = \{z : p^0(g(z)) = \infty\}$$

and assume for the moment that $\pi(N) = 0$. Using [1, p. 170] we can obtain a sequence of disjoint compact sets (K_n) of positive measure and disjoint from N such that $\pi(Z - \bigcup K_n) = 0$, $g|_{K_n}$ is weakly continuous, and $p^0 \circ g|_{K_n}$ is continuous. Let $\varepsilon > 0$ be given. By the first part of the proof we can find, for each n, an $f_n \in \Gamma(E)$ with Supp $f_n \subseteq K_n$ such that $p \circ f_n \leq |a|$ and

$$\int_{K_n} |a| p^0 \circ g \, d\pi \leq \int_{K_n} \langle f_n, g \rangle \, d\pi + \varepsilon 2^{-n}.$$

Then

$$\int |a| p^0 \circ g \, d\pi = \sum_{j=1}^{\infty} \int_{K_j} |a| p^0 \circ g \, d\pi$$

$$\leq \operatorname{Sup}_n \left| \sum_{j=1}^{n} \int \langle f_j, g \rangle \, d\pi \right| + \varepsilon.$$

This establishes (1).

For (2), set $f = \sum_{j=1}^{\infty} f_n$. Then $f \in \Delta(E)$ and

$$\int |a| p^0 \circ g \, d\pi \leq \int |\langle f, g \rangle| \, d\pi + \varepsilon.$$

This proves (2).

It remains to prove (1) when $\pi(N) > 0$. In this case $\int |a| p^0 \circ g \, d\pi = \infty$ so it is sufficient to show that given $M \geq 0$ there is an $f \in \Gamma(E)$ with $p \circ f \leq a$ such that $|\int \langle f, g \rangle \, d\pi| \geq M$. It is not hard to find a compact set $K \subseteq N$ of positive measure and a $\delta > 0$ such that $g|_K$ is weakly continuous and $|a(z)| \geq \delta$ on K. The construction of an $f \in \Gamma(E)$ with the desirable properties now proceeds as in the first part of the proof. ∎

LEMMA 5.3. *Suppose $p \in P$, $f \in \Omega(E)$, and $b \in \Omega$. Then:*

(1) $\int p \circ f |b| \, d\pi = \operatorname{Sup} \{ |\int \langle f, g \rangle \, d\pi| : g \in \Gamma(F)$ *and* $p^0 \circ g \leq |b| \}$.

(2) *If for every $g \in \Delta(F)$ with $p^0 \circ g \leq |b|$ we have $\int |\langle f, g \rangle| \, d\pi < \infty$, then $\int p \circ f |b| \, d\pi < \infty$.*

Proof. First we claim that if K is a compact set of positive measure and $f \in \Gamma(E)$, then (1) is valid if we only integrate over K. The proof is similar to that of the first part of Lemma 5.2. We now claim that if $f \in \Omega(E)$ and $b|_K$ is continuous then (1) is valid if we only integrate over K. This is so since we now know (1) to be true for $f \in \Gamma(E)$, since both sides of the equality are continuous as functions of $f \in \Omega(E)$, and since $\Gamma(E)$ is dense in $\Omega(E)$ (for $\Gamma(E)$ separates points of $\Omega(E)' = \bar{\Phi}(E')$). Now (1) and (2) follow as in Lemma 5.2. ∎

6. Integrals of elements of $\Omega(E)$ and $\bar{\Omega}(F)$

For a relatively compact measurable set R and $f \in \Omega(E)$ we define $\int_R f \, d\pi$ to be the element of the algebraic dual of F defined by

$$\left\langle \int_R f \, d\pi, y \right\rangle = \int_R \langle f, y \rangle \, d\pi$$

(cf. [2, p. 8]). Similarly, for $g \in \bar{\Omega}(F)$ we define $\int_R g\, d\pi$ to be that element of the algebraic dual of E defined by

$$\left\langle x, \int_R g\, d\pi \right\rangle = \int_R \langle x, g,\rangle\, d\pi.$$

THEOREM 6.1. (1) *If* $f \in \Omega(E)$, *then* $\int_R f\, d\pi \in \hat{E}$ *and*

$$p\left(\int_R f\, d\pi\right) \leq \int_R p \circ f\, d\pi.$$

(2) *If* $g \in \bar{\Omega}(F)$, $\int_R g\, d\pi \in E'$.

Proof. (1) If $f \in \Gamma(E)$ then straight from the definition we have $\int_R f\, d\pi \in E$. Now for any $f \in \Gamma(E)$ and $p \in P$, the inequality of (1) holds [2, p. 12]. Thus the map from $\Gamma(E)$ into E given by $f \to \int_R f\, d\pi$ is continuous. Therefore it has a continuous extension $\bar{\int}_R f\, d\pi$, from $\Omega(E)$ into $\hat{E}$ ($\Gamma(E)$ is dense in Ω $(E$ since it separates points of $\Omega(E)' = \bar{\Phi}(E')$). Let $f \in \Omega(E)$ and let $(f_\alpha) \subseteq \Gamma(E)$ be a net such that $f_\alpha \to f$. Then,

$$\left\langle \bar{\int}_R f\, d\pi, y \right\rangle = \lim_\alpha \left\langle \bar{\int}_R f_\alpha\, d\pi, y \right\rangle = \lim_\alpha \int_R \langle f_\alpha, y\rangle\, d\pi$$

$$= \int_R \langle f, y\rangle\, d\pi = \left\langle \int_R f\, d\pi, y \right\rangle$$

where the first equality follows from the fact that

$$f \to \left\langle \bar{\int}_R f\, d\pi, y \right\rangle$$

is continuous on $\Omega(E)$ and the next to last from the fact that $c(R)y \in \Omega(E)'$. Thus

$$\int_R f\, d\pi = \bar{\int}_R f\, d\pi \in \hat{E}.$$

The inequality of (1) was shown above on the dense subspace $\Gamma(E)$ of $\Omega(E)$ and so is valid on $\Omega(E)$.

(2) Set $g|_R = bg_0$ where $b \in \Omega$ and $p^0 \circ g_0 \leq 1$. Then for $x \in U$,

$$\left|\left\langle x, \int_R g\, d\pi \right\rangle\right| \leq \int_R |b|\, d\pi < \infty.$$

Thus $\int_R g\, d\pi$ is bounded on a neighborhood in E and so is continuous. ∎

PROPOSITION 6.2. *If* $f \in \Omega(E)$ *and* $\int_K f\, d\pi = 0$ *for every compact set* K, *then* $f = 0$.

Proof. For a compact set K and $p \in P$, set

$$Q(K, p) = \{g \in \Gamma(F) : p^0 \circ g \leq c(K)\}.$$

Now

$$Q(K, p)^0 = \{f \in \Omega(E) \colon \operatorname{Sup}_{g \in Q(K,p)} \mid \int \langle f, g \rangle \, d\pi \mid \leq 1\}$$

$$= \{f \in \Omega(E) \colon \int_K p \circ f \, d\pi \leq 1\}$$

since by Lemma 5.3,

$$\operatorname{Sup}_{g \in Q(K,p)} \left| \int \langle f, g \rangle \, d\pi \right| = \int_K p \circ f \, d\pi.$$

Then

$$\Gamma(F)^0 = (\cup_{K,p} Q(K, p))^0$$
$$= \cap_{K,p} Q(K, p)^0$$
$$= \cap_{K,p} \left\{ f \in \Omega(E) : \int_K p \circ f \, d\pi \leq 1 \right\}$$
$$= \{0\}$$

since the last expression is the intersection of sets forming a base of neighborhoods in $\Omega(E)$. Thus given an $f \neq 0$ in $\Omega(E)$ there is a $g \in \Gamma(F)$ such that $\int \langle f, g \rangle \, d\pi \neq 0$. The result now follows easily. ▮

If Z is second countable we can do even better than the above result.

COROLLARY 6.3. *If Z is second countable, there is a countable collection $\mathcal{K} = \{K_n\}$ of compact sets such that if $f \in \Omega(E)$ and $\int_{K_n} f \, d\pi = 0$ for all n, then $f = 0$.*

Proof. Suppose $\{O_n\}$ is a countable base of open sets. By [1, p. 154], each O_i can be expressed as a union: $O_i = \cup_{j=1}^{\infty} L_{ij} \cup N_i$ where L_{ij} is compact and $\pi(N_i) = 0$. Let $\mathcal{K}$ be the set of finite unions of the L_{ij}. Now let $f \neq 0$ in $\Omega(E)$ be given and by the proposition choose a compact K and a $y \in F$ so that $\int_K \langle f, y \rangle \, d\pi \neq 0$. By the regularity of the measure we may choose an open set $O \supseteq K$ so that $\int_O \langle f, y \rangle \, d\pi \neq 0$ and then by the construction of $\mathcal{K}$ choose a $K' \in \mathcal{K}$ so that $\int_{K'} \langle f, y \rangle \, d\pi \neq 0$.

7. A Radon-Nikodym theorem

A *vector valued Köthe function space* (v.f.s.) will be a subspace $S(E)$ of $\Omega(E)$ containing $\Gamma(E)$ or a subspace $T(F)$ of $\bar{\Omega}(F)$ containing $\Gamma(F)$.

A set $A \subseteq \Omega(E)$ is said to be *solid* if for every $f \in A$ and $a \in L^{\infty}$ with $\| a \|_{\infty} \leq 1$, $af \in A$. The *solid hull* of a set $A \subseteq \Omega(E)$ is the smallest solid solid set containing A. A topology on a v.f.s. $S(E)$ will be called *solid* if it has a base at the origin of solid sets. Similar definitions apply to subsets of $\bar{\Omega}(F)$.

The following is a theorem of the Radon-Nikodym type in that it produces a function from rather simple properties.

THEOREM 7.1. *Let $S(E)$ be a solid v.f.s. and ϕ a linear functional on it.*

Then there is a $g \in \bar{\Omega}(E')$ *such that* $\phi(f) = \int \langle f, g \rangle \, d\pi$ *is equivalent to the following:*

(1) *if* $f \in S(E)$ *and if* $R_1 \subseteq R_2 \subseteq \cdots$ *is a sequence of measurable sets such that* $\bigcup R_j = R$ *(we write* $R_j \uparrow R$*), then* $\phi(f|_{R_j}) \to \phi(f|_R)$*, and*

(2) *for every compact set* $K \subseteq Z$*, there is a* $p \in P$ *such that the set*

$$\{\phi(f) : f \in S(E) \text{ and } p \circ f \leq c(K)\}$$

is bounded.

Proof. ($\Rightarrow$) (1) follows from the dominated convergence theorem. To see (2), let K be given and write $g|_K = bg_0$ with $b \in \Omega$ and $p^0 \circ g_0 \leq 1$. Then if $p \circ f \leq c(K)$,

$$|\phi(f)| = \left| \int \langle f, g \rangle \, d\pi \right| \leq \int_K |b| \, d\pi < \infty.$$

($\Leftarrow$) The proof is long and is divided into several steps.

(a) For a fixed relatively compact measurable set R, the linear functional $m(R)$ on E defined by

$$\phi(c(R)x) = \langle x, m(R) \rangle$$

is, by (2), in E'.

(b) Let a compact set $K \subseteq Z$ be given and let $p \in P$ be associated with K by (2). For any measurable set $R \subseteq K$ define

$$|m|_K(R) = \operatorname{Sup} \left| \textstyle\sum_i \langle x_i, m(R_i) \rangle \right|$$

where the supremum is taken over all countable partitions $\{R_i\}$ of R and $p(x_i) \leq 1$. By an appropriate choice of a_i we have

$$\begin{aligned} \textstyle\sum_i |\langle x_i, m(R_i) \rangle| &= \textstyle\sum_i a_i \langle x_i, m(R_i) \rangle = \textstyle\sum_i \phi(a_i c(R_i) x_i) \\ &= \lim_{n \to \infty} \textstyle\sum_{i=1}^n \phi(a_i c(R_i) x_i) \end{aligned}$$

which is finite by (2). Thus $|m|_K(R) < \infty$. We now show that $|m|_K$ is countably additive. First note that

$$|m|_K(R) = \operatorname{Sup} \textstyle\sum_i |\langle x_i, m(R_i) \rangle|.$$

Let $R \subseteq K$ be measurable and (R_i) be a sequence of disjoint measurable sets whose union is R. Then

$$\begin{aligned} \textstyle\sum_i |m|_K(R_i) &= \textstyle\sum_i \operatorname{Sup} \{ \textstyle\sum_j |\langle x_{ij}, m(R_{ij}) \rangle| : p(x_{ij}) \leq 1, \; R_{ij} \subseteq R_i \} \\ &\leq |m|_K(R). \end{aligned}$$

To see the reverse inequality, let $\delta > 0$ be given and let $\{S_j\}$ and $\{x_j\}$ satisfy

$$|m|_K(R) \leq \left| \textstyle\sum_j \langle x_j, m(S_j) \rangle \right| + \delta.$$

We have

$$\begin{aligned} \langle x_j, m(S_j) \rangle &= \phi(c(S_j)x_j) = \lim_{n \to \infty} \phi\left(\textstyle\sum_{i=1}^n c(R_i \cap S_j) x_j\right) \\ &= \textstyle\sum_{i=1}^\infty \langle x_j, m(R_i \cap S_j) \rangle \end{aligned}$$

where the middle equality follows from (1). Thus

$$|m|_K(R) \leq |\textstyle\sum_j \langle x_j, m(S_j)\rangle| + \delta = |\textstyle\sum_{ij} \langle x_j, m(R_i \cap S_j)\rangle| + \delta$$
$$\leq \textstyle\sum_i |m|_K(R_i) + \delta.$$

Since δ is arbitrary, $|m|_K$ is countably additive.

Note that $|m|_K$ is absolutely continuous with respect to π. By the classical Radon-Nikodym theorem, there is a π-measurable function $b_K(z)$ such that $|m|_K(R) = \int_R b_K \, d\pi$. Set $\mu = b_K \pi$. For any simple function $a = \sum_i a_i c(R_i)$ with $R_i \subseteq K$ and the R_i's disjoint, define $\mathbf{m}(a) = \sum_i a_i m(R_i)$. Give the simple functions the L^1 norm (with respect to μ on K), $\|\cdot\|_1$. Then

$$(*) \qquad |\langle x, \mathbf{m}(a)\rangle| = |\textstyle\sum_i a_i \langle x, m(R_i)\rangle| \leq \textstyle\sum_i |a_i| \, |\langle x, m(R_i)\rangle|$$
$$\leq p(x) \textstyle\sum_i |a_i| \, |m|_K(R_i) = p(x) \|a\|_1.$$

Thus the map $a \to \mathbf{m}(a)$ is continuous into the weak topology $\sigma(E', E)$. We can thus extend $\mathbf{m}$ to a map from $L^1(\mu)$ which is the completion of the π-simple functions. Furthermore, the inequality $(*)$ shows that the image under the extension of the unit ball of L^1 is contained in the weakly complete set U^0. By [2, p. 46] and [7, p. 544], there is a scalarly μ-measurable function $g'_K : K \to U^0$ such that $\mathbf{m}(a) = \int a g'_K \, d\mu$. In particular, for $a = c(R)$ this implies that

$$
\begin{aligned}
\phi(c(R)x) &= \langle x, m(R)\rangle \\
&= \langle x, \mathbf{m}(c(R))\rangle \\
&= \left\langle x, \int c(R) g'_K \, d\mu \right\rangle \\
(**) \qquad &= \int \langle c(R)x, g'_K\rangle \, d\mu \\
&= \int \langle c(R)x, g'_K\rangle b_K \, d\pi \\
&= \int \langle c(R)x, g_K\rangle \, d\pi.
\end{aligned}
$$

The function g_K above is defined as $b_K g''_K$ where $g''_K(z) = g'_K(z)$ whenever $b_K(z) \neq 0$ and $g''_K(z) = 0$ otherwise. Then g''_K is scalarly π-measurable since $\langle x, g''_K(z)\rangle$ is the quotient of the π-measurable functions $\langle x, g'_K(z)\rangle b_K(z)$ and $b_K(z)$ (here $0/0 = 0$).

(c) Now let $K_1 \subseteq K_2 \subseteq \cdots$ be a sequence of compact sets such that $K_n \subseteq K^0_{n+1}$ and $\bigcup_{n=1}^{\infty} K_n = Z$. Define $g : Z \to E'$ by $g|_{K_1} = g_{K_1}$ and $g|_{K_n - K_{n-1}} = g_{K_n}$ where the g_{K_i} are constructed as in (b). Since every compact set in Z is contained in some K_i we have $g \in \bar{\Omega}(E')$. Furthermore, it follows from $(**)$ that for $f \in \Gamma(E)$, $\phi(f) = \int \langle f, g\rangle \, d\pi$.

(d) We now extend the representation of ϕ to a larger class of functions.

Let $f \in S(E)$. Given a compact set K, let $p \in P$ be associated with it by (2). Let $K' \subseteq K$ be such that $p^0(g(z))|_{K'} \leq M$ and $\theta_p \circ f|_{K'}$ is continuous (into the Banach space E_p). We claim that $\phi(f|_{K'}) = \int \langle f|_{K'}, g\rangle\, d\pi$. By [1, p. 181], there is a sequence $(f_n) \subseteq \Gamma(E)$ with Supp $f_n \subseteq K'$ such that $p \circ (f|_{K'} - f_n) \to 0$ uniformly. We know from (c) that $\phi(f_n) = \int \langle f_n, g\rangle\, d\pi$. Now

$$\left| \int \langle f|_{K'}, g\rangle\, d\pi - \int \langle f_n, g\rangle\, d\pi \right| \leq \int_{K'} p \circ (f|_{K'} - f_n) M\, d\pi \to 0.$$

Choose a sequence $c_n \to \infty$ such that $p \circ (c_n(f|_{K'} - f_n)) \leq 1$. By (2), the set

$$\{\phi(c_n(f|_{K'} - f_n)) : n = 1, 2, \cdots\}$$

is bounded and so $\phi(f_n) \to \phi(f|_{K'})$ which establishes the claim.

(e) We now extend the representation to any $f \in S(E)$ satisfying $\langle f, g\rangle \geq 0$. Let a compact set K be given. Let p be associated with K by (2). Since $\theta_p \circ f$ is measurable, we may for each n find a compact set $K_n \subseteq K$ such that $K_n \subseteq K_{n+1}$, $\pi(K - K_n) \leq 1/n$, $\theta_p \circ f|_{K_n}$ is continuous, and $p^0 \circ g|_{K_n}$ is bounded.

By (d), $\phi(f|_{K_n}) = \int \langle f|_{K_n}, g\rangle\, d\pi$. By (1), $\phi(f|_{K_n}) \to \phi(f|_K)$. By the monotone convergence theorem,

$$\int \langle f|_{K_n}, g\rangle\, d\pi \to \int \langle f|_K, g\rangle\, d\pi.$$

Thus $\phi(f|_K) = \int \langle f|_K, g\rangle\, d\pi$. Now we assert that $\phi(f) = \int \langle f, g\rangle\, d\pi$. If Z is compact we have already shown this. If Z is not compact, let $K_1 \subseteq K_2 \subseteq \cdots$ be a sequence of compact sets such that $\bigcup_{n=1}^{\infty} K_n = Z$. By the above,

$$\phi(f|_{K_n}) = \int \langle f|_{K_n}, g\rangle\, d\pi.$$

By (1), $\phi(f|_{K_n}) \to \phi(f)$. By the monotone convergence theorem,

$$\int \langle f|_{K_n}, g\rangle\, d\pi \to \int \langle f, g\rangle\, d\pi.$$

(e) We now extend the representation to any $f \in S(E)$. For $f \in S(E)$, set

$$f' = [|\langle f, g\rangle|/\langle f, g\rangle] f \quad (\text{here } 0/0 = 0).$$

We have $\langle f', g\rangle \geq 0$ and so by (d), $\phi(f') = \int \langle f', g\rangle\, d\pi$. But $\langle f', g\rangle = |\langle f, g\rangle|$ and so $\int |\langle f, g\rangle|\, d\pi < \infty$. With this established we can repeat the arguments of (d), using the dominated convergence theorem to get $\phi(f) = \int \langle f, g\rangle\, d\pi$. This completes the proof. ∎

We now state a version of the above theorem in the language of vector valued measures. We only indicate the proof since we shall not use the result in what follows.

Corollary 7.2. *Let* $\mathbf{m}$ *be a function defined on the relatively compact measurable sets in* Z *and taking values in* E'. *Then there is a* $g \in \bar{\Omega}(E')$ *such that* $\mathbf{m}(R) = \int_R g\, d\pi$ *is equivalent to the following:*

(1) *for every* $x \in E$, $\langle x, \mathbf{m}(\cdot)\rangle$ *is countably additive on the* σ*-ring of relatively compact measurable sets,*

(2) *if* $\pi(R) = 0$, *then* $\mathbf{m}(R) = 0$, *and*

(3) *for every compact set* K, *there is a* $p \in P$ *such that*

$$\text{Sup}\,\{\textstyle\sum_i p^0(\mathbf{m}(R_i))\} < \infty$$

where the supremum is taken over all countable partitions $\{R_i\}$ *of* K *consisting of measurable sets.*

Proof. If $f = \sum_{i=1}^n c(R_i)x_i \in \Gamma(E)$ with the R_i's disjoint, define $\phi(f) = \sum_{i=1}^n \langle x_i, \mathbf{m}(R_i)\rangle$. The result can now be deduced from Theorem 7.1. ($\Gamma(E)$ is not solid and so Theorem 7.1 does not apply, but an inspection of the proof will show that it is valid for $\Gamma(E)$.) ▮

Remark. Gregory [5] gives several examples which show that various hypotheses of various theorems in the present series of papers cannot be dropped even in the case that Z is the set of natural numbers and π is the counting measure.

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