ON THE WEYL SPECTRUM

BY

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Abstract

In this paper we give some continuity properties of the Weyl spectrum of a continuous linear operator on a Banach space and show that the Weyl's theorem holds for a spectral operator of finite type although the theorem fails for a spectral operator in general.

1. Preliminaries

Throughout this paper X will denote a complex Banach space and $\mathcal{L}(X)$ the space of continuous linear operators on X considered with the norm topology. For $T \in \mathcal{L}(X)$ let $\sigma(T)$, $\rho(T)$ and $\pi_{00}(T)$ be respectively the spectrum, the resolvent set and the isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity. Let $\mathfrak{N}(T)$ and $\mathfrak{R}(T)$ be respectively the null space and the range space of T. Let $\mathfrak{F}$ be the class of Fredholm operators on X ($T \in \mathfrak{F}$ if and only if $\mathfrak{R}(T)$ is closed and dimension $\mathfrak{N}(T)$ and co-dimension $\mathfrak{R}(T)$ are both finite) and let $\mathfrak{F}_0$ be the class of Fredholm operators of index 0 (i.e., dimension $\mathfrak{N}(T)$ = co-dimension $\mathfrak{R}(T)$). Let $\mathcal{C}(X)$ be the ideal of compact operators on X and let $\hat{T}$ be the image of T under the canonical mapping of $\mathcal{L}(X)$ into the quotient algebra $\mathcal{L}(X)/\mathcal{C}(X)$. Finally, let $\mathcal{C}$ be the set of complex numbers.

Definition 1. The Weyl spectrum $\omega(T)$ of $T \in \mathcal{L}(X)$ is defined by

$$\omega(T) = \{\lambda \in \mathcal{C} : \lambda I - T \notin \mathfrak{F}_0\}.$$

It is well known (see e.g., [1]) that

(i) $T \in \mathfrak{F}$ if and only if $0 \in \rho(\hat{T})$, and
(ii) $\sigma(\hat{T}) \subset \omega(T) \subset \sigma(T)$.

In particular if X is infinite dimensional then $\omega(T)$ is a non-empty compact subset of $\mathcal{C}$.

2. Continuity of $\omega(T)$

In this section we define upper and lower semi-continuity of the mapping $T \to \omega(T)$ and show that this mapping is upper semi-continuous while it may not be lower semi-continuous.

Definition 2. Let (G_n) be a sequence of compact subsets of $\mathcal{C}$. The *limit inferior*, lim inf G_n is the set of all λ in $\mathcal{C}$ such that every neighbourhood

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of λ has a non-empty intersection with all but finitely many G_n. The *limit superior*, lim sup G_n, is the set of all λ in $\mathcal{C}$ such that every neighbourhood of λ intersects infinitely many G_n. If *lim inf* G_n = lim sup G_n then lim G_n is said to exist and is equal to this common limit.

A mapping τ defined on $\mathcal{L}(X)$ whose values are compact subsets of $\mathcal{C}$ is said to be *upper semi-continuous* at T when if $T_n \rightarrow T$ then $\limsup \tau(T_n) \subset \tau(T)$. τ is *lower semi-continuous* at T if $\tau(T) \subset$ *lim inf* $\tau(T_n)$. If τ is both upper and lower semi-continuous at T then it is said to be *continuous* at T and in this case $\lim \tau(T_n) = \tau(T)$.

THEOREM 1. *The mapping $T \rightarrow \omega(T)$ is upper semi-continuous at T.*

Proof. Let $\lambda \notin \omega(T)$ so that $\lambda I - T$ is a Fredholm operator of index 0. By [4; Theorem 4.5.17] there exists an $\eta > 0$ such that if $S \in \mathcal{L}(X)$ and $\| \lambda I - T - S \| < \eta$ then $S \in \mathfrak{F}_0$.

There exists an integer N such that

$$\| \lambda I - T - (\lambda I - T_n) \| < \eta/2 \quad \text{for } n \geq N.$$

Let V be an open $(\eta/2)$ neighbourhood of λ. We have, for $\mu \in V$ and $n \geq N$

$$\| \lambda I - T - (\mu I - T_n) \| < \eta$$

so that $(\mu I - T_n) \in \mathfrak{F}_0$. This implies that $\lambda \notin \limsup \omega(T_n)$. Thus

$$\limsup \omega(T_n) \subset \omega(T)$$

and the theorem is proved.

The standard example (see e.g., [6; p. 282]) to show that the mapping $T \rightarrow \sigma(T)$ is in general not lower semicontinuous may be used to show that the mapping $T \rightarrow \omega(T)$ need not be lower semi-continuous.

THEOREM 2. *Let $T_n \rightarrow T$. Then if* $\lim \sigma(\hat{T}_n) = \sigma(\hat{T})$ *then* $\lim \omega(T_n) = \omega(T)$.

Proof. In the presence of Theorem 1 it is enough to show that $\omega(T) \subset \liminf \omega(T_n)$.

Suppose $\lambda \notin \liminf \omega(T_n)$ so that there is a neighbourhood V of λ that does not intersect infinitely many $\omega(T_n)$. Since $\sigma(\hat{T}_n) \subset \omega(T_n)$, V does not intersect infinitely many $\sigma(\hat{T}_n)$, i.e., $\lambda \notin \lim \sigma(\hat{T}_n) = \sigma(\hat{T})$. This shows that $(\lambda I - T) \in \mathfrak{F}$. By using [4; Theorem 4.5.17] it is easy to see that index $(\lambda I - T) = 0$ so that $\lambda \notin \omega(T)$.

COROLLARY. *Let $T_n \rightarrow T$. Then* $\lim \omega(T_n) = \omega(T)$ *in each one of the following cases.*

(i) *$T_n T = T T_n$ for all n.*
(ii) *$\sigma(T)$ is totally disconnected.*
(iii) *X is a Hilbert space and T, T_n are normal operators.*

Proof. Each one of the above conditions implies $\lim \sigma(\hat{T}_n) = \sigma(\hat{T})$ (see [5] for details).

3. Weyl's theorem

Let $T \in \mathfrak{L}(X)$. If

$$(^*) \qquad \omega(T) = \sigma(T) \sim \pi_{00}(T)$$

then we say that Weyl's theorem holds for T. If X is finite dimensional then, of course, Weyl's theorem holds for each $T \in \mathfrak{L}(X)$. There are several classes of operators including normal and hyponormal operators on a Hilbert space (see e.g., [1] and [2]) for which Weyl's theorem holds. In this section we show that if T is a spectral operator, in the sense of Dunford, of finite type (for definitions we refer to [3: Chapter XV]), then Weyl's theorem holds for T.

The following simple example shows that Weyl's theorem need not hold for a spectral operator.

Example. Let $X = l_2$. Define T by

$$T(x_1, x_2, \cdots) = (\tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \cdots).$$

T is a quasi-nilpotent operator and hence a spectral operator. $0 \in \pi_{00}(T)$ and also $0 \in \omega(T)$. Thus T does not satisfy the relation $(^*)$.

In what follows T will denote a spectral operator on X, S and N will denote its scalar and radical parts respectively, and $E(\cdot)$ will denote its resolution of the identity. The following results which will be used in the proof of Theorems 3 and 4 are given in [3] as Theorems XV.8.2 and XV.7.14.

LEMMA 1. *For an $x \in X$ and a non-negative integer n, $(\lambda I - T)^n x = 0$ if and only if $E(\{\lambda\})x = x$ and $N^n x = 0$.*

LEMMA 2. *The operator T has a closed range if and only if*

(i) *the point $\lambda = 0$ is either in $\rho(T)$ or is an isolated point of $\sigma(T)$, and*

(ii) *the operator $TE(\{0\})$ has a closed range.*

Remark. Lemma 1 shows that $\mathfrak{N}(S) = E(\{0\})X$ and if T is replaced by S in Lemma 2 then (i) implies (ii) (since, in this case $SE(\{0\}) = 0$) so that the condition (ii) is superfluous for a scalar type operator.

THEOREM 3. *Let S be a scalar type operator on X. Then Weyl's theorem holds for S.*

Proof. We have to show that $\lambda \in \pi_{00}(S)$ if and only if $\lambda \in \sigma(S) \sim \omega(S)$. Without loss of generality we may assume that $\lambda = 0$.

Let $0 \in \pi_{00}(S)$ so that $\mathfrak{N}(S)$ is finite dimensional and by Lemma 2, $\mathfrak{R}(S)$ is closed. Lemma 1 shows thst $S^2 x = 0$ if and only if $Sx = 0$. Hence

$$\mathfrak{R}(S) \cap \mathfrak{N}(S) = \{0\}.$$

Also, from the relation $\sigma(S \mid E(\Delta)X) \subset \bar{\Delta}$ for a Borel subset Δ of $\mathcal{C}$ it is easy to see that

$$\mathcal{R}(S) \oplus \mathfrak{N}(S) = X.$$

Thus dimension $\mathfrak{N}(S)$ = codimension $\mathcal{R}(S)$ so that $S \in \mathfrak{F}_0$ i.e., $0 \notin \omega(S)$.

Conversely suppose $0 \in \sigma(S) \sim \omega(S)$. Since $\mathcal{R}(S)$ is closed, 0 is an isolated point of $\sigma(S)$. Also $\mathfrak{N}(S)$ is finite dimensional and non-zero so that $0 \in \pi_{00}(S)$.

LEMMA 3. *Let T be a spectral operator of finite type so that for some non-negative integer m, $N^m = 0$. Then $\pi_{00}(S) = \pi_{00}(T)$.*

Proof. We need only to show that $0 \in \pi_{00}(S)$ if and only if $0 \in \pi_{00}(T)$.

Let $0 \in \pi_{00}(S)$. It is immediate that if $Sx = 0$ then $T^m x = 0$. Thus 0 is an eigenvalue of T. From the relation $\mathfrak{N}(T) \subset \mathfrak{N}(S)$ it follows that $0 \in \pi_{00}(T)$.

Conversely let $0 \in \pi_{00}(T)$ so that 0 is also an eigenvalue of S. Since $\mathfrak{N}(T)$ is a finite-dimensional subspace of $\mathfrak{N}(S)$ we may write

$$\mathfrak{N}(S) = \mathfrak{N}(T) \oplus Y.$$

If $y \in Y$ then $Sy = 0$ so that $T^m y = 0$ i.e., $T^{m-1}y \in \mathfrak{N}(T)$. This implies that Y and hence $\mathfrak{N}(S)$ is finite dimensional showing thereby that $0 \in \pi_{00}(S)$.

THEOREM 4. *Let T be a spectral operator of finite type. Then Weyl's theorem holds for T.*

Proof. We have

$$\omega(S) = \sigma(S) \sim \pi_{00}(S) = \sigma(T) \sim \pi_{00}(T).$$

Hence the theorem follows if we show that $\omega(S) = \omega(T)$. It is enough to show that $0 \in \omega(S)$ if and only if $0 \in \omega(T)$.

Let $0 \notin \omega(S)$ so that $S \in \mathfrak{F}_0$. Since $\mathcal{R}(S)$ is closed, either $0 \in \rho(S) = \rho(T)$, or 0 is an isolated point of $\sigma(S) = \sigma(T)$ and $\mathfrak{N}(S) = E(\{0\})X$ is finite dimensional. Therefore $TE(\{0\})X$ is finite dimensional and hence a closed subspace of X. By Lemma 2, $\mathcal{R}(T)$ is closed. Let

$$X = \mathfrak{N}(S) \oplus Y \quad \text{where } Y = \mathcal{R}(S) = E(\mathcal{C} \sim \{0\})X. \tag{1}$$

Also, let

$$\mathfrak{N}(S) = \mathfrak{N}(T) \oplus \operatorname{span}\{x_1, x_2, \cdots x_r\}. \tag{2}$$

where $x_1, \cdots x_r$ are linearly independent. It is easy to verify that $Tx_1, \cdots Tx_r$ are linearly independent. We assert that

$$\mathcal{R}(T) = Y \oplus \operatorname{span}\{Tx_1, \cdots Tx_r\}. \tag{3}$$

Since $0 \notin \sigma(T \mid Y)$, $TY = Y$. If possible let $Tx_1 = y \in Y$ for some i $(1 \leq i \leq r)$. Since S is injective on Y we have

$$0 \neq Sy = STx_i = TSx_i = 0$$

which is a contradiction. In fact no non-zero linear combination of Tx_i can belong to Y.

This proves our assertion. Relations (1), (2) and (3) together with the fact that $S \in \mathfrak{F}_0$ show that $T \in \mathfrak{F}_0$ i.e., $0 \notin \omega(T)$.

The converse assertion viz., if $0 \notin \omega(T)$ then $0 \notin \omega(S)$ follows in exactly the same fashion.

We conclude this paper with the following conjecture.

Let $T \in \mathfrak{L}(X)$ and let N be a nilpotent operator commuting with T. Then if Weyl's Theorem holds for T it also holds for $T + N$.

Added in proof. The above conjecture is true. However, if N is not assumed to commute with T then the conjecture is false. The proofs will appear elsewhere.

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