A PERTURBATION THEOREM FOR EVOLUTION EQUATIONS AND SOME APPLICATIONS

BY

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Abstract

The following well-known perturbation theorem is of fundamental importance in semigroup theory: Let A be m-dissipative (i.e., A generates a (C_0) contraction semigroup). If P_1 is dissipative and A bounded with relative A bound less than one, and if P_2 is bounded, then $A + P_1 + P_2$ generates a (C_0) semigroup. This result is generalized to allow A, P_1, P_2 all to depend on a real parameter t. Thus in many cases, establishing the well-posedness of the Cauchy problem for

$$u'(t) = (A(t) + P_1(t) + P_2(t))u(t) \quad (' = d/dt)$$

is reduced to proving the well-posedness of the Cauchy problem for $u'(t) = A(t)u(t)$. Applications are given to temporally inhomogeneous scattering theory and to second order evolution equations of the form

$$u''(t) + B(t)u'(t) + C(t)u(t) = 0;$$

here both $B(t)$ and $C(t)$ can be unbounded. Some concrete examples are given, including mixed problems for

$$u_{tt} = \alpha u_{xx} + \beta u_x + \gamma u_{tx} + \delta u_t + \varepsilon u + \phi$$

with time dependent boundary conditions; here all the coefficients are smooth functions on $\{(t, x) : -\infty < t < \infty, 0 \leq x \leq 1\}$, α is positive and γ is sufficiently small.

1. Introduction

One of the fundamental perturbation theorems in the theory of semigroups of linear operators is the following result of Kato and Gustafson [11], [9], which generalizes earlier work of Rellich.

THEOREM 1. *Let A be an m-dissipative operator on a Banach space X. Let P_1 be a dissipative operator on X and let P_2 be a bounded operator on X. Suppose that the domain of P_1 contains that of A and*

$$\| P_1 f \| \leq a \| Af \| + b \| f \|$$

for some constants $a < 1$, $b \geq 0$, and for all f in the domain of A. Then, $A + P_1 + P_2 - kI$ is m-dissipative for some real k.

In fact, k may be taken to be $\| P_2 \|$. For a short proof see Goldstein [4,

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p. 32]. See Chernoff [3] for a related result and Kato [14] for a nonlinear generalization.

According to the Lumer-Phillips form of the Hille-Yosida generation theorem (cf. [16], [4]), A is the infinitesimal generator of a (C_0) contraction semigroup iff A is m-dissipative. Thus Theorem 1 tells us that in many cases solving the Cauchy problem for $du/dt = (A + P_1 + P_2)u$ reduces to solving the Cauchy problem for $du/dt = Au$.

The purpose of this paper is to generalize Theorem 1 to the case where A, P_1, P_2 are all functions of a real variable and to give some applications. The perturbed Cauchy problem that we shall treat is thus the Cauchy problem for the time dependent evolution equation $du/dt = (A(t) + P_1(t) + P_2(t))u$.

The main results are stated in Section 2 and proved in Section 3. Applications to temporally inhomogeneous scattering theory and to second order evolution equations are given in Sections 4 and 5.

2. The perturbation theorem

Let X be a Banach space. The domain of a (linear) operator A on X will be denoted by $D(A)$. $\mathcal{B}(X)$ is the space of all bounded linear operators on X. $\mathbf{R}^+ = [0, \infty)$, $\mathbf{R} = (-\infty, \infty)$. Let

$$T(\cdot) : \mathbf{R}^+ \to \mathcal{B}(X)$$

(or $T(\cdot) : \mathbf{R} \to \mathcal{B}(X)$). Write $T(\cdot) \in \mathfrak{M}(X)$ iff there is a strongly measurable function $S(\cdot) : \mathbf{R}^+$ (or $\mathbf{R}$) $\to \mathcal{B}(X)$ such that

(i) $T(t)f - T(0)f = \int_0^t S(s)f\,ds$ whenever $t \in \mathbf{R}^+$ (or $\mathbf{R}$), $f \in X$,
(ii) for each $\tau \in \mathbf{R}^+$, there is a constant m_τ such that

$$\| S(s) \| \leq m_\tau \quad \text{for} \quad | s | \leq \tau.$$

For $T(\cdot) : \mathbf{R}^+$ (or $\mathbf{R}$) $\to \mathcal{B}(X)$ write $T(\cdot) \in \operatorname{Lip}(X)$ iff for each $\tau \in \mathbf{R}^+$ there is a constant m_τ such that $\| T(t) - T(s) \| \leq m_\tau | t - s |$ whenever $| s |, | t | \leq \tau$. It is well known that $\mathfrak{M}(X) \subset \operatorname{Lip}(X)$ with equality holding when X is reflexive.

Concerning semigroups we shall use the standard terminology of Hille-Phillips [10].

Our basic tool is the Kato existence theorem ([12], cf. also [7]).

THEOREM 2. *Assume*

(A1) *For each* $\tau \in \mathbf{R}^+$ *there exists a* $Q(t) \in \mathcal{B}(X)$ *such that* $Q(t)^{-1} \in \mathcal{B}(X)$ *and* $Q(t)A(t)Q(t)^{-1}$ *is m-dissipative; moreover* $Q(\cdot) \in \mathfrak{M}(X)$.

(A2) *There is a function* $R(\cdot) : \mathbf{R}^+ \to \mathcal{B}(X)$ *having a strong derivative* $R'(\cdot)$ *such that* $R'(\cdot) \in \mathfrak{M}(X)$, $R(t)^{-1} \in \mathcal{B}(X)$ *for each* $t \in \mathbf{R}^+$, *and*

$$B(t) = R(t)A(t)R(t)^{-1}$$

has domain independent of t. It follows that

$$C(\cdot) = B(\cdot)(I - B(0))^{-1} : \mathbf{R}^+ \to \mathcal{B}(X);$$

suppose $C(\cdot) \in \mathfrak{M}(X)$.

Then for any $f \in D(A(0))$ there exists a unique function $u(\cdot): \mathbf{R}^+ \to X$ having a strongly continuous strong derivative and satisfying

$$u'(t) = A(t)u(t) \ (t \in \mathbf{R}^+), \quad u(0) = f.$$

Our main result is the following perturbation theorem.

THEOREM 3. *Let the hypotheses of Theorem 2 hold. Further assume:*

(A3) *For each $t \in \mathbf{R}^+$, $Q(t)P_1(t)Q(t)^{-1}$ is dissipative, $D(P_1(t)) \supset D(A(t))$, and there are constants $a(t) < 1$, $b(t) \geq 0$ such that*

$$\| Q(t)P_1(t)f \| \leq a(t) \| Q(t)A(t)f \| + b(t) \| f \|$$

for each $f \in D(A(t))$; moreover, $P_1(\cdot)R(\cdot)^{-1}(I - B(0))^{-1} \in \mathfrak{M}(X)$.

(A4) $P_2(\cdot) \in \mathfrak{M}(X)$.

Then for any $f \in D(A(0))$ there exists a unique function

$$u(\cdot) : \mathbf{R}^+ \to X$$

having a strongly continuous strong derivative and satisfying

$$u'(t) = (A(t) + P_1(t) + P_2(t))u(t) \ (t \in \mathbf{R}^+), \quad u(0) = f.$$

Some results similar to Theorem 3 occur in the literature; cf. Phillips [18], Goldstein [5], [6], Kato [13]. However, all these authors treat the case when $P_1(t) = 0$, so that the perturbation $(P_2(t))$ is a one parameter family of bounded operators; the point of the present paper is to consider unbounded perturbations.

We shall call P a *Kato perturbation* of A iff $D(P) \supset D(A)$ and for each $a > 0$ there exists a $b_a \geq 0$ such that

$$\| Pf \| \leq a \| Af \| + b_a \| f \| \quad \text{for each} \quad f \in D(A).$$

COROLLARY 4. *In the hypotheses of Theorem 3 let (A3) be replaced by:*

(A3′) *For each $t \in \mathbf{R}^+$, $Q(t)P_1(t)Q(t)^{-1}$ is dissipative, $P_1(t)$ is a Kato perturbation of $A(t)$, and*

$$P_1(\cdot)R^{-1}(\cdot)(I - B(0))^{-1} \in \mathfrak{M}(X).$$

Then the conclusion of Theorem 3 holds.

COROLLARY 5. *For each $t \in \mathbf{R}$ let $H(t)$ be a self-adjoint operator on a complex Hilbert space X having domain D independent of t, and suppose*

$$H(\cdot)(iI - H(0))^{-1} \in \mathrm{Lip}\,(X).$$

For each $t \in \mathbf{R}$ let $V(t)$ be a symmetric operator satisfying $D(V(t)) \supset D$ and there exist constants $a(t) < 1$, $b(t) \geq 0$ such that

$$\| V(t)f \| \leq a(t) \| H(t)f \| + b(t) \| f \|$$

for each $f \in D$. Finally suppose $V(\cdot)(iI - H(0))^{-1} \in \mathrm{Lip}\,(X)$. Then for *each $f \in D$, $\tau \in \mathbf{R}^+$, there is a unique function $u(\cdot) : \mathbf{R} \to X$ having a strongly continuous strong derivative and satisfying*

$$u'(t) = i(H(t) + V(t))u(t) \ (t \in \mathbf{R}), \quad u(\tau) = f.$$

Moreover, the operator $U(t, \tau)$ sending f into $u(t)$ is unitary, for all $t, \tau \in \mathbf{R}$.

3. Proofs

Proof of Theorem 3. To solve

$$(1) \qquad du/dt = (A(t) + P_1(t) + P_2(t))u(t)u(0) = f \in D(A(0))$$

uniquely for $t \in \mathbf{R}^+$, it suffices to solve it uniquely for $t \in [0, \tau]$ with $\tau > 0$ fixed but otherwise arbitrary. For $0 \leq t \leq \tau$, (A1), (A3), (A4) and Theorem 1 together imply that

$$Q(t)(A(t) + P_1(t) + P_2(t) - kI)Q(t)^{-1}$$

is m-dissipative, where

$$k = \max\{\| Q(t) \| \| P_2(t) \| \| Q(t)^{-1} \| : 0 \leq t \leq \tau\} \in \mathbf{R}^+.$$

Let $A_1(t) = A(t) + P_1(t) + P_2(t) - kI$.

$$D(R(t)A_1(t)R(t)^{-1}) = D(R(t)A(t)R(t)^{-1})$$

does not depend on t by (A2), and if

$$B_1(t) = R(t)A_1(t)R(t)^{-1},$$

then

$$\begin{aligned} B_1(t)(I - B_1(0))^{-1} & \\ = B(t)(I - B(0))^{-1}&[(I - B(0))(I - B_1(0))^{-1}] \\ &+ R(t)P_1(t)R(t)^{-1}(I - B(0))^{-1}[(I - B(0))(I - B_1(0))^{-1}] \\ &+ R(t)(P_2(t) - kI)R(t)^{-1}(I - B_1(0))^{-1}. \end{aligned}$$

Consequently $B_1(\cdot)(I - B_1(0))^{-1} \in \mathfrak{M}(X)$ by (A2), (A3), (A4), since $R(\cdot)^{-1} \in \mathfrak{M}(X)$, and since $\mathfrak{M}(X)$ is closed under linear combinations and composition.

Thus (A1), (A2) hold for $A_1(\cdot)$ on $[0, \tau]$ with the same $Q(\cdot)$, $R(\cdot)$ as for $A(\cdot)$. Thus

$$dv/dt = A_1(t)v \ (0 \leq t \leq \tau), \quad v(0) = f \in D(A(0))$$

has a unique solution on $[0, \tau]$; $u(t) = e^{kt}v(t)$ is thus the unique solution of (1) on $[0, \tau]$. This completes the proof of Theorem 3.

Proof of Corollary 4. Let (A3′) hold and let $t \in \mathbf{R}^+$. Then for any $a > 0$

there is a $b_a \geq 0$ such that

$$\| Q(t)P_1(t)f \| \leq a \| Q(t) \| \| A(t)f \| + b_a \| Q(t) \| \| f \|$$
$$\leq a \| Q(t) \| \| Q(t)^{-1} \| \| Q(t)A(t)f \| + b_a \| Q(t) \| \| f \|,$$

and $a \| Q(t) \| \| Q(t)^{-1} \| < 1$ if a is chosen sufficiently small. Hence (A3) holds, and Corollary 4 follows from Theorem 3.

Proof of Corollary 5. This is based on the analog of Theorem 1 for self-adjoint operators, due to Rellich (cf. [11, p. 187]), which states that if A is self-adjoint, P symmetric, $D(P) \supset D(A)$, and $\| Pf \| \leq a \| Af \| + b \| f \|$ for constants $a < 1$, $b \geq 0$ and all $f \in D(A)$, then $A + P$ is self-adjoint. Corollary 5 follows from this result, Stone's theorem (cf. e.g. [4, p. 22], [10, p. 598]), Theorem 2 (with $Q(t) \equiv I$, $R(t) \equiv I$), and some routine computations which we omit.

4. Temporary inhomogeneous scattering

THEOREM 6. *Let X be a complex Hilbert space. For each $t \in [-\infty, \infty]$ let $H_0(t)$ be a self-adjoint operator with domain D independent of t satisfying $H_0(\cdot)(iI - H_0(0))^{-1} \in \mathrm{Lip}\,(X)$. For each $t \in [-\infty, \infty]$ let $V(t)$ be a symmetric operator with $D(V(t)) \supset D$ and suppose*

$$\| V(t)f \| \leq a(t) \| H_0(t)f \| + b(t) \| f \|$$

for constants $a(t) < 1, b(t) \geq 0$, and all $f \in D$. Suppose

$$V(\cdot)(iI - H(0))^{-1} \in \mathrm{Lip}\,(X).$$

Let $H_1(t) = H_0(t) + V(t)$. For $j = 0, 1$ suppose that the subspace $\mathfrak{A}_j$ of absolute continuity for $H_j(t)$ does not depend on t (cf. [11, *p*, 516]). *Let*

$$U_j = \{U_j(t, s) : t, s \in \mathbf{R}\}$$

be the family of evolution operators governing $H_j(\cdot)$, i.e., *$u(t) = U_j(t, s)f$ is the unique strongly continuously differentiable solution at time t of the time dependent Schrödinger equation $du/dt = iH_j(t)u$, $u(s) = f \in D$. Suppose there exists a $\tau > 0$ such that for $t > \tau$,*

$$H_0(\pm t) - H_0(\pm\infty), \qquad V(\pm t) - V(\pm\infty) \in \mathfrak{B}(X)$$

and

$$\int_\tau^\infty \| H_0(\pm t) - H_0(\pm\infty) \| \, dt + \int_\tau^\infty \| V(\pm t) - V(\pm\infty) \| \, dt < \infty.$$

Finally suppose that $[V(\pm\infty)^* V(\pm\infty)]^{1/4}(\lambda_\pm I - H_0(\pm\infty))^{-1}$ is a Hilbert-Schmidt operator for some non-real complex number $\lambda_\pm$. Then for $s \in \mathbf{R}$ the temporally inhomogeneous wave operators

$$W_\pm(s) = \text{strong}\lim_{t\to\pm\infty} U_1(s, t)\,U_0(t, s)P_0$$

exist, where P_0 is the orthogonal protection onto $\mathfrak{A}_0$; and the temporally inhomogeneous scattering operators

$$S(s) = W_+(s)^* W_-(s)$$

are unitary operators on $\mathfrak{A}_0$.

Proof. That U_1 exists as a family of unitary evolution operators follows from Corollary 5. The temporally homogeneous wave operators

$$\Omega^{\pm} = \text{strong} \lim_{t \to \pm\infty} \exp\{-itH_1(\pm\infty)\} \exp\{itH_0(\pm\infty)\}$$

exist and are complete by our Hilbert-Schmidt hypothesis, according to a result of Kuroda [15]. The theorem now follows from results of Monlezun [17]. Note that the Hilbert-Schmidt hypothesis holds if $V(\pm\infty)$ is in the trace class.

5. Second order evolution equations

Of concern here is the Cauchy problem

$$(2) \quad u''(t) + (P_1(t) + P_2(t))u'(t) + (A(t) + P_0(t))u(t) = 0 \quad (t \in \mathbf{R}),$$

$$(3) \quad u(0) = f_1, \quad u'(0) = f_2$$

in a Hilbert space. Below we present a general existence theorem for (2), (3) together with some examples.

For W, X Banach spaces, $\mathfrak{B}(W, X)$ is the space of all bounded linear operators from W to X. Write $T(\cdot) \in \text{Lip}\,(W, X)$ iff

$$T(\cdot) : \mathbf{R} \to \mathfrak{B}(W, X)$$

and for each $\tau \in \mathbf{R}^+$ there is a constant m_τ such that

$$\| T(t) - T(s) \| \leq m_\tau | t - s | \quad \text{for} \quad |t|, \quad |s| \leq \tau.$$

THEOREM 7. *Let X be a Hilbert space with inner product $(\cdot, \cdot)$. Assume:*

(B1) *For each $t \in \mathbf{R}$, $A(t)$ is a self-adjoint operator on X satisfying*

$$(A(t)f, f) \geq c_1(t)(f, f) \quad \textit{for all} \quad f \in D = D(A(t)),$$

independent of t, where $c_1(\cdot) : \mathbf{R} \to (0, \infty)$ is bounded away from 0 on bounded intervals; $A(\cdot)A(0)^{-1} \in \text{Lip}\,(X)$.

(B2) $P_2(\cdot) \in \text{Lip}\,(X)$; $P_0(\cdot) \in \text{Lip}\,(W, X)$, *where $W = D(A(0)^{1/2})$ is given the norm $\| f \|_W = \| A(0)^{1/2} f \|$.*

(B3) *For each $t \in \mathbf{R}$, $-P_1(t)$ is dissipative, $D(P_1(t)) \supset W$,*

$$\| P_1(t)f \| \leq a(t) \| A(0)^{1/2} Q_0(t) f \| + b(t) \| f \|$$

for constants $a(t) < 1$, $b(t) \geq 0$, and all $f \in W$; and $P_1(\cdot) \in \text{Lip}\,(W, X)$.[2]

[2] $Q_0(t)$ will be defined in the course of the proof. It is an operator on W which is determined by $A(0)$ and $A(t)$.

Then for any $f_1 \in D$, $f_2 \in W$ *there exists a unique function* $u(\cdot) : \mathbf{R} \to X$ *having a strongly continuous second strong derivative and satisfying* (2) *and* (3).

Proof. First assume $P_j(\cdot) \equiv 0$, $j = 0, 1, 2$. Then the theorem follows from the main result in [6]. We recall the outline of the proof. Consider the Hilbert space $Y = W \times X$ where the norm in Y is given by

$$\left\| \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\|_Y = \{\| A(0)^{1/2} f_1 \|^2 + \| f_2 \|^2\}^{1/2}.$$

Let

$$B(t) = \begin{pmatrix} 0 & I \\ -A(t) & 0 \end{pmatrix}$$

with domain $D(B(t)) = D \times W$.

In [6] we constructed $Q(\cdot) : \mathbf{R} \to \mathcal{B}(Y)$ of the form

$$Q(t) = \begin{pmatrix} Q_0(t) & 0 \\ 0 & I \end{pmatrix}$$

such that (A1), (A2) of Theorem 2 hold with $R(\cdot) \equiv I$. $Q_0(t)$ was constructed as follows. $A(t)A(0)^{-1}$, considered as a member of $\mathcal{B}(W)$, is an invertible positive self-adjoint operator; $Q_0(t)$ is its (unique) positive square root. (This is the $Q_0(t)$ appearing in (B3).) The desired conclusion of Theorem 7 with $P_j \equiv 0, j = 0, 1, 2$, follows from Theorem 2.

Now we can prove Theorem 7 as a consequence of Theorem 3. Let

$$G_2(t) = \begin{pmatrix} 0 & 0 \\ P_0(t) & -P_2(t) \end{pmatrix}.$$

Then $G_2(\cdot) \in \mathrm{Lip}\,(Y)$ by (B2). Let

$$G_1(t) = \begin{pmatrix} 0 & 0 \\ 0 & -P_1(t) \end{pmatrix}$$

with domain $D(G_1(t)) = W \times D(P_1(t))$. For $t \in \mathbf{R}$,

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in D(G_1(t)),$$

$\mathrm{Re}\,(G_1(t)f, f)_Y = \mathrm{Re}\,(-P_1(t)f_2, f_2) \le 0$ by (B3), whence $G_1(t)$ is dissipative; that $Q(t)G_1(t)Q(t)^{-1}$ is dissipative follows easily. Next,

$$G_1(t)B(0)^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = G_1(t) \begin{pmatrix} 0 & -A(0)^{-1} \\ I & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -P_1(t) & f_1 \end{pmatrix},$$

and $G_1(\cdot)(I - B(0)^{-1} \in \mathrm{Lip}\,(Y)$ follows easily from $P_1(\cdot) \in \mathrm{Lip}\,(W, X)$ (by (B3)).

Finally

$$\begin{aligned} \left\| Q(t)G_1(t) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\|_Y &= \| P_1(t) f_2 \| \\ &\le a(t) \| A(0)^{1/2} Q_0(t) f_2 \| + b(t) \| f_2 \| \quad \text{by (B3)} \\ &\le a(t) \left\| Q(t)B(t) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\|_Y + b(t) \left\| \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\|_Y \end{aligned}$$

for each

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in D \times W.$$

Theorem 7 now follows from Theorem 3.

Remark 8. Suppose that for each $t \in \mathbf{R}$, $-P_1(t)$ is dissipative, $D(P_1(t)) \supset W$, and $P_1(\cdot) \in \mathrm{Lip}\,(W, X)$. Then by the closed graph theorem there are constants $c(t)$, $d(t)$ such that

$$\| P_1(t)f \| \leq c(t) \, \| A(0)^{1/2} Q_0(t) f \| + d(t) \, \| f \|$$

for each $f \in W$. Consequently there is an $\alpha_0(t) \,(= (c(t))^{-1}) > 0$ such that $\beta(\cdot)P_1(\cdot)$ satisfies (B3) whenever $0 \leq \beta(t) < \alpha_0(t)$ for each $t \in \mathbf{R}$, and $\beta(\cdot)$ is Lipschitzian. If we only want to solve (2), (3) for t in a compact interval J, then it is easily shown that $\alpha_0(t)$ may be chosen to be independent of t for $t \in J$, and we may consider simply $\beta P_1(\cdot)$, where $0 \leq \beta < \alpha_0$.

Example 9. *Let Ω be a bounded region in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. For $i, j \in \{1, \cdots, n\}$ let α_{ij}, β_i, γ_i, δ, ε be smooth (i.e., C^∞) complex-valued functions on $\mathbf{R} \times \bar{\Omega}$ with the matrix $(\alpha_{ij}(t, x))$ being hermitian and positive definite for each $(t, x) \in \mathbf{R} \times \bar{\Omega}$. Let $\sigma, \tau \in \mathbf{R}^+$. Then there exists a $c_2 > 0$ such that given $f_1, f_2 \in C_0^\infty(\Omega)$, the problem*

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\alpha_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} \beta_i \frac{\partial u}{\partial x_i} + c \sum_{i=1}^{n} \gamma_i \frac{\partial^2 u}{\partial x_i \partial t} + \delta \frac{\partial u}{\partial t} + \varepsilon u \tag{4}$$

$$u(0, x) = f_1(x), \qquad (\partial u/\partial t)(0, x) = f_2(x), \tag{5}$$

$$u(t, x) = 0 \quad \mathit{for} \quad x \in \partial\Omega \tag{6}$$

has a unique strong solution on $[-\sigma, \tau]$ for each real c with $|c| < c_2$.

Before sketching the proof we remark that the smoothness hypotheses on the coefficients $\alpha_{ij}, \cdots, \varepsilon$ can be substantially weakened; we omit the statement of the minimal smoothness requirements on the coefficients. Also, using results of Browder [2], certain unbounded domains Ω can be considered (cf. [6]).

We are now ready to sketch the proof of the assertion of Example 9. Some of the omitted details can be found in [5], [6]. Let X be the complex Hilbert space $L^2(\Omega)$. Let $A(t)$ be the distributional differential operator

$$(A(t)f)(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\alpha_{ij}(t, x) \frac{\partial f}{\partial x_j}(x)\right)$$

with domain $D(A(t)) = (\mathcal{H}_2(\Omega) \cap \mathcal{H}_1^0(\Omega)) \times \mathcal{H}_1^0(\Omega)$.

We are using the Sobolev space notation described in [5]. Let

$$(P_0(t)f)(x) = \sum_{i=1}^{n} \beta_i(t, x) \frac{\partial f}{\partial x_i}(x) + \varepsilon(t, x) f(x),$$

$D(P_0(t)) = \mathcal{H}_1^0(\Omega)$. For $f \in C_0^\infty(\Omega)$, integration by parts shows that

$$2 \int_\Omega \gamma_i(t, x) \frac{\partial f}{\partial x_i}(x) \bar{f}(x)\, dx = -\int_\Omega \frac{\partial \gamma_i}{\partial x_i}(t, x) \, |f(x)|^2 \, dx;$$

hence if

$$(P_1(t)f)(x) = \sum_{i=1}^{n} \gamma_i(t, x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial \gamma_i}{\partial x_i}(t, x) f(x),$$

$D(P_1(t)) = \mathcal{H}_1^0(\Omega)$, then $\pm P_1(t)$ is dissipative on X. Finally let

$$(P_2(t)f)(x) = \left[-\frac{1}{2} \sum_{i=1}^{n} \frac{\partial \gamma_i}{\partial x_i}(t, x) + \delta(t, x) \right] f(x),$$

$D(P_2(t)) = X$. Then Theorem 7 and Remark 8 imply that the problem (4), (5), (6) has a unique strong solution if

$$f_1 \in D = \mathcal{H}_2(\Omega) \cap \mathcal{H}_1^0(\Omega), \qquad f_2 \in W = \mathcal{H}_1^0(\Omega)$$

and if $|c|$ is sufficiently small; (6) follows from $u(\cdot) : \mathbf{R} \to D$, which also follows from Theorem 7.

A version of Theorem 7 can be given in which the domain of $A(t)$ varies with t. The resulting theorem, whose statement we omit, generalizes the main result of [8]. The idea of the proof, as in [8], is to let $u(\cdot)$ be the solution described in Theorem 7 and to let $v(t) = T(t)u(t)$, where

$$T(\cdot) : \mathbf{R} \to \mathcal{B}(X), \qquad T''(\cdot), T(\cdot)^{-1} \in \text{Lip}(X).$$

Then v is the unique solution of the Cauchy problem for an equation of the form

$$v''(t) + \tilde{P}(t)v'(t) + \tilde{A}(t)v(t) = 0,$$

and $D(\tilde{A}(t))$ can vary with t.

As a consequence of this theorem we can generalize Example 5.2 of [8] (which corresponds to Example 10 below with $\gamma = 0$).

Example 10. *The following mixed problem is well posed.*

$$\frac{\partial^2 u}{\partial t^2} = \alpha \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial u}{\partial x} + c\gamma \frac{\partial^2 u}{\partial t \partial x} + \delta \frac{\partial u}{\partial t} + \varepsilon u + \phi$$

$(t \in [-\sigma, \tau], 0 \le x \le 1)$,

$$u(0, x) = f_1(x), \qquad (\partial u/\partial t)(0, x) = f_2(x),$$

$$(\partial u/\partial x)(t, 0) = a(t)u(t, 0), \qquad (\partial u/\partial x)(t, 1) = b(t)u(t, 1).$$

Here $\alpha, \beta, \gamma, \delta, \varepsilon, \phi$ are smooth real-valued functions on $\mathbf{R} \times [0, 1]$ with α positive; a, b are smooth real-valued functions on $\mathbf{R}$; f_1, f_2 are smooth complex-valued functions on $[0, 1]$ with

$$f_1'(0) = a(0)f_1(0), \qquad f_1'(1) = b(0)f_1(1),$$

$$f_2'(0) = a(0)f_2(0) + a'(0)f_1(0),$$

$$f_2'(1) = b(0)f_2(1) + b'(0)f_1(1);$$

$\sigma, \tau \ge 0$; and c is real with $|c| < c_3$, where c_3 is a positive constant depending only on $\alpha, \gamma, \sigma, \tau$.

The proof is omitted.

6. Concluding remarks

Remark 11. Let $\varepsilon > 0$. Theorem 7 enables one to solve a class of initial value problems of the form

$$\varepsilon u''(t) + L(t)u'(t) + M(t)u(t) = 0 \quad (t \in J), \tag{7}$$

$$u(0) = f_1, \qquad u'(0) = f_2, \tag{8}$$

where J is an interval in $\mathbf{R}$ containing 0 and $L(t), M(t)$ are suitable unbounded operators on X. Denoting the unique solution of (7), (8) by u_ε, it would be of interest to show that $u_0(t) = \lim_{\varepsilon \to 0} u_\varepsilon(t)$ exists and satisfies the (time dependent) Sobolev equation

$$L(t)u_0'(t) + M(t)u_0(t) = 0 \quad (t \in J), \qquad u(0) = f_1.$$

(See Showalter [19] for an extensive bibliography on the Sobolev equation.) We believe that the singular perturbation technique suggested above is capable of yielding significant new results for the Sobolev equation; however, this program remains to be carried out.

We note that using existing singular perturbation results, some special cases of the above program can be carried out. For instance, using results of Bobisud and Hersh [1], one can treat the case where $L(t)$, $M(t)$ are suitable polynomials in a finite number of commuting (C_0) group generators, with coefficients depending on t.

Remark 12. This is a remark on condition (A1). Recall that the *duality map* $\mathcal{J}$ for a Banach space X is defined as follows: for $\phi \in X^*$, $f \in X$, $\phi \in \mathcal{J}f$ iff $\operatorname{Re} \phi(f) = \| f \|^2 = \| \phi \|^2$. A *section* of $\mathcal{J}$ is a map $J : X \to X^*$ such that $Jf \in \mathcal{J}f$ for each $f \in X$. An operator A on X is *dissipative* iff $\| (I - \lambda A)f \| \geq \| f \|$ for each $\lambda > 0$ and each $f \in D(A)$ iff $\operatorname{Re} (Jf)(Af) \leq 0$ for some section J of $\mathcal{J}$ and all $f \in D(A)$. In the latter case one says A is *dissipative with respect to J*. A dissipative operator A is *m-dissipative* iff $D((I - A)^{-1}) = X$. Any m-dissipative operator A is dissipative with respect to J for each section J of $\mathcal{J}$.

Let $Q \in \mathcal{B}(X)$ be such that $Q^{-1} \in \mathcal{B}(X)$, and let J be a section of the duality map $\mathcal{J}$ of X. If A is dissipative with respect to J [resp. m-dissipative] and if $Q^*J = JQ^{-1}$, then QAQ^{-1} is dissipative with respect to J [resp. m-dissipative].

Proof. Write $\langle f, \phi \rangle$ for $\phi(f)$ where $f \in X$, $\phi \in X^*$. For each $f \in D(QAQ^{-1})$,

$$\begin{aligned} \operatorname{Re} \langle QAQ^{-1}f, Jf \rangle &= \operatorname{Re} \langle AQ^{-1}fQ^*Jf \rangle \\ &= \operatorname{Re} \langle A(Q^{-1}f), J(Q^{-1}f) \rangle \leq 0, \end{aligned}$$

whence QAQ^{-1} is dissipative with respect to J. Given $g \in X$ let $h = Q^{-1}g$. If A is m-dissipative, there is an $f_1 \in D(A)$ such that $f_1 - Af_1 = h$. Let $f = Qf_1$. Then

$$f - QAQ^{-1}f = Q(f_1 - Af_1) = Qh = g,$$

and so A is m-dissipative.

Note that if X is a Hilbert space, then $\mathcal{J} = J = I$, and $Q^*J = JQ^{-1}$ means that Q is unitary.

Finally, we note that the assertion of Remark 12 holds even if A is nonlinear and multi-valued; see [14] the notion of dissipative in this case.

Added in proof. Example 10 is not a hyperbolic problem if $c\gamma \neq 0$. It is closely related to a "parabolic regularization" of a hyperbolic problem in the sense of Lions-Magenes [20, pp. 280–282].

A result of the type discussed in Remark 11 has been obtained by J. L. Lions [21].

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