MODULES OVER SHEAVES OF HOLOMORPHIC FUNCTIONS WITH DIFFERENTIABLE BOUNDARY VALUES[1]

BY

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The object of this paper is to develop an analogue of the Oka-Cartan theory of coherent analytic sheaves, in the case in which the structure sheaf $\mathcal{O}$ of germs of holomorphic functions on an analytic space is replaced by the sheaf of germs of holomorphic functions on a plane domain which are continuous, or differentiable of some order, up to the boundary. To be precise, let $W \subset\subset \mathbf{C}$ be a relatively compact domain whose boundary is a finite union of simple closed m-times continuously differentiable curves. Let $\mathcal{A}$ denote the sheaf of germs of Whitney C^m-functions on $\bar{W}$ which are holomorphic on W. It is known that $H^q(\bar{W}, \mathcal{A}) = (0)$ for $q \geq 1$, and in this paper we obtain analogues of Cartan's theorems A and B for a certain class of sheaves of modules over $\mathcal{A}$.

It turns out that the class of coherent sheaves of modules over $\mathcal{A}$, defined in the usual way, is too small, for this class is not preserved under the usual algebraic operations on sheaves. For example, we show in section III that the kernel of a sheaf map $\mu : \mathcal{A}^p \to \mathcal{A}$ need not be locally finitely generated. The reason for this is that the stalks of the sheaf $\mathcal{A}$, at points on the boundary of W, fail to be Noetherian rings, and hence the classical definition of coherence is not appropriate.

In Section I we introduce new notions of a *globally Oka* sheaf of rings $\mathcal{R}$, and a *globally coherent* sheaf of modules $\mathcal{F}$ over $\mathcal{R}$ on a topological space X, which turn out to be more useful in the study of differentiable boundary values of holomorphic functions. The definition of globally coherent sheaf has built into it an analogue of Cartan's theorem A: global sections generate the stalks at each point. But we then show that under certain hypotheses on the sheaf of rings $\mathcal{R}$ and the space X, we obtain an analogue of Cartan's theorem B: every globally coherent sheaf has vanishing higher cohomology. We also show that the class of globally coherent sheaves of modules is preserved under the usual algebraic operations on sheaves. In Section II, we show that the sheaf $\mathcal{A}$ is a globally Oka sheaf of rings over $\bar{W}$, and that the theory developed in Section I applies. In particular, we obtain a vanishing theorem for cohomology of globally coherent sheaves over $\mathcal{A}$. In Section III, we show that coherent sheaves over $\mathcal{A}$ in the classical sense are in fact globally coherent, but that there are globally coherent sheaves which are not coherent. We also show that ideal sheaves of varieties, defined in the usual way, are globally

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coherent. Finally, in Section IV we prove a variant of Cartan's lemma on holomorphic matrices which is used in Section III.

I. Global coherence

In this section we study analogues of the classical notions of coherent sheaf of modules, and Oka sheaf of rings (see Gunning and Rossi [2], pages 128–133). These analogues seem appropriate in the study of sheaves of holomorphic functions on a plane domain having continuous or differentiable boundary values, but we present the results of this section in a quite general setting.

We shall use the following notation. Let $\mathcal{S}$ be any sheaf of abelian groups on a topological space X. For any $x \in X$, $\mathcal{S}_x$ denotes the stalk of $\mathcal{S}$ at x. For any set $Y \subset X$, $\Gamma(Y, \mathcal{S})$ denotes the group of sections of $\mathcal{S}$ over Y. If $f \in \Gamma(Y, \mathcal{S})$, and if $x \in Y$, $[f]_x$ denotes the germ determined by f in $\mathcal{S}_x$. $H^q(X, \mathcal{S})$ denotes the q-th Čech cohomology group of X with coefficients in $\mathcal{S}$.

We now define the analogues of coherent sheaf of modules, and Oka sheaf of rings.

DEFINITION 1.1. Let $\mathcal{R}$ be a sheaf of rings on a topological space X, and let $\mathcal{F}$ be a sheaf of $\mathcal{R}$-modules on X.

(a) $\mathcal{F}$ is *globally generated* if for every $x \in X$ and every $f \in \mathcal{F}_x$, there exist $F_1, \cdots, F_m \in \Gamma(X, \mathcal{F})$ and $r_1, \cdots, r_m \in \mathcal{R}_x$ so that $f = \sum_{i=1}^m r_i[F_i]_x$.

(b) If $F_1, \cdots, F_m \in \Gamma(X, \mathcal{F})$, the *sheaf of relations* among $F_1, \cdots, F_m$ is

$$R[F_1, \cdots, F_m] = \{(r_1, \cdots, r_m) \in \mathcal{R}^m \mid \textstyle\sum_{i=1}^m r_i F_i = 0\}.$$

(c) The sheaf $\mathcal{F}$ is *globally coherent* if $\mathcal{F}$ is globally generated, and if for every finite set $\{F_1, \cdots, F_m\} \subset \Gamma(X, \mathcal{F})$, the sheaf $R[F_1, \cdots, F_m]$ is also globally generated.

DEFINITION 1.2. Let $\mathcal{R}$ be a sheaf of rings on a topological space X. Then $\mathcal{R}$ is a *globally Oka* sheaf of rings if for every sheaf map $\mu : \mathcal{R}^p \to \mathcal{R}^q$ over X, the kernel of μ is globally generated.

The following result gives a sufficient condition for a sheaf of rings to be globally Oka. The proof is a simple modification of the induction step in the proof of the Oka coherence theorem, and is omitted (see for example Gunning and Rossi [2], page 136).

LEMMA 1.3. *Let $\mathcal{R}$ be a sheaf of rings on a topological space X, and suppose that for every sheaf map $\mu : \mathcal{R}^p \to \mathcal{R}$ over X, the kernel is globally generated. Then $\mathcal{R}$ is a globally Oka sheaf of rings.*

We shall also need the following analogue of an easy standard result. Again the proof is omitted (see Gunning and Rossi [2], Chapter IV, Corollary B9).

LEMMA 1.4. *Let $\mathcal{R}$ be a globally Oka sheaf of rings on a topological space X, and let $\mathcal{F} \subset \mathcal{R}^m$ be a globally generated sheaf of modules. Then $\mathcal{F}$ is globally coherent.*

The definition of global coherence has built into it an analogue of Cartan's theorem A: the global sections of a globally coherent sheaf generate the stalks of each point. The first main result of this section is that if $\mathcal{R}$ is a globally Oka sheaf of rings on X with vanishing higher cohomology groups, then under suitable conditions on the space X, we obtain an analogue of Cartan's theorem B for globally coherent sheaves of modules over $\mathcal{R}$. We first recall the following:

DEFINITION 1.5. A topological space X has covering dimension $N < \infty$ if every open cover of X has a refinement so that every point of X lies in at most $N + 1$ sets of the refinement.

THEOREM 1.6. *Let $\mathcal{R}$ be a sheaf of rings on a topological space X. Suppose that*

(i) *X is compact Hausdorff and has finite covering dimension,*
(ii) *$\mathcal{R}$ is a globally Oka sheaf of rings,*
(iii) *$H^q(X, \mathcal{R}) = (0)$ for all $q \geq 1$.*

Then for any globally coherent sheaf $\mathfrak{F}$ of $\mathcal{R}$-modules on X, $H^q(X, \mathfrak{F}) = (0)$ for all $q \geq 1$.

Proof. Suppose that the covering dimension of X is $N < \infty$. It follows that for any sheaf $\mathcal{S}$ on X, $H^q(X, \mathcal{S}) = (0)$ for all $q \geq N + 1$. We now argue by inverse induction on q, and assume that for all $q \geq k + 1$, $H^q(X, \mathcal{G}) = (0)$ for every globally coherent sheaf $\mathcal{G}$. Let $\mathfrak{F}$ be a globally coherent sheaf of $\mathcal{R}$-modules on X, and let $\xi \in H^k(X, \mathfrak{F})$. If we can prove that $\xi = 0$, this will complete the induction step, and hence prove the theorem.

Since X is compact, we can represent ξ as a k-cocycle on a finite open cover $\{V_i\}$ of X. Since X is normal, we can find a refinement $\{U_i\}$ of this cover so that $U_i \subset \bar{U}_i \subset V_i$ for each i. Then if ξ is the k-cocycle $\{\xi_{i_0,\cdots,i_k}\}$, $\xi_{i_0,\cdots,i_k}$ is a section of $\mathfrak{F}$ on an open neighborhood of the compact set $\bar{U}_{1_0} \cap \cdots \cap \bar{U}_{i_k}$. Note that since $\{U_i\}$ is finite, there are only finitely many such intersections.

Let $x \in \bar{U}_{i_0} \cap \cdots \cap \bar{U}_{i_k}$. Since $\mathfrak{F}$ is globally generated, we can find a neighborhood W_x of x, sections $r_1, \cdots, r_m \in \Gamma(W_x, \mathcal{R})$, and global sections $F_1, \cdots, F_m \in \Gamma(X, \mathfrak{F})$ so that

$$(1) \qquad [\xi_{i_0,\cdots,i_k}]_x = \textstyle\sum_{i=1}^m [r_i]_x[F_i]_x.$$

Since equation (1) gives an equality of germs of sections of $\mathfrak{F}$, we can find an open neighborhood $N_x \subset W_x$ of x so that for all $y \in N_x$,

$$(2) \qquad [\xi_{i_0,\cdots,i_k}]_y = \textstyle\sum_{i=1}^m [r_i]_y[F_i]_y.$$

We can now find a finite number of these neighborhoods N_x which cover the finite number of sets $\{\bar{U}_{i_0} \cap \cdots \cap \bar{U}_{i_k}\}$, and for each such neighborhood we get a finite number of global sections of $\mathfrak{F}$. Let $F_1, \cdots, F_n$ be the set of all these global sections, and let $\mathcal{G} \subset \mathfrak{F}$ be the subsheaf generated by $F_1, \cdots, F_n$.

Equation (2) now shows that $\xi_{i_0,\cdots,i_k}$ is a section of $\mathcal{G}$ over a neighborhood of $\bar{U}_{i_0} \cap \cdots \cap \bar{U}_{i_k}$, and hence the k-cocycle $\{\xi_{i_0,\cdots,i_k}\}$ on the open cover $\{U_i\}$ represents an element $\xi^* \in H^k(X, \mathcal{G})$. Moreover, if

$$i^* : H^k(X, \mathcal{G}) \to H^k(X, \mathcal{F})$$

is the map induced by inclusion, then by construction $i^*(\xi^*) = \xi$.

Define a map $\mu : \mathcal{R}^n \to \mathcal{F}$ by $\mu(r_1, \cdots, r_n) = \sum_{i=1}^m r_i F_i$. Then the image of μ is $\mathcal{G}$. Let $\mathcal{K}$ be the kernel of μ. Then $\mathcal{K} = R[F_1, \cdots, F_n]$, and since $\mathcal{F}$ was assumed to be globally coherent, $\mathcal{K}$ is globally generated. Since $K \subset \mathcal{R}^n$, it follows from Lemma 1.4 that $\mathcal{K}$ is globally coherent.

We now have a short exact sequence of sheaves

$$(0) \to \mathcal{K} \to \mathcal{R}^n \xrightarrow{\mu} \mathcal{G} \to (0).$$

The long exact sequence in cohomology gives

$$\cdots \to H^k(X, \mathcal{R}^n) \to H^k(X, \mathcal{G}) \to H^{k+1}(X, \mathcal{K}) \to H^{k+1}(X, \mathcal{R}^n) \to \cdots$$

but since $H^q(X, \mathcal{R}) = (0)$ for $q \geq 1$, we get $H^k(X, \mathcal{G}) \cong H^{k+1}(X, \mathcal{K})$. But $H^{k+1}(X, \mathcal{K}) = (0)$ by induction, so $\xi^* = 0$, and hence $\xi = 0$. This completes the proof.

The class of globally coherent sheaves is closed under the usual algebraic operations on sheaves, at least under the hypothesis of Theorem 1.6. In fact, we have the following more general result:

THEOREM 1.7. *Let $\mathcal{R}$ be a sheaf of rings on a topological space X, and suppose that $H^1(X, \mathcal{F}) = (0)$ for every globally coherent sheaf $\mathcal{F}$ of $\mathcal{R}$-modules. Let*

$$(0) \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to (0)$$

be a short exact sequence of sheaves of $\mathcal{R}$-modules. Then if any two sheaves in the sequence are globally coherent, so is the third.

Proof. The proof of this result is a tedious diagram chase, which is very similar to the classical argument for coherent analytic sheaves (see Gunning and Rossi [2], Chapter 4, Proposition B 13). The only difference is that one needs the hypothesis that $H^1(X, \mathcal{E}) = (0)$ when $\mathcal{E}$ is globally coherent to insure that the global sections map $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G})$ is surjective. We omit details.

COROLLARY 1.8. *Suppose that X and $\mathcal{R}$ satisfy the hypotheses of Theorem 1.6. If $\mathcal{F}$ is a sheaf of $\mathcal{R}$-modules over X and a sequence*

$$\mathcal{R}^p \xrightarrow{\mu} \mathcal{R}^q \to \mathcal{F} \to (0)$$

is exact on X, then $\mathcal{F}$ is globally coherent.

Proof. Let $\mathcal{G}$ be the image of μ. By Lemma 1.4, $\mathcal{G}$ is globally coherent, and we have an exact sequence $(0) \to \mathcal{G} \to \mathcal{R}^q \to \mathcal{F} \to (0)$. By Theorem 1.7, it follows that $\mathcal{F}$ is globally coherent.

II. The sheaves $\mathfrak{A}(m, T)$

In this section we study specific examples of sheaves of rings on plane domains. While we are primarily interested in the case of holomorphic functions with differentiable boundary valves, for technical reasons that will appear in Section III we consider the following slightly more general situation. Fix an element m of the set $\{0, 1, 2, \cdots, \infty\}$. Let $W \subset\subset \mathbf{C}$ be a relatively compact plane domain whose boundary ∂W consists of a finite number of simple closed m-times continuously differentiable curves. Let $T \subset \partial W$ be a finite collection of open intervals on the boundary of W. Then let $\mathfrak{A} = \mathfrak{A}(m, T)$ denote the sheaf of germs of Whitney C^m-functions on the closure $\bar{W}$ of W which are holomorphic on $W \cup T$.

THEOREM 2.1. *$\mathfrak{A} = \mathfrak{A}(m, T)$ is a globally Oka sheaf of rings.*

Proof. By Lemma 1.3, it suffices to show that if $\mu : \mathfrak{A}^p \to \mathfrak{A}$ is a sheaf map over $\bar{W}$, and if $\mathfrak{F}$ is the kernel of μ, then $\mathfrak{F}$ is globally generated. Let μ be given by the map $\mu(a_1, \cdots, a_p) = \sum_{i=1}^{m} a_i \mu_i$ where $\mu_i \in \Gamma(\bar{W}, \mathfrak{A})$. We may assume that the map μ is not identically zero, and hence without loss of generality that μ_1 is not identically zero. Let $Z(\mu_1) = \{z \in \bar{W} \mid \mu_1(z) = 0\}$. Then $Z(\mu_1)$ is compact, $Z(\mu_1) \cap W$ is discrete, and it follows from the Riemann mapping theorem and Jensen's inequality that $Z(\mu_1) \cap \partial W$ is totally disconnected.

Let $z_0 \in \bar{W}$, and let $(f_1, \cdots, f_p) \in \mathfrak{F}_{z_0}$ so that $\sum_{i=1}^{p} f_i \mu_1 = 0$. We distinguish three possibilities.

(i) Suppose $z_0 \in \bar{W} - Z(\mu_1)$. Then $(\mu_1)^{-1} \in \mathfrak{A}_{z_0}$, and hence

$$f_1 = -\sum_{i=2}^{p} ((\mu_1)^{-1} f_i)\mu_i.$$

Then

$$(f_1, \cdots, f_p) = \sum_{i=2}^{p} ((\mu_1)^{-1} f_i) k_i \tag{1}$$

where k_i is the p-tuple $(-\mu_j, 0, \cdots, 0, \mu_1, 0, \cdots, 0)$ where μ_1 appears in the i-th place. But each k_i is a global section of $\mathfrak{F}$, and thus equation (1) shows that $\mathfrak{F}_{z_0}$ is in fact generated by a finite number of global sections of $\mathfrak{F}$.

(ii) Suppose $z_0 \in W \cup T$. Then there is an integer $n \geq 0$ so that

$$\mu_1(z) = (z - z_0)^n \nu_i(z) \quad \text{for} \quad i = 1, \cdots, p$$

where $\nu_i \in \Gamma(\bar{W}, \mathfrak{A})$, and so that $\nu_{j_0}(z_0) \neq 0$ for at least one j_0. Let $\nu : \mathfrak{A}^p \to \mathfrak{A}$ be defined by $\nu(a_1, \cdots, a_p) = \sum_{i=1}^{p} a_i \nu_i$. Then the kernel of ν is precisely the sheaf $\mathfrak{F}$, and by treating ν_{j_0} as we did μ_1 in case (i), we see that $\mathfrak{F}_{z_0}$ is generated by a finite number of global sections of $\mathfrak{F}$.

(iii) Suppose $z_0 \in \partial W \cap Z(\mu_1)$. Choose a relatively open neighborhood V_1 of z_0 in $\bar{W}$ so that all the germs $\{f_i\}$ have representations as functions on $\bar{V}_1$. Choose V_2 a relatively open neighborhood of $\bar{W} - V_1$. Since $Z(\mu_1) \cap W$ is discrete and $Z(\mu_1) \cap \partial W$ is totally disconnected, we can find relatively open sets $U_i \subset V_i$, $i = 1, 2$ so that $U_1 \cup U_2 = \bar{W}$ and so that

(a) the boundaries of U_1, U_2, and $U_1 \cap U_2$ consist of a finite number of simple closed m-times continuously differentiable curves,

(b) $(\bar{W} - U_1) \cap (\bar{W} - U_2) = \emptyset$,

(c) $(U_1 \cap U_2)^- \cap Z(\mu_1) = \emptyset$.

Because of property (c), $(\mu_1)^{-1} \in \Gamma((U_1 \cap U_2)^-, \mathfrak{A})$. Thus as in case (i), we can write

$$(f_1, \cdots, f_p) = \sum_{i=2}^p ((\mu_1)^{-1} f_i) k_i$$

where $(\mu_1)^{-1} f_i \in \Gamma((U_1 \cap U_2)^-, \mathfrak{A})$. In particular, each $(\mu_1)^{-1} f_i$ is a Whitney C^m-function on $(U_1 \cap U_2)^-$ which is holomorphic on the interior of $U_1 \cap U_2$. By Lemma 4.1, we can write $(\mu_1)^{-1} f_i = g_i - h_i$ where g_i is a Whitney C^m-function on $\bar{U}_2$, holomorphic on the interior, and h_i is a Whitney C^m-function on $\bar{U}_1$, holomorphic on the interior. Thus

$$(f_1, \cdots, f_p) + \sum_{i=2}^p h_i k_i = \sum_{i=2}^p g_i k_i \tag{2}$$

on $U_1 \cap U_2$. In equation (2), the right hand side is a global section of $\mathfrak{F}$ over U_2, while the left hand side is a global section of $\mathfrak{F}$ over U_1. Hence we can define a global section of $\mathfrak{F}$ over all of $\bar{W}$ by

$$\begin{aligned} (t_1, \cdots, {}_x t_p) &= (f_1, \cdots, f_p) + \textstyle\sum_{i=2}^p h_i k_i && \text{on} \quad U_1 \\ &= \textstyle\sum_{i=2}^p g_i k_i && \text{on} \quad U_2. \end{aligned}$$

But then $(f_1, \cdots, f_p) = (t_1, \cdots, t_p) - \sum_{i=2}^p h_i k_i$ and this shows that $\mathfrak{F}_{z_0}$ is generated by global sections of $\mathfrak{F}$. This completes the proof of the theorem.

THEOREM 2.2. *Let $\bar{W}$ and $\mathfrak{A}$ be as in Theorem 2.1. If $\mathfrak{F}$ is a globally coherent sheaf of $\mathfrak{A}$-modules over $\bar{W}$. Then $H^q(\bar{W}, \mathfrak{F}) = (0)$ for $q \geq 1$.*

Proof. We simply show that $\bar{W}$ and $\mathfrak{A}$ verify the conditions of Theorem 1.6. $\bar{W}$ is a compact Hausdorff space of covering dimension 2 (see [3], Theorem IV 3). By Theorem 2.1, $\mathfrak{A}$ is a globally Oka sheaf of rings. Finally, if $T = \emptyset$, the fact that $H^q(\bar{W}, \mathfrak{A}) = (0)$ for $q \geq 1$ is in [4, Theorem 1.9]. (The condition there that ∂W be $(m + 1)$-times continuously differentiable is easily modified since $\bar{W} \subset \mathbf{C}$). If $T \neq \emptyset$, we can find a sequence of domains $W_i \subset \mathbf{C}$, $i = 1, 2, \cdots$, each having m-times continuously differentiable boundary curves, so that $\bar{W} = \bigcap_{i=1}^\infty W_i$, $\bar{W}_{i+1} \subset \bar{W}_i$, and

$$\partial W \cap \partial W_i = \partial W - T.$$

If we define sheaves $\mathfrak{A}_i$ of germs of Whitney C^m functions on $\bar{W}_i$ which are holomorphic on W_i, then $H^q(\bar{W}_i, \mathfrak{A}_i) = (0)$ for $q \geq 1$. But $H^q(\bar{W}, \mathfrak{A}) = \lim_{\rightarrow} H^q(\bar{W}_i, \mathfrak{A}_i) = (0)$. Thus Theorem 2.2 follows from Theorem 1.6.

THEOREM 2.3. *Let $\mu : \mathfrak{A}^p \rightarrow \mathfrak{A}^q$ be a sheaf map over $\bar{W}$, and let $\mathfrak{F}$ denote either the kernel, cokernel, or image sheaf of μ. Then $H^q(\bar{W}, \mathfrak{F}) = (0)$ for $q \geq 1$.*

Proof. This follows from Theorems 1.8 and 2.2.

III. Globally coherent sheaves over $\mathfrak{A}$

In this section we study examples of globally coherent sheaves of modules over the sheaf of rings $\mathfrak{A} = \mathfrak{A}(m, \emptyset)$ studied in Section II. Our definition of global coherence involves global behavior of the sheaf of modules, while the classical definition of coherence only involves local behavior. We show that in the case of sheaves of modules over $\mathfrak{A}$, coherence in the classical sense implies global coherence. The proof of this fact uses a variant of Cartan's theorem on holomorphic matrices, which is presented in Section IV. We next show that if varieties and ideal sheaves are defined in the usual way relative to the sheaf $\mathfrak{A}$, then ideal sheaves are globally coherent. Finally, we show that there are globally coherent sheaves over $\mathfrak{A}$ which are not coherent in the classical sense, and hence the introduction of the notion of global coherence is not artificial.

We recall first the classical definition of coherence:

DEFINITION 3.1. A sheaf $\mathfrak{F}$ of modules over $\mathfrak{A}$ is *coherent* if for each $z \in \bar{W}$, there exists a relatively open neighborhood U of z, and an exact sequence

$$\mathfrak{A}^p \to \mathfrak{A}^q \to \mathfrak{F} \to (0)$$

over U.

The main step in proving that coherence implies global coherence is then the following:

LEMMA 3.2. *Let $\bar{W} = U_1 \cup U_2$ where U_1, U_2 are relatively open subsets of $\bar{W}$ such that*

(a) *the boundaries of U_1, U_2, and $U_3 = U_1 \cap U_2$ each consist of a finite number of simple closed m-times continuously differentiable curves,*

(b) $(\bar{W} - U_1) \cap (\bar{W} - U_2) = \emptyset$,

(c) *U_3 consists of a finite number of disjoint, simply connected regions. Suppose that $\mathfrak{F}$ is a sheaf of $\mathfrak{A}$-modules on $\bar{W}$, and that for $j = 1, 2$, there is an exact sequence over $\bar{U}_j$:*

$$\mathfrak{A}^{p_j} \xrightarrow{\nu_j} \mathfrak{A}^{p_j} \to \mathfrak{F} \to (0). \tag{1}$$

Then there is an exact sequence over all of $\bar{W}$:

$$\mathfrak{A}^r \xrightarrow{\tau} \mathfrak{A}^s \to \mathfrak{F} \to (0).$$

Proof. Let $\mathfrak{A}^\sim$ be the sheaf obtained by restricting the sheaf $\mathfrak{A}$ to the set $\bar{U}_3 = \bar{U}_1 \cap \bar{U}_2$. Then $\mathfrak{A}^\sim$ is the sheaf of germs of Whitney C^m-functions on $\bar{U}_3$ which are holomorphic on the interior of U_3 and on $\partial U_3 \cap W$. But $\partial U_3 \cap W$ is just a finite collection of open intervals on ∂U_3, and so $\mathfrak{A}^\sim$ is a sheaf of the type covered by Theorem 2.1. Hence $\mathfrak{A}^\sim$ is a globally Oka sheaf of rings on $\bar{U}_3$, and on $\bar{U}_3$ we have exact sequences

$$\mathfrak{A}^{\sim p_j} \to \mathfrak{A}^{\sim q_j} \to \mathfrak{F} \to (0) \quad \text{for} \quad j = 1, 2.$$

By Corollary 1.8, $\mathfrak{F}$ is a globally coherent sheaf of $\mathfrak{A}^{\sim}$ modules. By Lemma 1.4, the images of the maps ν_j are globally coherent, and hence by Theorem 2.2, the induced sequences of global sections are exact:

$$\Gamma(\bar{U}_3, \mathfrak{A}^{\sim p_j}) \rightarrow \Gamma(\bar{U}_3, \mathfrak{A}^{\sim q_j}) \rightarrow \Gamma(\bar{U}_3, \mathfrak{F}) \rightarrow (0) \quad \text{for} \quad j = 1, 2.$$

It now follows, as in Gunning and Rossi [2, Chapter VI, Theorem F3], that we can modify the sequences (1) as follows: we can find integers r and s, and exact sequences:

$$\mathfrak{A}^r \xrightarrow{\sigma_j} \mathfrak{A}^s \rightarrow \mathfrak{F} \rightarrow (0)$$

over $\bar{U}_j$ so that over $\bar{U}_3$ we have a commutative diagram

$$\begin{array}{ccccccc} \mathfrak{A}^r & \xrightarrow{\sigma_1} & \mathfrak{A}^s & \rightarrow & \mathfrak{F} & \rightarrow & (0) \\ \uparrow \mu & & \uparrow \lambda & & \uparrow \text{id} & & \\ \mathfrak{A}^r & \xrightarrow{\sigma_2} & \mathfrak{A}^s & \rightarrow & \mathfrak{F} & \rightarrow & (0). \end{array}$$

Moreover, the sheaf maps $\lambda : \mathfrak{A}^s \rightarrow \mathfrak{A}^s$, and $\mu : \mathfrak{A}^r \rightarrow \mathfrak{A}^s$ over $\bar{U}_3$ are invertible.

We interpret the sheaf maps μ and λ as non-singular $r \times r$ and $s \times s$ matrices whose entries are sections of $\mathfrak{A}$ over $\bar{U}_3$; i.e. $\mu = \{\mu_{ij}\}$, $\lambda = \{\lambda_{ij}\}$ where μ_{ij} and λ_{ij} are Whitney C^m-functions on $\bar{U}_3$ which are holomorphic on the interior of U_3. By Theorem 4.2 we can write

$$\lambda = (\lambda_1)(\lambda_2)^{-1} \quad \text{and} \quad \mu = (\mu_1)(\mu_2)^{-1}$$

where μ_i, λ_i are non-singular $r \times r$ and $s \times s$ matrices over U_i with entries which are sections of $\mathfrak{A}$. In the obvious way we let the matrices λ_i and μ_i define sheaf maps

$$(\lambda_1)^{-1} : \mathfrak{A}^s \rightarrow \mathfrak{A}^s, \qquad (\mu_1)^{-1} : \mathfrak{A}^r \rightarrow \mathfrak{A}^r$$

over U_1, and

$$\lambda_2 : \mathfrak{A}^s \rightarrow \mathfrak{A}^s, \qquad \mu_2 : \mathfrak{A}^r \rightarrow \mathfrak{A}^r$$

over U_2. We then have exact sequences

$$\mathfrak{A}^r \xrightarrow{(\lambda_1)^{-1}\sigma_1 \mu_1} \mathfrak{A}^s \rightarrow \mathfrak{F} \rightarrow (0)$$

over U_1 and

$$\mathfrak{A}^r \xrightarrow{(\lambda_2)^{-1}\sigma_2 \mu_2} \mathfrak{A}^s \rightarrow \mathfrak{F} \rightarrow (0)$$

over U_2. But over $U_3 = U_1 \cap U_2$

$$(\lambda_1)^{-1} \circ \sigma_1 \circ \mu_1 = (\lambda_2)^{-1} \circ \sigma_2 \circ \mu_2.$$

Hence

$$\begin{aligned} \tau &= (\lambda_1)^{-1} \circ \sigma_1 \circ \mu_1 \quad \text{on} \quad U_1 \\ &= (\lambda_2)^{-1} \circ \sigma_2 \circ \mu_2 \quad \text{on} \quad U_2 \end{aligned}$$

defines a global sheaf map over $\bar{W}$, and we obtain the required exact sequence

$$\mathfrak{A}^r \to \mathfrak{A}^s \to \mathfrak{F} \to (0).$$

We now can prove

THEOREM 3.3. *Let $\mathfrak{F}$ be a coherent sheaf of modules over $\mathfrak{A}$. Then $\mathfrak{F}$ is globally coherent.*

Proof. By Corollary 1.8, it suffices to show that there is an exact sequence $\mathfrak{A}^r \to \mathfrak{A}^s \to \mathfrak{F} \to (0)$ over $\bar{W}$. Suppose there is no such sequence. By Lemma 3.2, it follows that if $\bar{W} = U_1 \cup U_2$ with U_1, U_2 relatively open in $\bar{W}$, satisfying conditions (a), (b), and (c) of Lemma 3.2, then there is no exact sequence $\mathfrak{A}^{p_i} \to \mathfrak{A}^{q_i} \to \mathfrak{F} \to (0)$ over $\bar{U}_i$ for at least one of $\bar{U}_1$, $\bar{U}_2$. By continuing to subdivide in this way, we find that there is a point $z \in \bar{W}$ such that in no open neighborhood of z is there an exact sequence of the required type. But this contradicts the definition of coherence. Hence $\mathfrak{F}$ is globally coherent.

Next we study the notions of variety and ideal sheaf.

DEFINITION 3.4. A subset $V \subset \bar{W}$ is a *variety* relative to the structure sheaf $\mathfrak{A}$ if for every $z_0 \in \bar{W}$, there is a relatively open neighborhood U of z_0 in $\bar{W}$, and a collection of functions $\{f_i \in \Gamma(U, \mathfrak{A}),\ i \in I\}$ so that

$$V \cap U = \{z \in U \mid f_i(z) = 0 \quad \text{for all} \quad i \in I\}.$$

The *ideal sheaf* of the variety V is the sheaf

$$\{f \in \mathfrak{A} \mid f(z) = 0 \quad \text{for all} \quad z \in V\}.$$

Before proving that ideal sheaves are globally coherent, we need the following covering lemma.

LEMMA 3.5. *Let $Z \subset \bar{W}$ be a compact set such that $Z \cap W$ is discrete and $Z \cap \partial W$ is totally disconnected. Suppose that $\{U_i\} i = 1, \cdots, n$ is a finite open cover of $\bar{W}$. Then there exists a finite refinement $\{\tilde{U}_j\} j = 1, \cdots, m$ of the cover $\{U_i\}$ so that for all $i \neq j$, $\tilde{U}_i \cap \tilde{U}_j \cap Z = \emptyset$.*

Proof. This is a purely topological result, and hence by taking a homeomorphic image of $\bar{W}$, we may assume that the boundary of $\bar{W}$ consists of a finite number of circles.

Choose $\delta_0 > 0$ so that for all $z_0 \in \bar{W}$, there exists an element of the cover U_i so that $\{z \in \mathbf{C} \mid |z - z_0| < \delta_0\} \subset U_i$, (i.e. δ_0 is the Lebesgue number of the covering $\{U_i\}$). Let X and Y be the projections of Z on the real and imaginary axes. Then X and Y are compact, and because $\bar{W}$ has nice boundary, X and Y are totally disconnected. Hence we can choose real numbers $\{a_j\} j = 1, \cdots, n$, $\{b_k\} k = 1, \cdots, m$ so that:

(i) $a_j < a_{j+1}$; $b_k < b_{k+1}$; $|a_{j+1} - a_j| < \frac{1}{2}\delta_0$; $|b_{k+1} - b_k| < \frac{1}{2}\delta_0$.

(ii) $a_j \notin X$ for all j, and $ib_k \notin Y$ for all k.

(iii) For all $z \in \bar{W}$, if $z = x + iy$ then

$$a_0 < x < a_n \, b_0 < y < b_m.$$

Let $\delta_1 = \min_j \text{dist}\,(a_j, X)$, $\delta_2 = \min_k \text{dist}\,(b_k, Y)$. Then $\delta_1, \delta_2 > 0$ since X and Y are closed. Let $\varepsilon = \frac{1}{2}\min\,(\delta_0, \delta_1, \delta_2)$. Put

$$N_{jk} = \{(x + iy) \in \mathbf{C} \mid a_{j-1} - \varepsilon < x < a_j + \varepsilon, b_{k-1} - \varepsilon < y < b_k + \varepsilon\}.$$

Then $N_{jk}\, j = 1, \cdots, n$, $h = 1, \cdots, m$ form a cover of $\bar{W}$, and by construction, each N_{jk} is contained in some U_i, and $N_{j_1,k_1} \cap N_{j_2\,k_2} \cap Z = \emptyset$ if $(j_1, k_1) \neq (j_2, k_2)$. The cover $\{N_{jk}\}$ is the required refinement.

THEOREM 3.6. *If $V \subset \bar{W}$ is a variety then $\mathcal{I}_v$ is globally coherent.*

Proof. Since $\mathcal{I}$ is a subsheaf of $\mathcal{A}$, by Lemma 1.4, it suffices to show that $\mathcal{I}_v$ is globally generated. Let $z_0 \in \bar{W}$ and let $f_0 \in (\mathcal{I})_{z_0}$. Let U_0 be a relatively open neighborhood of z_0 in $\bar{W}$ on which f_0 is defined. For each $z \in \bar{W} - \{z_0\}$ choose an element $f_z \in (\mathcal{I})_z$ and a relatively open neighborhood U_z of z in $\bar{W} - \{z_0\}$ in which f_z is defined. The sets U_0, and U_z, $z \in \bar{W} - \{z_0\}$ cover $\bar{W}$ so we can find a finite subcover $U_0, \cdots, U_N$ with functions $f_j \in \Gamma(U_j, \mathcal{I})$. Since $\bar{W}$ is normal, we can find a refinement of this cover, $U_0', \cdots, U_N'$ so that $\bar{U}_j' \subset U_j$. Let

$$Z_j = \{z \in \bar{U}_j' \mid f_j(z) \in 0\}$$

and let $Z = \bigcup_{j=1}^N Z_j$. Then Z is compact, $Z \cap W$ is discrete, and $Z \cap \partial W$ is totally disconnected.

By Lemma 3.5 we can find a refinement $N_0, \cdots, N_m$ of $\{U_j\}, j = 1, \cdots, N$ so that if $i \neq j$, $N_i \cap N_j \cap Z = \emptyset$. Thus f_i and f_j have no zeros in $N_i \cap N_j$, so $g_{ij} = f_i/f_j$ is an invertible section of $\mathcal{A}$ over $N_i \cap N_j$. Thus $\{g_{ij}\}$ represents an element of $H'(\bar{W}, \mathcal{A}^*)$, where $\mathcal{A}^*$ is the sheaf of invertible elements of $\mathcal{A}$. But $H^1(\bar{W}, \mathcal{A}^*) \cong H^2(\bar{W}, \mathbf{Z}) = (0)$ (see [4], Theorem 20). Hence on some refinement $\{\tilde{N}_i\}$ of the cover $\{N_i\}$ we can find non-zero functions g_i so that $f_{ij} = g_j/g_i$. Hence $f_i g_i = f_j g_j$ on $\tilde{N}_i \cap \tilde{N}_j$ and so $F = f_i g_i$ on N_i defines a global section of $\mathcal{I}_v$. Moreover on $\tilde{N}_0$, $f_0 = g_0^{-1}F$ so the global sections of $\mathcal{I}_v$ generate $(\mathcal{I})_{z_0}$. Thus $\mathcal{I}$ is globally coherent.

The final object of this section is to show that there are examples of globally coherent sheaves which are not coherent. For simplicity, we consider the case in which W is the open unit disc, and $\mathcal{A}$ is the sheaf of germs of continuous functions on $\bar{W}$ which are holomorphic on W. The crucial fact is that for $z \in \partial W$, $\mathcal{A}_z$ is not a Noetherian ring.

LEMMA 3.7. *The ideal $I = \{f \in \mathcal{A}_1 \mid f(1) = 0\}$ is not finitely generated.*

Proof. Suppose that I were generated by $f_1, \cdots, f_p$. Choose a neighborhood N_1 of 1 in which all the f_j are defined. Let $h(z) = \exp((z + 1)/(z - 1))$. Then $hf_j \in I$ for $j = 1, \cdots, p$, and so we can find $a_{ij} \in \mathcal{A}$, so that

$$hf_u = \sum_{i=1}^p a_{ij} f_i \; j = 1, \cdots, p.$$

Since $(z - 1) \in I$, there are $b_j \in \mathfrak{A}$, $j = 1, \cdots, p$ so that $(z - 1) = \sum_{i=1}^{p} b_i f_i$. Choose a neighborhood $N_2 \subset N_1$ in which all the a_{ij}, b_i are defined.

We have $\{z \in N_2 \mid f_j(z) = 0\ j = 1, \cdots, p\} = \{1\}$ since $z - 1 \neq 0$ except for $z = 1$. But on N_2 we also have the system

$$(a_{11} - h)f_1 + a_{12} f_2 + \cdots + a_{1p} f_p = 0$$
$$\vdots$$
$$a_{p1} f_1 + a_{p2} f_2 + \cdots + (a_{pp} - h)f_p = 0.$$

Hence on $N_2 - \{1\}$,

$$\det \begin{bmatrix} a_{11} - h, a_{12}, \cdots a_{1p} \\ a_{p1}, a_{p2}, \cdots a_{pp} - h \end{bmatrix} = 0.$$

Expanding this determinant, we obtain

$$h^p = \alpha_1 h^{p-1} + \cdots + \alpha_p.$$

where each $\alpha_j \in I$. But then $h^p \in I$ is impossible. Hence I is not finitely generated.

Example 3.8. Let $V \subset \bar{W}$ be the set $\{1\}$. Then V is a variety, and by Theorem 3.6, $\mathcal{I}_v$ is globally generated. But $(\mathcal{I}_v)_1$ is not finitely generated, so $\mathcal{I}_v$ is not locally finitely generated, and hence not coherent.

Example 3.9. Define a map $\mu : \mathfrak{A}^2 \to \mathfrak{A}$ by

$$\mu(a_1, a_2) = (z - 1)a_1 + (z - 1) \exp (z + 1)/(z - 1))a_2.$$

An easy calculation shows that the stalk of the kernel of μ at 1 is isomorphic to the ideal I, and hence is not finitely generated. Hence the kernel of μ is not coherent. Yet by Theorem 1.7, the kernel of μ is globally coherent.

IV. A generalization of Cartan's lemma

In Section IV, we needed a variant of Cartan's lemma on holomorphic matrices for the case in which the entries of the matrix are m-times continuously differentiable up to the boundary, and the region on which the matrix is defined is not connected. The proof of the case $m < \infty$ is a simple modification of the case $m = 0$ which appears in Douady [1, Chapter 6, Proposition 2]. The proof of the case $m = \infty$ is then a standard approximation argument. For completeness, however, we present the proofs here.

Let U_1, U_2 and $U_3 = U_1 \cap U_2$ be relatively compact domains in **C** and let $U = U_1 \cup U_2$. Assume that

(a) the boundaries of U, U_1, U_2, and U_3 consist of a finite number of simple closed m-times continuously differentiable curves,

(b) $(U - U_1)^- \cap (U - U_2)^- = \emptyset$.

Let B_m^k be the algebras of $n \times n$ matrices whose entries are Whitney C^m-functions on $\bar{U}_k$ and are holomorphic on U_k. Let $G_m^k \subset B_m^k$ be the groups of invertible elements in B^k_m; i.e. the sets of nonsingular matrices with appropriate entries. Note that for $m < \infty$, B^k_m is a Banach algebra under the usual norm, and G^k_m is an open subset of B^k_m. Define maps

$$\pi : G^1_m \times G^2_m \to G^3_m, \qquad \rho : B^1_m \times B^2_m \to B^3_m$$

by

$$\pi(\lambda_1, \lambda_2) = (\lambda_1)(\lambda_2)^{-1}, \qquad \rho(\lambda_1, \lambda_2) = \lambda_1 - \lambda_2.$$

We need the following preliminary result, which was used in Section III.

LEMMA 5.1. *Under assumptions* (a) *and* (b), *there is a continuous linear map*

$$T : B^3_m \to B^1_m \times B^2_m$$

so that $\rho \circ T =$ *identity.*

Proof. By assumption (b) we can find a C^∞-function η defined on $\bar{U}$ so that $0 \leq \eta \leq 1$, $\eta = 0$ in a neighborhood of $\bar{U}_1$, $\eta = 1$ in a neighborhood of $\bar{U}_2$. Choose $\lambda \in B^3_m$. Since the entries of λ are holomorphic on U_3,

$$(\partial/\partial\bar{z})(\eta\lambda) = (\partial/\partial z)((1-\eta)\lambda) \quad \text{on} \quad U_3.$$

Then

$$\begin{aligned} \mu(z) &= (\partial/\partial\bar{z})(\eta\lambda) && \text{on } \bar{U}_1 \\ &= (\partial/\partial\bar{z})((1-\eta)\lambda) && \text{on } \bar{U}_2 \end{aligned}$$

defines a matrix whose entries are Whitney C^m-functions on $\bar{U}$. Put

$$S(\lambda)(z) = \frac{1}{2\pi i}\iint_{\bar{U}} \frac{\mu(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}.$$

It follows from Vekua [5, Theorem 1.32] that the entries of $S(\lambda)$ are of Whitney class C^m on $\bar{U}$, and that $(\partial/\partial\bar{z})(S(\lambda)) = \mu$. Finally put

$$T(\lambda) = (\eta\lambda - S(\lambda), (1-\eta)\lambda - S(\lambda)).$$

It is clear that T satisfies the requirements of the lemma.

THEOREM 5.2. *Suppose that U, U_1, U_2, and U_3 satisfy conditions* (a) *and* (b), *and in addition*

(c) *U_3 consists of a finite disjoint union or simply connected regions.*

Then the map $\pi : G^1_m \times G^2_m \to G^3_m$ is surjective.

Proof. Consider first the case $m < \infty$. Then π is a continuously differentiable mapping of an open set of the Banach space $B^1_m \times B^2_m$ to the Banach space B^3_m, and the differential of π at the point (1.1) is just the map ρ. It follows from Lemma 5.1 and the open mapping theorem for Banach spaces that the image of π contains an open neighborhood V of $1 \in G^3_m$.

Next, we show that the image under π of $G^1_m \times \{1\}$ is dense in G^3_m. The closure H of $\pi(G^1_m \times \{1\})$ is a closed subgroup of G^3_m. But by condition (c) and Runge's theorem, the restriction of B^1_m to $\bar{U}_3$ is dense in B^3_m, and hence H

contains exp (B_m^3). Thus H contains an open neighborhood of $1 \in G_m^3$, and hence is open. Since each component of U_3 is simply connected, every element of G_m^3 is connected by a path in G_m^3 to a matrix whose entries are constant on each component of U_3. Since the group $GL(n, \mathbf{C})$ is connected, it follows that G_m^3 is connected, and hence $H = G_m^3$.

Finally, let $\lambda \in G_m^3$. We can find $\mu_1 \in G_m^1$ so that $(\mu_1)^{-1}\lambda \in V$. Hence $(\mu_1)^{-1}\lambda = \lambda_1(\lambda_2)^{-1}$ where $\lambda_j \in G_m^j$. But then $\lambda = (\mu_1 \lambda_1)(\lambda_2)^{-1}$, so π is surjective if $m < \infty$.

Next suppose that $\lambda \in G_\infty^3$. For each $m < \infty$ we can find $\lambda_m^k \in G_m^k$ so that $\lambda = (\lambda_m^1)(\lambda_m^2)^{-1}$. Put

$$\begin{aligned} \tau_m &= (\lambda_{m+1}^1)^{-1}(\lambda_m^1) \quad \text{on} \quad \bar{U}_1 \\ &= (\lambda_{m+1}^2)^{-1}(\lambda_m^2) \quad \text{on} \quad \bar{U}_2. \end{aligned}$$

Then τ_m is a nonsingular matrix whose entries are Whitney C^m-functions on $\bar{U}$ and are holomorphic on U. We can approximate τ_m by a nonsingular matrix σ_m which is holomorphic in a neighborhood of $\bar{U}$ so that

$$\| \sigma_m - \tau_m \|_m < (1/2m)(\max_k (\| \lambda_{m+1}^k)^{-1} \|_{m+1}).$$

Then $\lambda_{m+1}^k \sigma_m \in G_{m+1}^k$,

$$\| \lambda_{m+1}^k \sigma_m - \lambda_m^k \|_m < 1/2m$$

and $(\lambda_{m+1}, \sigma_m)(\lambda_{m+1}^2 \sigma_m)^{-1} = \lambda$. Hence in choosing λ_{m+1}^k we may assume that

$$\| \lambda_m^k - \lambda_{m+1}^k \|_m < 1/2m.$$

Put $\lambda^k = \lambda_1 + \sum_{m=1}^{\infty} (\lambda_{m+1}^k - \lambda_m^k) = \lim_{m\to\infty} \lambda_m^k$. Then $\lambda^k \in G_\infty^k$, and $(\lambda^1)(\lambda^2)^{-1} = \lambda$. This completes the proof.

References

1. A. Douady, *Le problème des modules pour les sousespaces analytiques compacts d'un espace analytique donné*, Ann. Inst. Fourier (Grenoble), vol. 16 (1966), pp. 1–95.
2. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, N. J., 1965.
3. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton University Press, Princeton, N. J., 1948.
4. A Nagel, *Cohomology of sheaves of holomorphic functions satisfying boundary conditions on product domains*, Trans. Amer. Math. Soc., vol. 172 (1972), pp. 133–146.
5. I. N. Vekua, *Generalized analytic functions*, Addison-Wesley, Reading, Mass., 1962.

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