ON THE EXISTENCE OF LOGARITHMS IN MIKUSINSKI'S OPERATIONAL CALCULUS FOR A FINITE INTERVAL[1]

BY

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1. Introduction

In 1962, L. Máté published a paper [5] containing a theorem which characterized a certain set of logarithms in Mikusinski's Operational Calculus. This same theorem appears again in [6]. Unfortunately, the theorem is incorrect as stated, given Máté's definitions and assumptions. Part of the problem lies in the fact that the proof utilizes assumptions which do not appear in the hypotheses of the theorem. The proof itself is essentially an adaptation to the operational calculus setting of the proof of the Hille-Yosida Theorem as it appears in the book of Hille and Phillips (see [4]).

This notwithstanding, it is clear that Máté's result is essentially correct. With a little care, a true version of his theorem can be proved without too much difficulty.

The results of this present article were obtained independently and without knowledge of Máté's prior work.[2] Our methods are entirely different from Máté's. He works entirely within the field of Mikusinski operators with its notion of convergence. The present article views Mikusinski operators as linear transformations on a Banach space and applies the Hille-Yosida Theorem directly. Our arguments are, on the whole, more economical and reveal the connection between (Mikusinski) operators and the more traditional linear transformations on a Banach space.

Let $L^1[0, T]$ denote the Banach algebra of integrable functions on $[0, T]$. Addition and scalar multiplication are defined as usual and multiplication is defined by convolution, i.e. if $f, g \in L^1[0, T], fg(x) = \int_0^x f(x - t)g(t)\, dt$. The norm of an element $f \in L^1[0, T]$ is $\int_0^T | f(t) |\, dt$. In this article, we shall, as above, denote convolution of functions by justaposition and in place of $L^1[0, T]$ we shall usually write L^1.

In the first part of this paper we shall show that there is a natural identification between a certain subset of Mikusinski's operators (on the finite interval $[0, T]$) and a certain class of closed multipliers on L^1. By a closed multiplier on L^1 we mean a closed linear map $A : D \to L^1$, where D is an ideal in L^1 and $A(fg) = fA(g)$ for all $g \in D$ and $f \in L^1$. An example of such a multiplier would be the map A defined by $A(f) = f'$, where $D(A) = \{f \mid f \text{ is absolutely continuous and } f(0) = 0\}$.

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The remainder of the paper is devoted to showing how, by redefining the notion of logarithm in Mikusinski's operational calculus, one can use the above identification together with some results of T. K. Boehme [1] on bounded multipliers and the classical Hille-Yosida Theorem on the generation of semigroups to produce necessary and sufficient conditions for the existence of logarithms. We assume that the reader is familiar with the ideas and notation in Mikusinski's paper and book. (See [7] and [8].) However, for the sake of completeness we include some of the basics.

2. Mikusinski operators

Mikusinski's operators (or convolution quotients) are pairs of functions $(p, q) = p/q$ where p and q are locally integrable on $[0, T)$ and q does not vanish identically on any neighborhood of the origin. We shall denote the set of all such functions q by X_0. Equality of operators is defined by $p/q = p_1/q_1$ if and only if $pq_1 = p_1 q$. The sum and product of operators is defined by $p/q + r/s = (ps + qr)/qs$ and $p/q \cdot r/s = pr/qs$, respectively. By an important theorem of Titchmarsh, $q, s \in X_0$ imply $qs \in X_0$. Hence, the operations are closed. It is easy to verify that they are well defined.

Following Mikusinski, we shall sometimes write $\{a(t)\}$ for the function $a = a(t)$, $0 \leq t < T$. The symbol l will be used exclusively to mean the function whose value at each point of $[0, T)$ is 1, i.e. $l = \{1\}$. $\delta = l/l$ will denote the unit in the ring of operators. Thus, an operator p/q has an inverse q/p if and only if $p \in X_0$. The symbol s will denote l^{-1} (the differential operator). A complex number α can be identified with the operator $\{\alpha\}/l$ and if a is locally integrable on $[0, T)$, a can be identified with $l\{a(t)\}/l$ so that the ring of (Mikusinski) operators contains the set of complex numbers as well as the locally summable functions.

A function of two variables, $\chi(\lambda, t)$, continuous on the rectangle $\lambda_1 \leq \lambda \leq \lambda_2$, $0 \leq t < T$, is termed parametric, and we write $\chi(\lambda) = \{\chi(\lambda, t)\}$. Parametric functions are special cases of operator-valued functions, i.e. functions of a real variable whose values are Mikusinski operators. If $\chi(\lambda, t)$ has a continuous partial derivative $(\partial/\partial\lambda)\chi(\lambda, t)$ on the rectangle, then the parametric function $\chi(\lambda) = \{\chi(\lambda, t)\}$ is said to have a continuous derivative $\chi'(\lambda) = \{(\partial/\partial\lambda)\chi(\lambda, t)\}$. An operator-valued function $\chi(\lambda)$, $\lambda_1 \leq \lambda \leq \lambda_2$, has a continuous derivative $\chi'(\lambda)$ if and only if there exists a parametric function $a(\lambda)$ having a continuous derivative $a'(\lambda)$ such that $\chi(\lambda) = a(\lambda)/b$ and $\chi'(\lambda) = a'(\lambda)/b$ where $b \in X_0$. (See Mikusinski [8, pp. 230–231].)

3. Mikusinski operators and closed multipliers on L^1

Mikusinski operators give rise to closed multipliers in the following way:

Let $A = p/q$, $D(A) = \{f \in L^1 \mid fp/q \in L^1\}$ and define $A(f) = fp/q$ for $f \in D(A)$.

It is clear that A is unambiguously defined, linear on its domain and that $A(fg) = fA(g)$ for all $f \in L^1$ and $g \in D(A)$. Finally, A is closed, for if $\{f_n\}$ is a sequence in $D(A), f_n \to f$ and $A(f_n) = g_n \to g$, then $pf_n = g_n q$ so that $pf = gq$, i.e. $f \in D(A)$ and $A(f) = g$.

Denote by Ω the function as defined above which assigns closed multipliers to operators, define a Mikusinski operator p/q to be of finite type if and only if there is an operator u/v with $u, v \in L^1$ such that $p/q = u/v$, and let M_0 be the class of closed multipliers A such that $D(A) \cap X_0 \neq \emptyset$. We then have the following theorem.

THEOREM 3.1. *If Ω is restricted to the set of operators of finite type, then the range of Ω is M_0 and Ω is one-to-one.*

Proof. If p/q is of finite type, then there exist $u, v \in L^1$ such that $p/q = u/v$. But $\{vf \mid f \in L^1\} \subseteq D(A)$ and by Titchmarsh's theorem if $v \in X_0$ and $f \in X_0$, then $vf \in X_0$. Thus, $D(A) \cap X_0 \neq \emptyset$. This implies also that $D(A)$ is dense, since by a theorem due to C. Foias [3] if $v \in X_0$, then $\{vf \mid f \in L^1\}$ is dense in L^1.

Conversely, let $A \in M_0$ and let $f_0 \in D(A) \cap X_0$. Set $A(f_0) = p, f_0 = q$. We claim that $\Omega(p/q) = A$. To see this, let $B = \{f \mid pf/q \in L^1\}$ and take $\phi \in D(A)$. Then $A(\phi f_0) = f_0 A(\phi) = \phi A(f_0)$ so that $A(f_0)\phi/f_0 = A(\phi)$, i.e. $\phi \in B$. Thus $D(A) \subseteq B$. If $\phi \in B, \phi p/q = h \in L^1$. Since $D(A)$ is dense, it must contain an approximate identity $\{e_n\}$. Since $\phi A(f_0) = \phi p = hq = hf_0$, $\phi A(f_0)e_n = hf_0 e_n$ or $f_0 A(\phi e_n) = hf_0 e_n$. However, since $f_0 \in X_0$, the last equation implies by Titchmarsh's theorem that $A(\phi e_n) = he_n$. But $\phi e_n \to \phi$ and $A(\phi e_n) \to h$ so that $\phi \in D(A)$ and $A(\phi) = h$ since A is closed. Therefore $B = D(A)$ and $A(f_0)f/f_0 = A(f)$ for every $f \in D(A)$. Thus, $\Omega(p/q) = A$. It is a straight forward matter to prove that Ω is one-to-one.

THEOREM 3.2. *Let $\Omega(p/q) = A$. Then $(p/q)^{-1}$ exists (as a Mikusinski operator) if and only if A^{-1} exists. In either case $\Omega((p/q)^{-1}) = A^{-1}$.*

Proof. If $(p/q)^{-1}$ exists, then $p \in X_0$, so that $(p/q)^{-1} = q/p$. Letting $B = \Omega(q/p)$ we have, if $f \in D(A)$, $A(f) = (p/q)f$ so that $qA(f)/p = f$, i.e. $A(f) \in D(B)$ and $BA(f) = f$. Similarly, $AB(f) = f$ for all $f \in D(B)$.

Conversely, assume A^{-1} exists. Since $p/q = A(f_0)/f_0$ for some $f_0 \in D(A) \cap X_0$, it must be the case that $A(f_0) \in X_0$. Otherwise, if $A(f_0)$ vanishes on $[0, a]$, say, we could find functions ϕ and ψ such that $\phi \neq \psi$ but but $A(f_0)\phi = A(f_0)\psi$. This would mean that $A(f_0 \phi) = A(f_0 \psi)$ which in turn implies that $f_0 \phi = f_0 \psi$. Hence, $\phi = \psi$ since $f_0 \in X_0$, and this is a contradiction. Therefore, $A(f_0) \in X_0$ so that $(p/q)^{-1}$ exists.

4. Logarithms

There are certain operators in the operational calculus which are of special importance. They are the so-called logarithms. An operator ω is a logarithm if ω satisfies a differential equation of the form $\chi'(\lambda) = \omega\chi(\lambda)$, $\chi(0) = \delta$, where $\chi(\lambda)$ is an operator-valued function defined in some real interval. It turns out that for the case treated here (i.e. when the interval is finite) there are three types of logarithms depending on the domain of the function $\chi(\lambda)$. As mentioned earlier, we propose to redefine the notion of logarithms. Our definition will be somewhat narrower than Mikusinski's but will still include the most important logarithms. Moreover, it will be possible to obtain necessary and sufficient conditions for the existence of logarithms (in this narrower sense).

We begin by stating Mikusinski's definition:

DEFINITION 4.1 (Mikusinski). The operator ω is a right (left) logarithm if and only if there exists an operator-valued function $\chi(\lambda)$ such that $\chi'(\lambda) = \omega\chi(\lambda)$, $\chi(0) = \delta$, $0 \leq \lambda < \infty$ $(-\infty < \lambda \leq 0)$ and $\chi(\lambda)$ cannot be continued to the left (right). ω is a bilateral logarithm if and only if $\chi'(\lambda) = \omega\chi(\lambda)$, $\chi(0) = \delta$ for $-\infty < \lambda < \infty$. All three cases can occur. (See [7, p. 235].)

We propose to replace the above by the following definition.

DEFINITION 4.2. The operator ω is a right logarithm if and only if there exists an operator-valued function $\chi(\lambda)$ such that

(i) $\chi'(\lambda) = \omega\chi(\lambda)$, $\chi(0) = \delta$, $0 \leq \lambda < \infty$,

(ii) each $\chi(\lambda)$ is a bounded operator, i.e. the multiplier determined by $\chi(\lambda)$ is bounded,

(iii) $\lim_{\lambda\to 0} \chi(\lambda)f = f$ for each $f \in L^1$ (the convergence here is norm convergence in L^1).

Similar definitions hold for left and bilateral logarithms. If ω is a logarithm we shall write $e^{\lambda\omega}$ for $\chi(\lambda)$.

For example, $-s$ is a right logarithm. As a matter of fact $e^{-\lambda s} = h^\lambda$, $\lambda \geq 0$ where h^λ denotes translation to the right by λ. $-\sqrt{s}$, where $(\sqrt{s})^{-1} = \{1/\sqrt{(\pi t)}\}$, is a right logarithm since it satisfies $\chi'(\lambda) = -(\sqrt{s})\chi(\lambda)$ for $\lambda \geq 0$ with

$$\begin{aligned} \chi(\lambda) &= \delta && \text{for } \lambda = 0 \\ &= (\lambda/2\sqrt{(\pi t^3)})e^{(\lambda^2/4t)} && \text{for } \lambda > 0 \end{aligned}$$

Actually $-\sqrt{s}$ is a bilateral logarithm since it can be continued to the left. (See [7, pp. 233–235] for further details.)

In what follows, we will consider only right logarithms. However, similar conclusions hold for left and bilateral logarithms.

THEOREM 4.1. *If ω is a right logarithm, then ω has the semigroup property, i.e.* $\chi(\lambda_1)\chi(\lambda_2) = \chi(\lambda_1 + \lambda_2)$, $\lambda_1, \lambda_2 \geq 0$.

Proof. Fix $\lambda_1 \geq 0$ and put $y(\lambda) = \chi(\lambda)\chi(\lambda_1) - \chi(\lambda + \lambda_1)$. Then $y(\lambda)$ is differentiable and

$$y'(\lambda) = \chi'(\lambda)\chi(\lambda_1) - \chi'(\lambda + \lambda_1) = \omega\chi(\lambda)\chi(\lambda_1) - \omega\chi(\lambda + \lambda_1) = \omega y(\lambda).$$

Moreover, $y(0) = 0$. However, there is only one function satisfying the conditions on $y(\lambda)$ and this function is $y(\lambda) \equiv 0$. (See [7, p. 232].) This proves the assertion.

THEOREM 4.2. *Let ω be a logarithm of finite type. Then the multiplier determined by ω is the infinitesimal generator of the semigroup $\chi(\lambda)$.*

Proof. Let A be the infinitesimal generator of $\chi(\lambda)$. By definition $A(f) = \lim_{\lambda\to 0} (\chi(\lambda)f - f)/\lambda$ in L^1 norm. However,

$$\lim_{\lambda\to 0} (\chi(\lambda)f - f)/\lambda = \chi'(0)f = \omega\chi(0)f$$

in the operational sense and hence, these limits are the same. Therefore $A(f) = \omega f$ for all $f \in D(A)$.

THEOREM 4.3. *Let ω be an operator of finite type. Then in order that ω be a logarithm, it is necessary and sufficient that there exist numbers a and M such that*

(i) *For each $\lambda > a$, there is a function ν_λ of bounded variation on $[0, T]$ with the property that $(\lambda - \omega)^{-1} = s\nu_\lambda$.*

(ii) $\operatorname{Var}(s^{n-1}\nu_\lambda^{*n}) + | s^{n-1}\nu_\lambda^{*n}(0) | \le M(\lambda - a)^{-n}, n = 1, 2, 3, \cdots$.

We assume here that each ν_λ is normalized so that $\nu_\lambda(t) = \nu_\lambda(t^-)$ for $t > 0$ and $\nu_\lambda(0) = \nu_\lambda(0^+)$. Also ν_λ^{*n} denotes n-fold convolution.

Proof. If ω is a logarithm, then by the Hille-Phillips-Yosida theorem there are real numbers a and M such that $(\lambda - \omega)^{-1}$ exists for $\lambda > a$ as a bounded multiplier. By a theorem of T. K. Boehme [1, p. 226], if T is a bounded multiplier on L^1, then T has the form $T(g) = s\nu g$ for all $g \in L^1$ where ν is a function of bounded variation on $[0, T]$ and s is the differential operator. If ν is normalized, then $\| T \| = \operatorname{Var}(\nu) + | \nu(0) |$. Hence for $\lambda > a$ we have $(\lambda - \omega)^{-1} = s\nu_\lambda$. Moreover,

$$\| (\lambda - \omega)^{-1} \| \le M(\lambda - a)^{-n} \quad \text{for} \quad n = 1, 2, 3, \cdots,$$

i.e. $\operatorname{Var}(s^{n-1}\nu_\lambda^{*n}) + | s^{n-1}\nu_\lambda^{*n}(0) | \le M(\lambda - a)^{-n}$.

Conversely, if ω is an operator of finite type such that conditions (i) and (ii) hold, then the multiplier determined by ω generates a strongly continuous semigroup of bounded linear operators, i.e. ω is a logarithm.

We conclude with two applications of this theorem.

Example 4.1. Let $\omega = -s$. Then

$$(\lambda + s)^{-n} = s\left\{\int_0^t \frac{\tau^{n-1}}{(n-1)!} e^{-\lambda\tau} d\tau\right\},$$

and it is easily seen that

$$\operatorname{Var}\left\{\int_0^\tau \frac{\tau^{n-1}}{(n-1)!} e^{-\lambda\tau} d\tau\right\} \le 1/\lambda^n$$

so that $-s$ is a right logarithm. We note, however, that s is not a right logarithm.

Example 4.2. Let $\omega = \delta/ - \{e^{at}\} = -\{e^{at}\}^{-1}$, where a is real. It is not hard to verify that $\delta/(\lambda + \omega) = \{e^{(a-\lambda)t}\}$, so that

$$(\lambda + \omega)^{-n} = s \int_0^{\tau} e^{(a-\lambda)\tau} \frac{\tau^{n-1}}{(n-1)!} d\tau.$$

$$\mathrm{Var} \left\{ \int_0^{\tau} e^{(a-\lambda)\tau} \frac{\tau^{n-1}}{(n-1)!} d\tau \right\} \leq 1/(\lambda - a)^n \quad \text{for } \lambda > a, n = 1, 2, 3, \cdots.$$

Hence, $\{e^{at}\}^{-1}$ is a right logarithm for each real a.

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