CONTINUOUS SPECTRA OF AN EVEN ORDER DIFFERENTIAL OPERATOR

BY
DON HINTON

We consider here a differential operator l of order $2n$ defined by

$$(1) \qquad l(y) = (1/w)\{(-1)^n(ry^{(n)})^{(n)} - qy\}.$$

The coefficients w, r, and q are real continuous functions defined on a ray $[a, \infty)$ and w and r are positive. Associated with l is the Hilbert space H of all complex-valued, measurable functions f satisfying

$$\int_a^\infty w|f|^2\,dx < \infty.$$

We recall that l determines a certain minimal closed operator L_0 in H in the following way. Let $\mathfrak{D}$ be the set of all $y \in H$ such that (i) $y, y', \cdots, y^{(n-1)}$, $(ry^{(n)})', \cdots, (ry^{(n)})^{(n-1)}$ are absolutely continuous on compact subintervals of $[a, \infty)$ and (ii) $l(y) \in H$. Define $\mathfrak{D}_0'$ as the set of all $y \in \mathfrak{D}$ which have compact support interior to (a, ∞), and let L and L_0' be the restrictions of l to $\mathfrak{D}$ and $\mathfrak{D}_0'$, respectively. Then L_0' is a densely defined symmetric operator in H; hence admits a closure L_0 with domain $\mathfrak{D}_0$. As in [8, Section 17], it may be shown that $L_0^* = L$.

Since each of the equations $l(y) = \pm iy$ has at most $2n$ linearly independent solutions in H, the theory of symmetric operators [8, Section 14] yields that the dimension of $\mathfrak{D}$ modulo $\mathfrak{D}_0$ is finite. Furthermore, if T is a symmetric extension of L_0, then the continuous spectrum $C(T)$ of T is equal to $C(L_0)$. Thus the continuous spectrum of all self-adjoint operators generated by l in H is $C(L_0)$. In this paper we give conditions for $C(L_0)$ to be $(-\infty, \infty)$ or $[0, \infty)$.

Our basic tool for determining $C(L_0)$ will be to use a theorem from the theory of symmetric operators. For each complex number λ we define n_λ to be the dimension of the orthogonal complement (in H) of the range of $L_0 - \lambda I$, where I denotes the identity transformation. Since $L_0^* = L$, an alternate calculation of n_λ is

$$(2) \qquad n_\lambda = \dim\{y \mid Ly = \bar{\lambda}y\}.$$

As in [8, Sections 14, 17] where $w \equiv 1$, it may be shown that n_λ is actually the same for all non-real λ and that $n_\lambda \geq n$ when $\operatorname{im}\lambda \neq 0$. The cases $\lambda = \pm i$ are called the deficiency indices of L_0. Furthermore, L_0 can have no eigenvalues since all of $y, y', \cdots, y^{(n-1)}, (ry^{(n)}), \cdots (ry^{(n)})^{(n-1)}$ have value 0 at a

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for all $y \in \mathfrak{D}_0$. The result of symmetric operators [8, pp. 42–43] we apply is: *If for some real* λ, $n_\lambda < n$, *then* λ *is in the continuous spectrum of* L_0.

The approach used for showing $n_\lambda < n$ will be to apply the asymptotic theory given in [5]. By contrast, constructive methods have recently been applied in the case $n = 1$ and $C(L_0) = (-\infty, \infty)$ [2], [7]. These constructive methods use directly the definition of continuous spectrum, but they appear cumbersome for higher order equations. A different approach using asymptotic methods for finding $C(L_0)$ is given in [8, p. 229]; however, a weight function is not present, and greater monotonicity is required of the coefficients r and q than we require here. We note also that M. V. Fedorjuk has developed asymptotic formulae in [3] for solutions of the $(n + 1)$-term even order equation

$$\sum_{k=0}^{n} \varepsilon^{2k}(-1)^k(P_{n-k}(x)y^{(k)})^{(k)} = 0.$$

When these results are applied to the 2-term equation (1) with $w = 1$, the conditions on the coefficients are very similar to those required in [5]. However, in applying his asymptotic theory to yield conditions for

$$C(L_0) = (-\infty, \infty)$$

[3, Th. 5.3], Fedorjuk requires the coefficient r in (1) to satisfy $r(x) \to 1$ as $x \to \infty$; again the effect of a weight function is not considered.

Some comprehensive asymptotic formulae for the fourth order equation

$$[(ry'')' - py']' + qy = \sigma y$$

have recently been given by P. W. Walker [9], [10]. It is likely that these formulae give extensions of Theorems 1 and 2 below for the case $n = 2$. The even order equation studied by Fedorjuk has also been investigated by A. Devinatz in [1] where asymptotic solutions are given. These solutions may too yield extensions of the results here.

LEMMA 1. *Suppose* f *is a continuously differentiable positive function on* $[a, \infty)$ *such that* $f'(t)/f^2(t) \to 0$ *as* $t \to \infty$. *If* $\varepsilon > 0$ *and* $K > 0$, *then there is a number* B *such that if* t *and* s *are* $\geq B$ *and* $|t - s| \leq K/f(s)$, *then*

$$|f(t)f^{-1}(s) - 1| < \varepsilon.$$

This is a special case of Lemma 2 of [6].

LEMMA 2. *If* f *is as in Lemma* 1 *except that* $f'(t)/f^2(t) = 0(1)$ *as* $t \to \infty$, *then* $\int_a^\infty f\,dt = \infty$.

Proof. Let $M > 0$ and t_0 be such that $f'(t)/f^2(t) \geq -M$ for $t \geq t_0$. An integration then yields

$$1/f(t_0) - 1/f(t) \geq -M(t - t_0),$$

and hence

$$f(t) \geq 1/(M(t - t_0) + f^{-1}(t_0))$$

which implies $\int_a^\infty f\,dt = \infty$.

THEOREM 1. *Suppose in* (1) *that r, q, and w are positive and twice continuously differentiable, and the following conditions hold.*

(i) $w/q \to 0$ as $t \to \infty$.

(ii) $(q/r)^{-1/2n}(q/w)(|q'|/q + |r'|/r + |w'|/w) = 0(1)$ *as* $t \to \infty$.

(iii) $\int_a^\infty (q/r)^{-1/2n}[(q'/q)^2 + (w'/q)^2 + (r'/r)^2 + |r''/r| + |q''/q| + |w''/q|]\,dt < \infty$.

Then $C(L_0) = (-\infty, \infty)$. *Moreover, for* $n = 1$ *and* λ *real,* $n_\lambda = 0$; *hence every self-adjoint extension of* L_0 *has a purely continuous spectrum.*

Proof. Let λ be a real number. By (2), n_λ is the number of linearly independent solutions y in H of $l(y) = \lambda y$. Let $Q = q + \lambda w$; then $l(y) = \lambda y$ can be written as

$$(ry^{(n)})^{(n)} + (-1)^{n-1}Qy = 0 \tag{3}$$

By (i), Q is eventually positive, say on $[b, \infty)$, and Theorem 1 of [5] is applicable if r and Q satisfy $\int_b^\infty (Q/r)^{1/2n}\,dt = \infty$ and each of

$$[(Q/r)^{-1/2n}r'/r]', \qquad [(Q/r)^{-1/2n}Q'/Q]', \qquad [(Q/r)^{-1/2n}(r'/r)^2],$$

and

$$[(Q/r)^{-1/2n}(Q'/Q)^2]$$

is in $\mathfrak{L}(b, \infty)$.

For $f_1 \equiv (q/r)^{1/2n}$, it follows that

$$f_1'/f_1^2 = (1/2n)(q/r)^{-1/2n}(q'/q - r'/r)$$

which by (i) and (ii) tends to 0 as t tends to infinity; hence Lemma 2 gives

$$\int_b^\infty (q/r)^{1/2n}\,dt = \infty.$$

Since $(q/r)^{1/2n} = (Q/r)^{1/2n}[1 + o(1)]$, we have $\int_b^\infty (Q/r)^{1/2n}\,dt = \infty$. We also have

$$(Q/r)^{-1/2n}(Q'/Q)^2 = (q/r)^{-1/2n}(q'/q + \lambda w'/q)^2[1 + o(1)],$$

$$(Q/r)^{-1/2n}(r'/r)^2 = (q/r)^{-1/2n}(r'/r)^2[1 + o(1)],$$

$$\left[\left(\frac{Q}{r}\right)^{-1/2n}\frac{r'}{r}\right]' = \left(\frac{q}{r}\right)^{-1/2n}\left[\frac{r''}{r} - \left(\frac{r'}{r}\right)^2\right][1 + o(1)]$$

$$- \left(\frac{1}{2n}\right)\left(\frac{q}{r}\right)^{-1/2n}\left(\frac{r'}{r}\right)\left\{\frac{q' + \lambda w'}{q}[1 + o(1)] - \frac{r'}{r}[1 + o(1)]\right\},$$

and

$$\left[\left(\frac{Q}{r}\right)^{-1/2n}\frac{Q'}{Q}\right]'$$

$$= \left(\frac{q}{r}\right)^{-1/2n} \left\{\frac{q'' + \lambda w''}{q} [1 + o(1)] - \left(\frac{q' + \lambda w'}{q}\right)^2 [1 + o(1)]\right\}$$
$$- \left(\frac{1}{2n}\right)\left(\frac{q}{r}\right)^{-1/2n} \left(\frac{q' + \lambda w'}{q}\right)\left\{\frac{q' - \lambda w'}{q} [1 + o(1)] - \frac{r'}{r} [1 + o(1)]\right\}.$$

Application of (iii) now yields that the left hand side of each of the above equations is in $\mathcal{L}(b, \infty)$; hence Theorem 1 of [5] applies to yield solutions y_τ $(\tau = 1, \cdots, 2n)$ of (3) satisfying as $t \to \infty$,

$$y_\tau(t) = \left\{Q(t)^{(1-2n)/4n} r(t)^{-1/4n} \exp\left[\lambda_\tau \int_b^t (Q/r)^{1/2n}\right]\right\} \{1 + o(1)\} \tag{4}$$

where

$$\lambda_\tau = \exp[\pi i(\tau - 1)/n], \qquad n \text{ even}.$$
$$= \exp[\pi i(2\tau - 1)/2n], \qquad n \text{ odd}.$$

Note that in either case, $\pm i$ are two such λ_τ.

For $f_2 = (w/Q)(Q/r)^{1/2n}$, we have

$$f_2'/f_2^2 = (Q/w)(Q/r)^{-1/2n}[w'/w - Q'/Q + (1/2n)(Q'/Q - r'/r)]$$

which by (i) and (ii) is $O(1)$ as $t \to \infty$; hence by Lemma 2,

$$\int_b^\infty (w/Q)(Q/r)^{1/2n}\, dt = \infty. \tag{5}$$

From (4), we have for $\operatorname{Re} \lambda_\tau \geq 0$,

$$w \mid y_T \mid^2 \geq (w/Q)(Q/r)^{1/2n}[1 + o(1)];$$

hence (5) implies that $\int^\infty w \mid y_T \mid^2 dt = \infty$ if $\operatorname{Re} \lambda_\tau \geq 0$. Since the set $\{y_\tau \mid \operatorname{Re} \lambda_\tau \geq 0\}$ consists of $n + 1$ linearly independent solutions of $l(y) = \lambda y$, we will have shown $n_\lambda \leq n - 1$, and thus $\lambda \in C(L_0)$, if we show that no linear combination of the y_τ ($\operatorname{Re} \lambda_\tau \geq 0$) is in H.

Let z_1 and z_2 be the two solutions (4) where λ_τ is i and $-i$ respectively. We first prove no linear combination $c_1 z_1 + c_2 z_2$ is in H. Since z_1 and z_2 are not in H, it is sufficient to suppose $c_1 \neq 0$ and $c_2 \neq 0$. Writing

$$z = c_1 z_1 + c_2 z_2 = c_2 z_2[1 + c(z_1/z_2)]; \qquad c = c_1/c_2,$$

and noting that $\mid z_1/z_2 \mid \to 1$ as $t \to \infty$, we have that $\mid c \mid \neq 1$ implies $z \notin H$. Consider now $\mid c \mid = 1$, say $c = -e^{-2i\theta}$ $(0 \leq \theta < \pi)$. Since $\int_b^\infty (Q/r)^{1/2n} = \infty$, we can choose increasing sequences $\{t_n\}$ and $\{s_n\}$ $(n \geq 1)$ so that

$$\int_b^t (Q/r)^{1/2n} = \pi n + \theta - \pi/4, \quad t = t_n$$
$$= \pi n + \theta + \pi/4, \quad t = s_n.$$

Then for $s_n \leq t \leq t_{n+1}$,

$$3\pi/2 \geq 2\left[\int_b^t (Q/r)^{1/2n} - \theta\right] \geq \pi/2 \pmod{2\pi}$$

and

$$|1 + cz_1(t)/z_2(t)| = \left|1 - [1 + o(1)] \exp 2i\left[-\theta + \int_b^t (Q/r)^{1/2n}\right]\right| > 1$$

for all sufficiently large n. Thus $z \notin H$ if

$$(6) \qquad \sum_{n=2}^{\infty} \int_{s_n}^{t_{n+1}} w|z_2|^2\,dt = \sum_{n=2}^{\infty} \int_{s_n}^{t_{n+1}} (w/Q)(Q/r)^{1/2n}[1 + o(1)]\,dt = \infty$$

As shown above, $f_1 = (q/r)^{1/2n}$ satisfies $f_1'/f_1^2 \to 0$ as $t \to \infty$. Applying Lemma 1 and $(Q/r)^{1/2n} = (q/r)^{1/2n}[1 + o(1)]$ yields a B such that if $s, t \geq B$ and $|t - s| \leq 20\,(Q(s)/r(s))^{-1/2n}$, then

$$(7) \qquad |[Q(t)/r(t)]^{1/2n}[Q(s)/r(s)]^{-1/2n} - 1| < 1/4.$$

Define $\Delta_n = (Q(s_n)/r(s_n))^{-1/2n}$. Now for $t_n \geq B$, $t_{n+1} - s_n < 10\,\Delta_n$ since otherwise (7) gives

$$\pi/2 = \int_{s_n}^{t_{n+1}} (Q/r)^{1/2n} \geq \int_{s_n}^{s_n+10\Delta_n} (Q/r)^{1/2n} \geq (10\Delta_n)(3/4\Delta_n^{-1}) = 15/2.$$

which is a contradiction. Similarly, $s_n - t_n < 10\Delta_n$ for $t_n \geq B$.

The function $f_2 = (w/Q)(Q/r)^{1/2n}$ satisfies, as shown above, $f_2' = O(f_2^2)$; let M be a bound for $|f_2' f_2^{-2}|$ on $[b, \infty)$. By (i) there is a $B' \geq B$ such that $w(t)/Q(t) < 1/100M$ for $t \geq B'$.

For $t_n \geq B'$, consider f_2 on $[t_n, t_{n+1}]$. Let the maximum and minimum values of f_2 occur at t' and t'' respectively. Then

$$(8) \qquad \begin{aligned} |f_2(t'')/f_2(t') - 1| &= \left|\int_{t'}^{t''} f_2'(t)/f_2(t')\,dt\right| \\ &\leq \int_{t_n}^{t_{n+1}} Mf_2^2(t)/f_2(t')\,dt \\ &\leq M(20\Delta_n)f_2(t') \\ &= 20M(\Delta_n[Q(t')/r(t')]^{1/2n})(w(t')/Q(t')) \\ &< 20M(5/4)(1/100M) \\ &= 1/4 \end{aligned}$$

where the last inequality uses (7).

Also by (7), it follows that

$$(9) \qquad \frac{1}{2} = \frac{(\pi/2)}{\pi} = \frac{\int_{s_n}^{t_{n+1}} (Q/r)^{1/2n}}{\int_{t_n}^{t_{n+1}} (Q/r)^{1/2n}} \leq \frac{(5/4)(t_{n+1} - s_n)\Delta_n^{-1}}{(3/4)(t_{n+1} - t_n)\Delta_n^{-1}} = \frac{5(t_{n+1} - s_n)}{3(t_{n+1} - t_n)}.$$

Finally, from (8) and (9) we conclude that

$$\frac{\int_{s_n}^{t_{n+1}} (w/Q)(Q/r)^{1/2n}}{\int_{t_n}^{t_{n+1}} (w/Q)(Q/r)^{1/2n}} \geq \frac{f_2(t'')(t_{n+1} - s_n)}{f_2(t')(t_{n+1} - t_n)} \geq (3/4)(3/10) > 0.$$

From this last inequality, (6), and $\int^\infty f_2\,dt = \infty$ we have $z \notin H$. This concludes proving no linear combination of z_1 and z_2 is in H.

Consider a general linear combination $z = \sum_{\mathrm{Re}\,\lambda_\tau \geq 0} c_\tau y_\tau$. Now either there is a unique $\tau = \tau_0$ so that $\mathrm{Re}\,\lambda_\tau$, $c_\tau \neq 0$, is a maximum or there are exactly two such τ's. In the former case the asymptotic behavior (4) yields $z/y_{\tau_0} \to c_{\tau_0}$ as $t \to \infty$; thus $z \notin H$. In the latter case the two such λ_τ's are complex conjugates and the argument used above for z_1 and z_2 is applicable. The proof is now complete.

The moreover part of the theorem follows by the observation that if $n = 1$, then we have shown $n_\lambda = 0$.

As an example consider when the coefficients are powers of t. For $r(t) = t^\alpha$, $q(t) = t^\beta$, and $w(t) = t^\delta$, the conditions of Theorem 1 are simply $\beta > \delta$, $1 + \delta \geq \beta + (\alpha - \beta)/2n$, and $\alpha - \beta < 2n$. For $\alpha = \delta = 0$ and $n = 1$, we obtain the familiar result: $0 < \beta \leq 2$ implies $C(L_0) = (-\infty, \infty)$.

THEOREM 2. *Suppose in* (1) *that* $r > 0$, $w > 0$, r, q, *and* w *are twice continuously differentiable, and the following conditions hold.*

(i) $q/w \to 0$ *as* $t \to \infty$.
(ii) $(w/r)^{-1/2n}(|w'|/w + |r'|/r) \to 0$ *as* $t \to \infty$.
(iii) $\int_a^\infty (w/r)^{-1/2n}[(q'/w)^2 + (w'/w)^2 + (r'/r)^2 + |r''|/r + |q''|/w + |w''|/w]\,dt < \infty$.

Then $C(L_0) = [0, \infty)$. *Moreover, for* $n = 1$, *the spectrum* $(0, \infty)$ *is purely continuous for every self-adjoint extension of* L_0.

Proof. For $\lambda > 0$, we can proceed as in the proof of Theorem 1 to show $n_\lambda \leq n - 1$. In this case

$$Q = q + \lambda w = \lambda w[1 + o(1)],$$

and condition (ii) is used to show $f \equiv (w/r)^{1/2n}$ satisfies $f'/f^2 \to 0$ as $t \to \infty$. In this case the functions corresponding to f_1 and f_2 coincide with $f = (w/r)^{1/2n}$. We omit the details, but arguments similar to those above show $(0, \infty) \subset C(L_0)$.

Since $C(L_0)$ is closed, the proof of Theorem 2 is complete if we show $(-\infty, -\varepsilon) \cap C(L_0) = \emptyset$ for each $\varepsilon > 0$. To establish this it is sufficient to prove that for each $\varepsilon > 0$ there exists an $N > a$ such that L_0' is bounded below by $-\varepsilon$ when restricted to those $y \in \mathfrak{D}_0'$ with support in $[N, \infty)$ (for $w \equiv 1$, see [4, p. 34]). Since $q/w \to 0$ as $t \to \infty$, we need only choose N so that

$|q(t)/w(t)| \leq \varepsilon$ for $t \geq N$. Then if $y \in \mathfrak{D}_0'$ has support $[c, d]$ in $[N, \infty)$, we have by integrating by parts that

$$\begin{aligned}\int_c^d w(L_0' y)y\,dt &= \int_c^d [(-1)^n(ry^{(n)})^{(n)} - qy]y\,dt \\ &= \int_c^d [r(y^{(n)})^2 - qy^2]\,dt \\ &\geq \int_c^d w(-q/w)y^2\,dt \\ &\geq -\epsilon\int_c^d wy^2\,dt;\end{aligned}$$

hence the lower bound for L_0' is established.

As an example, for $r(t) = t^\alpha$, $q(t) = t^\beta$, and $w(t) = t^\delta$, the conditions of Theorem 2 are $\delta > \beta$ and $(\alpha - \delta)/2n < 1$.

References

1. A. Devinatz, *The deficiency index of a certain class of ordinary self-adjoint differential operators*, Advances in Math., vol. 8 (1972), pp. 434–473.
2. M. S. P. Eastham and A. A. Ed-Deberky, *The spectrum of differential operators with large coefficients*, J. London Math. Soc. (2), vol. 2 (1970), pp. 257–266.
3. M. V. Fedorjuk, *Asymptotic methods in the theory of one-dimensional singular differential operators*, Translations of Transactions of Moscow Mathematical Society, vol. 15 (1966), pp. 333–386.
4. I. M. Glazman, *Direct methods of qualitative spectral analysis of singular differential operators*, Israel program for scientific translation, Jerusalem, 1965.
5. D. B. Hinton, *Asymptototic behavior of solutions of* $(ry^{(m)})^{(k)} \pm qy = 0$, J. Differential Equations, vol. 4 (1968), pp. 590–596.
6. ———, *Some stability conditions for a nonlinear differential equation*, Trans. Amer. Math. Soc., vol. 139 (1969), pp. 349–358.
7. ———, *Continuous spectra of second-order differential operators*, Pacific J. Math., vol. 33 (1970), pp. 641–643.
8. M. A. Naimark, *Linear differential operators*, Part II, Ungar, New York, 1968.
9. P. W. Walker, *Asymptotica of the solutions to* $[(ry'')' - py']' + qy = \sigma y$, J. Differential Equations, vol. 9 (1971), pp. 108–132.
10. ———, *Asymptotics for a class of fourth order differential operators*, J. Differential Equations, vol. 11 (1972), pp. 321–334.

The University of Tennessee
Knoxville, Tennessee