SUBSETS OF INFINITE DIMENSIONAL METRIC SPACES

BY

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Let Ω be a complete, separable metric space of non-σ-finite Λ^h-measure, where Λ^h denotes the Hausdorff measure corresponding to some continuous, increasing real function $h(t)$, defined for $t \geqq 0$ with $h(0) = 0$ and $h(t) > 0$ for $t > 0$. The following problem appears to offer some difficulty:

Does Ω necessarily contain a system of $\mathfrak{c}$ *disjoint closed subsets each of non-σ-finite Λ^h-measure?*

A. S. Besicovitch [1], [2] and [3] and R. O. Davies [4], [5] and [6] have shown that the answer is affirmative when Ω is a subset of a Euclidean space. In [8] D. G. Larman introduced the concept of a finite-dimensional compact metric space and in [9] he showed that the answer to the problem was affirmative in such spaces. Thus the problem has been solved in the case of "small" metric spaces. It is the purpose of this paper to obtain results connected with the problem, in the case of "large" metric spaces.

Let Ω be a complete, separable metric space. Then, for $x \in \Omega$ and positive real numbers r, $\alpha (\alpha < \frac{1}{2})$, we shall denote by $N(x, r, \alpha)$ the maximal number of closed spheres of radius αr whose corresponding open spheres are disjoint and which meet the closed sphere $S(x, r)$. Ω is said to be *spherically uniform* if there exist positive constants $\lambda_1 = \lambda_1(\alpha)$ and $\lambda_2 = \lambda_2(\alpha)$ such that, for any $r > 0$,

$$\lambda_2 \leqq N(x, r, \alpha)/N(y, r, \alpha) \leqq \lambda_1$$

for all $x, y \in \Omega$ and

$$N(x, r, \alpha) \geqq N(x, r', \alpha)$$

for all $x \in \Omega$ and $r \leqq r'$. We use the convention

$$\begin{aligned} \beta/\infty &= 0 \quad \text{if} \quad \beta \neq \infty \\ &= 1 \quad \text{if} \quad \beta = \infty. \end{aligned}$$

Larman's notion of dimension [8] can be defined as follows: if Ω is a complete, separable metric space then Ω is of *finite dimension* if and only if given α $(< \frac{1}{2})$ there exists a positive number R and an integer N such that $N(x, r, \alpha) \leqq N$ for all $x \in \Omega$ and for all $r < R$. Then we see that a complete, separable spherically uniform metric space Ω has infinite dimension if and only if there exists a positive number α $(< \frac{1}{2})$ such that given any positive integer N, there exists a real number R such that $N(x, r, \alpha) \geqq N$ for all $r \leqq R$ and for all $x \in \Omega$.

We shall now obtain results concerning "large" spherically uniform metric spaces. It would be interesting to see to what extent the "spherically uniform" hypothesis can be dropped. Unfortunately, I have been unable to solve this although, as will be seen, the restriction can be slackened somewhat for the result in Theorem 2.

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We shall need the following

LEMMA 1. *Let Ω be a spherically uniform metric space and let $x \in \Omega$ be such that $N(x, r, \alpha) \geqq 2$ for some r, α $(< \frac{1}{2})$. Then $d(S(x, r)) \geqq \alpha r\ (1 + 2\alpha)^{-1}$, where $d(C)$ denotes the diameter of the set C.*

Proof. Since Ω is spherically uniform, $N(x, r(1 + 2\alpha)^{-1}, \alpha) \geqq 2$. Let y, z be the centres of two closed spheres of radius $\alpha r(1 + 2\alpha)^{-1}$ which meet $S(x, r(1 + 2\alpha)^{-1})$ and whose corresponding open spheres are disjoint. Then $y, z \in S(x, r)$ and $\rho(y, z) \geqq \alpha r(1 + 2\alpha)^{-1}$, where ρ is the metric in Ω.

LEMMA 2. *If Ω is a finite dimensional, separable metric space then $\Lambda^n(\Omega) = 0$ for some positive integer n, where Λ^n denotes the Hausdorff measure corresponding to the function $h(x) = x^n$.*

Proof. Since Ω is finite dimensional there exist positive real numbers R and α $(\alpha < \frac{1}{2})$ and a positive integer N such that $N(x, r, \alpha) \leqq N$ for all $r < R$ and for all $x \in \Omega$. Thus any closed sphere $S(x, r)$ in Ω with $r \leqq R$ can be covered by N spheres of radius $2\alpha r$. Continuing this argument we see that any such sphere $S(x, r)$ can be covered by N^j spheres of radius $(2\alpha)^j r$. Now choose a positive integer n such that $N(2\alpha)^n < 1$. Then, since $(2\alpha)^j \to 0$ as $j \to \infty$ we have

$$\Lambda^n(S(x, r)) \leqq \lim_{j\to\infty} N^j(2\cdot(2\alpha)^j r)^n = 0,$$

for $r \leqq R$. Now since Ω is separable it can be covered by a countable collection of spheres of the form $S(x, r)$ with $r \leqq R$ and so $\Lambda^n(\Omega) = 0$ as required.

THEOREM 1. *Let Ω be a complete, separable, spherically uniform metric space of infinite dimension. Then Ω contains* c *disjoint, compact sets of infinite dimension.*

Proof. Let α $(< \frac{1}{2})$ be such that given any positive integer N, there exists a real number $R(N)$ such that $N(x, r, \alpha) \geqq N$ for all $r < R(N)$ and for all $x \in \Omega$.

Put $C = S(x, R)$ for some $x \in \Omega$ and some $R < R(2)$. Then let $2n_1$ be the largest even integer such that $S(x, R(1 + 2\alpha)^{-1})$ meets $2n_1$ closed spheres of radius $\alpha R(1 + 2\alpha)^{-1}$ and whose corresponding open spheres are disjoint. If there is no such largest integer put $n_1 = 1$. We note that, in either case, since Ω is spherically uniform and $R < R(2)$ we must have $n_1 \geqq 1$. We now choose $2n_1$ such spheres, say

$$S(x(i_1), \alpha R(1 + 2\alpha)^{-1}), \qquad i_1 = 1, 2, \cdots, 2n_1$$

and for each i_1 with $1 \leqq i_1 \leqq 2n_1$ put

$$C(i_1) = S(x(i_1), (\alpha R/4)(1 + 2\alpha)^{-1}).$$

We note that $C(i_1) \subset C$ for all i_1. Now assume that for $j = 1, 2, \cdots, m - 1$ we have defined integers $n_j \geqq 1$ and for any $(m - 1)$-tuple $(i_1, i_2, \cdots, i_{m-1})$

with $1 \leqq i_j \leqq 2n_j$ for $j = 1, 2, \cdots, m-1$ a sphere

$$C(i_1, i_2, \cdots, i_{m-1})$$

of the form

$$S(x(i_1, i_2, \cdots, i_{m-1}), (\alpha^{m-1}R/4^{m-1})(1+2\alpha)^{-(m-1)})$$

with the following properties: each of the $2n_{m-1}$ spheres

$$C(i_1, i_2, \cdots, i_{m-2}, k), \qquad k = 1, 2, \cdots, 2n_{m-1}$$

is contained in the sphere $C(i_1, i_2, \cdots, i_{m-2})$ and

$$\rho(C(i_1, i_2, \cdots, i_{m-2}, k), C(i_1, i_2, \cdots, i_{m-2}, q)) \geqq (2\alpha^{m-1}R/4^{m-1})(1+2\alpha)^{-(m-1)}$$

if $k \neq q$. Now choose the largest even integer $2n_m$, say, such that each

$$S(x(i_1, i_2, \cdots, i_{m-1}), (\alpha^{m-1}R/4^{m-1})(1+2\alpha)^{-m})$$

meets $2n_m$ closed spheres of radius $(\alpha^m R/4^{m-1})(1+2\alpha)^{-m}$ and whose corresponding open spheres are disjoint. If there is no such largest integer put $n_m = 2n_{m-1}$. Again, in either case, we must have $n_m \geqq 1$. For each $C(i_1, i_2, \cdots, i_{m-1})$ we now choose $2n_m$ such spheres, say

$$S(x(i_1, i_2, \cdots, i_{m-1}, i_m)(\alpha^m R/4^{m-1})(1+2\alpha)^{-m}), \qquad i_m = 1, 2, \cdots, 2n_m$$

and for each i_m with $1 \leqq i_m \leqq 2n_m$ put

$$C(i_1, i_2, \cdots, i_{m-1}, i_m) = S(x(i_1, i_2, \cdots, i_{m-1}, i_m), (\alpha^m R/4^m)(1+2\alpha)^{-m}).$$

We note that $C(i_1, i_2, \cdots, i_{m-1}, i_m) \subset C(i_1, i_2, \cdots, i_{m-1})$ for $1 \leqq i_m \leqq 2n_m$. Also if $k \neq q$

$$\rho(C(i_1, i_2, \cdots, i_{m-1}, k), C(i_1, i_2, \cdots, i_{m-1}, q)) \geqq (2\alpha^m R/4^m)(1+2\alpha)^{-m}.$$

Hence we have defined, by induction, for each integer j an integer $n_j \geqq 1$ and for each m-tuple $(i_1, i_2, \cdots, i_m)$ with $1 \leqq i_j \leqq 2n_j$ for $j = 1, 2, \cdots, m$ a sphere $C(i_1, i_2, \cdots, i_m)$ of radius

$$(\alpha^m R/4^m)(1+2\alpha)^{-m}$$

and each of the $2n_m$ spheres $C(i_1, i_2, \cdots, i_{m-1}, j)$ is contained in the sphere $C(i_1, i_2, \cdots, i_{m-1})$. We also have

$$\rho(C(i_1, i_1, \cdots, i_{m-1}, k), C(i_1, i_2, \cdots, i_{m-1}, q)) \geqq (2\alpha^m R/4^m)(1+2\alpha)^{-m}$$

if $k \neq q$.

Now define

$$S(0) = \bigcup_{i_1=1}^{n_1} C(i_1) \quad \text{and} \quad S(1) = \bigcup_{i_1=n_1+1}^{2n_1} C(i_1).$$

Assume that $S(k_1, k_2, \cdots, k_m)$ has been defined for all m-tuples $(k_1, k_2, \cdots, k_m)$ of zeros and ones and that for some finite collection A of m-tuples,

$$S(k_1, k_2, \cdots, k_m) = \bigcup_{(i_1, i_2, \cdots, i_m) \in A} C(i_1, i_2, \cdots, i_m).$$

Then we put

$$S(k_1, k_2, \cdots, k_m, 0) = \bigcup_{(i_1, i_2, \cdots, i_m) \in A} \bigcup_{i_{m+1}=1}^{n_m} C(i_1, i_2, \cdots, i_m, i_{m+1})$$

and

$$S(k_1, k_2, \cdots, k_m, 1) = \bigcup_{(i_1, i_2, \cdots, i_m) \in A} \bigcup_{i_{m+1}=n_m+1}^{2n_m} C(i_1, i_2, \cdots, i_m, i_{m+1}).$$

Thus we have defined, by induction, closed sets $S(k_1, k_2, \cdots, k_m)$ for all m-tuples of zeros and ones and for $m = 1, 2, \cdots$. We note that

$$S(k_1, k_2, \cdots, k_m, k_{m+1}) \subset S(k_1, k_2, \cdots, k_m)$$

and

$$S(k_1, k_2, \cdots, k_m) \cap S(j_1, j_2, \cdots, j_m) = \emptyset \quad \text{if} \quad (k_1, k_2, \cdots, k_m) \neq (j_1, j_2, \cdots, j_m).$$

Let $\{k_m\}$ be an infinite sequence of zeros and ones and define

$$S(k_1, k_2, \cdots) = \bigcap_{m=1}^{\infty} S(k_1, k_2, \cdots, k_m).$$

Then we have **c** disjoint, non-empty, compact subsets of Ω.

We now turn our attention back to the integers n_j defined in the construction of the spheres $C(i_1, i_2, \cdots, i_m)$. We note that $n_j \to \infty$ as $j \to \infty$ since, otherwise, there would be an integer M and spheres $S(x, r)$ of arbitrarily small radius r such that $S(x, r)$ meets at most M closed spheres of radius αr whose corresponding open spheres are disjoint. This is impossible because of our choice of α.

Now let $\{k_m\}$ be an arbitrary sequence of zeros and ones and let p be an arbitrary positive integer. We show that $\Lambda^p(S(k_1, k_2, \cdots)) > 0$.

Since $n_j \to \infty$ as $j \to \infty$ we may choose N so large that for $j \geqq N$,

$$n_j > (32(1 + 2\alpha)/\alpha)^p.$$

Let $\{U_i\}_{i=1,2,\cdots,I}$ be an arbitrary finite open covering of $S(k_1, k_2, \cdots)$ with

$$d(U_i) < 2\alpha^N R(1 + 2\alpha)^{-N}/4^N \quad \text{for} \quad i = 1, 2, \cdots, I.$$

We may, and do, assume that each U_i contains at least two points of $S(k_1, k_2, \cdots)$. Then for each i, $1 \leqq i \leqq I$ there is an integer $m(i) > N$ such that U_i intersects only one sphere of $S(k_1, k_2, \cdots, k_{m(i)-1})$ but has points in common with at least two different spheres of $S(k_1, k_2, \cdots, k_{m(i)})$. So we must have

$$d(U_i) \geqq 2\alpha^{m(i)} R(1 + 2\alpha)^{-m(i)}/4^{m(i)} \geqq d(V_i),$$

where V_i is any one of the spheres of $S(k_1, k_2, \cdots, k_{m(i)})$ which U_i intersects Then

$$\sum_{i=1}^{i} (d(U_i))^p \geqq \sum_{i=1}^{i} (d(V_i))^p.$$

Now let $C(i_1, i_2, \cdots, i_{N-1})$ be any sphere of $S(k_1, k_2, \cdots, k_{N-1})$, we shall show that

$$\sum \{(d(V_i))^p : V_i \cap C(i_1, i_2, \cdots, i_{N-1}) \neq \emptyset\} \geqq (d(C(i_1, i_2, \cdots, i_{N-1})))^p.$$

For, assume otherwise; then using Lemma 1 and the fact that

$$n_N\left(\frac{\alpha^{N+2}R}{4^{N+1}}(1+2\alpha)^{-N-2}\right)^p \geqq \left(\frac{2\alpha^{N-1}R(1+2\alpha)^{-N+1}}{4^{N-1}}\right)^p,$$

we see that there must be a sphere

$$C(i_1, i_2, \cdots, i_{N-1}, i_N^*),$$

say, of $S(k_1, k_2, \cdots, k_N)$ such that

$$\sum\{(d(V_i))^p : V_i \cap C(i_1, i_2, \cdots, i_{N-1}, i_N^*) \neq \emptyset\} < \left(\frac{\alpha^{N+2}R}{4^{N+1}}(1+2\alpha)^{-N-2}\right)^p.$$

Thus if U_i is such that $U_i \cap C(i_1, i_2, \cdots, i_{N-1}, i_N^*) \neq \emptyset$ then $m(i) \geqq N+2$. Now assume that for some integer q there is a sphere

$$C(i_1, i_2, \cdots, i_{N-1}, i_N^*, i_{N+1}^*, \cdots, i_{N+q}^*)$$

of $S(k_1, k_2, \cdots, k_{N+q})$ such that

$$\sum\{(d(V_i))^p : V_i \cap C(i_1, i_2, \cdots, i_{N-1}, i_N^*, i_{N+1}^*, \cdots, i_{N+q}^*) \neq \emptyset\} < \left(\frac{\alpha^{N+q+2}}{4^{N+q+1}}(1+2\alpha)^{-N-q-2}\right)^p$$

and such that if

$$U_i \cap C(i_1, i_2, \cdots, i_{N-1}, i_N^*, i_{N+1}^*, \cdots, i_{N+q}^*) \neq \emptyset$$

then $m(i) \geqq N+q+2$. Then, as above, we deduce that there must be a sphere

$$C(i_1, i_2, \cdots, i_{N-1}, i_N^*, i_{N+1}^*, \cdots, i_{N+q}^*, i_{N+q+1}^*)$$

say, of $S(k_1, k_2, \cdots, k_{N+q+1})$ such that

$$\sum\{(d(V_i))^p : V_i \cap C(i_1, i_2, \cdots, i_{N-1}, i_N^*, i_{N+1}^*, \cdots, i_{N+q}^*, i_{N+q+1}^*) \neq \emptyset\} < \left(\frac{\alpha^{N+q+3}R}{4^{N+q+2}}(1+2\alpha)^{-N-q-3}\right)^p.$$

Thus if U_i is such that

$$U_i \cap C(i_1, i_2, \cdots, i_{N-1}, i_N^*, i_{N+1}^*, \cdots, i_{N+q}^*, i_{N+q+1}^*) \neq \emptyset$$

then $m(i) \geqq N+q+3$. Hence we have shown by induction that for $q = 0, 1, 2, \cdots$ there exists a sphere

$$C(i_1, i_2, \cdots, i_{N-1}, i_N^*, i_{N+1}^*, \cdots, i_{N+q}^*) \quad \text{of} \quad S(k_1, k_2, \cdots, k_{N+q})$$

such that if $U_i \cap C(i_1, i_2, \cdots, i_{N-1}, i_N^*, i_{N+1}^*, \cdots, i_{N+q}^*) \neq \emptyset$ then $m(i) \geqq N+q+2$. Now put

$$x = \bigcap_{q=0}^{\infty} C(i_1, i_2, \cdots, i_{N-1}, i_N^*, i_{N+1}^*, \cdots, i_{N+q}^*);$$

then $x \in S(k_1, k_2, \cdots)$. If $x \in U_i$ for some i then

$$U_i \cap C(i_1, i_2, \cdots, i_{N-1}, i_N^*, i_{N+1}^*, \cdots, i_{N+q}^*) \neq \emptyset$$

for all $q = 0, 1, 2, \cdots$. Hence $n(i)$ is not defined. This is impossible and so we have shown that

$$\sum \{(d(V_i))^p : V_i \cap C(i_1, i_2, \cdots, i_{N-1}) \neq \emptyset\} \geqq (d(C(i_1, i_2, \cdots, i_{N-1})))^p.$$

Hence by Lemma 1,

$$\sum_{i=1}^{i} (d(U_i))^p \geqq (\alpha^N R/4^{N-1})(1 + 2\alpha)^{-N}.$$

Now $\{U_i\}$ was an arbitrary open covering of $S(k_1, k_2, \cdots)$ and so $\Lambda^p(S(k_1, k_2, \cdots)) > 0$. Using Lemma 2 and the fact that p was an arbitrary integer we deduce that we have $\mathfrak{c}$ disjoint, infinite dimensional, compact sets as required.

We recall from [7] that a space S has *finite generalized Hausdorff dimension* if and only if there is a measure function h such that $\Lambda^h(S) = 0$. We say that a sequence $\{U_i\}$ of sets is a *fine repeated cover* of S is $S \subset \bigcup_{j=i}^{\infty} U_j$ for $i = 1, 2, \cdots$ and $d(U_i) \to 0$ as $i \to \infty$. In [7] it was shown that S has finite generalized Hausdorff dimension if and only if it has a fine repeated cover. We shall now deduce a result similar to that of Theorem 1, this time in the case of spaces of infinite generalized Hausdorff dimension.

A complete, metric space Ω is said to be *spherically monotonic* if

$$N(x, r, \alpha) \geqq N(y, r', \alpha)$$

whenever $\frac{1}{2} > \alpha > 0$ and $S(x, r) \subset S(y, r')$.

Let Ω be a complete, spherically monotonic metric space of infinite generalized Hausdorff dimension. Then Ω does not have a fine repeated cover. Assume that, for all positive numbers α ($< \frac{1}{2}$) and r, each closed sphere of radius r meets only finitely many closed spheres of radius αr whose corresponding open spheres are disjoint. Let $x \in \Omega$ and put $S(i) = S(x, i)$ for $i = 1, 2, \cdots$, so that $\Omega \subset \bigcup_{i=1}^{\infty} S(i)$ and $S(1) \subset S(2) \subset \cdots$. Put $U_1 = S(1)$. $S(2)$ can be covered by finitely many spheres of radius $\frac{1}{4}$, let these spheres be $U_2, U_3, \cdots, U_{n_1}$. Similarly, $S(3)$ can be covered by finitely many spheres of radius $\frac{1}{8}$, let these spheres be $U_{n_1+1}, U_{n_1+2}, \cdots, U_{n_2}$. Continuing in this manner we see that the sequence $\{U_i\}$ will form a fine repeated cover, which is impossible. Thus there exists an $x \in \Omega$ and positive numbers α, r such that $N(x, r, \alpha) = \infty$. We can now use this fact to prove

THEOREM 2. *Let Ω be a complete, spherically monotonic metric space of infinite generalized Hausdorff dimension. Then Ω contains* $\mathfrak{c}$ *disjoint, closed subsets each of infinite generalized Hausdorff dimension.*

Proof. Let $x \in \Omega$ and r, α be such that $N(x, r, \alpha) = \infty$. Then since Ω is spherically monotonic we can define points $c(i_1)$ for $i_1 = 1, 2, \cdots$ such that $\rho(c(i_1), c(j_1)) \geqq \alpha r(1 + 2\alpha)^{-1}$ for $i_1 \neq j_1$ and $\rho(c(i_1), x) \leqq r$ for $i_i = 1, 2, \cdots$.

Again, using the fact that Ω is spherically monotonic, for each $i_1 = 1, 2, \cdots$ define points $c(i_1, i_2)$ for $i_2 = 1, 2, \cdots$ such that

$$\rho(c(i_1, i_2), c(i_1, j_2)) \geqq \alpha^2 r/4(1 + 2\alpha)^2 \quad \text{for} \quad i_2 \neq j_2$$

and

$$\rho(c(i_1, i_2), c(i_1)) \leqq \alpha r/4(1 + 2\alpha) \quad \text{for} \quad i_1 = 1, 2, \cdots.$$

It is easy to see that, in general, we can define points $c(i_1, i_2, \cdots, i_n)$ where $i_j = 1, 2, \cdots$ for $j = 1, 2, \cdots, n$ such that

$$\rho(c(i_1, i_2, \cdots, i_{n-1}, i_n), c(i_1, i_2, \cdots, i_{n-1}, j_n)) \geqq \alpha^n r/4^{n-1}(1 + 2\alpha)^n \quad \text{for} \quad i_n \neq j_n$$

and

$$\rho(c(i_1, i_2, \cdots, i_{n-1}, i_n), c(i_1, i_2, \cdots, i_{n-1})) \leqq \alpha^{n-1} r/4^{n-1}(1 + 2\alpha)^{n-1} \quad \text{for} \quad i_n = 1, 2, \cdots.$$

Thus the $c(i_1, i_2, \cdots, i_n)$ may be defined for all n with $i_j = 1, 2, \cdots$ for $j = 1, 2, \cdots, n$.

Now let $\{i_n\}$ be any sequence of positive integers; then

$$\{c(i_1, i_2, \cdots, i_m)\}_{m=1,2,\cdots}$$

forms a Cauchy sequence. Denote the limit point by $c(i_1, i_2, \cdots)$. Then

$$\begin{aligned} \rho(c(i_1, i_2, \cdots, i_n, i_{n+1}, \cdots), c(i_1, i_2, \cdots, i_n, j_{n+1}, \cdots)) \\ \geqq \frac{\alpha^{n+1}(2 + 3\alpha)r}{4^n(1 + 2\alpha)^{n+1}(4 + 7\alpha)} \quad \text{for} \quad i_{n+1} \neq j_{n+1} \\ > \frac{\alpha^{n+1} r}{4^{n+1}(1 + 2\alpha)^{n+1}}. \end{aligned}$$

Let $\{k_n\}$ be a sequence of zeros and ones and put

$$S(k_1, k_2, \cdots) = \{c(i_1, i_2, \cdots) : i_m \equiv k_m \bmod (2) \ \text{for} \ m = 1, 2, \cdots\}.$$

Then the $S(k_1, k_2, \cdots)$ form $\mathbf{c}$ disjoint, closed subsets of Ω.

Assume that for some sequence $\{k_n\}$ of zeros and ones, $S(k_1, k_2, \cdots)$ has a fine repeated cover. That is, there exists a sequence $\{U_i\}$ of sets such that $S(k_1, k_2, \cdots) \subset \bigcup_{i=j}^{\infty} U_i$ for $j = 1, 2, \cdots$ and $d(U_i) \to 0$ as $i \to \infty$. For each $j = 1, 2, \cdots$ define $I(j)$ to be an integer so that

$$d(U_i) < \alpha^j r/4^j(1 + 2\alpha)^j \quad \text{for} \quad i \geqq I(j) \quad \text{and} \quad I(j + 1) > I(j).$$

Then if $i \geqq I(1)$, U_i cannot contain points $c(i_1, i_2, \cdots)$ and $c(j_1, j_2, \cdots)$ of $S(k_1, k_2, \cdots)$ with $i_1 \neq j_1$. Hence we may choose $i_1^* \equiv k_1 \bmod (2)$ such that $c(i_1, i_2, \cdots) \notin U_i$ for $I(1) \leqq i < I(2)$ if $i_1 = i_1^*$. It is easy to see that, in general, we may choose $i_n^* \equiv k_n \bmod (2)$ such that $c(i_1, i_2, \cdots) \notin U_i$ for

$I(n) \leqq i < I(n+1)$ if $i_j = i_j^*$ for $j = 1, 2, \cdots, n$. But then

$$c(i_1^*, i_2^*, \cdots) \in S(k_1, k_2, \cdots)$$

and

$$c(i_1^*, i_2^*, \cdots) \notin U_i \quad \text{for all} \quad i \geqq I(1).$$

Hence the sequence $\{U_i\}$ is not a fine repeated cover. Thus the $S(k_1, k_2, \cdots)$ are of infinite generalized Hausdorff dimension. This completes the proof of Theorem 2.

As we noted at the beginning of the work the problem has been solved in the case of Lorman's finite-dimensional spaces. In fact these spaces are essentially those with a geometry which behaves in a similar fashion to the geometry of Euclidean spaces. It is shown in [7] that in Banach spaces the notion of finite generalised Hausdorff dimension is exactly the same as the usual notion of finite dimension. So we see from the results of this paper that although the problem remains unsolved in "large" metric spaces, the difficulty is probably not the size of the space but more its lack of uniformity in the general case.

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