A CHARACTERIZATION OF CERTAIN FROBENIUS GROUPS

BY
MICHAEL ASCHBACHER

1. Introduction

Let $\mathfrak{F}$ be a collection of groups and G a finite group. Following B. Fischer, an *$\mathfrak{F}$-set* of G is a collection D of subgroups normalized by G and generating G, such that the subgroup generated by any pair of distinct members of D is isomorphic to a member of $\mathfrak{F}$.

Let p be a fixed odd prime and D an $\mathfrak{F}$-set of the nonabelian group G, such that each member of D has order p. Fischer has shown that if $\mathfrak{F} = \{G\}$, and G is solvable, then $G/Z(G)$ is a Frobenius group [4]. He has further shown that if $\mathfrak{F}$ is the collection of Frobenius groups with cyclic kernals, then G is a Frobenius group [5].

In this paper it is shown that:

THEOREM 1. *Let $\mathfrak{F}$ be the collection of groups F with $F/Z(F)$ Frobenius of odd order. Then $G \in \mathfrak{F}$, and $Z(G)$ is generated by the centers of 2-generator D-subgroups.*

As a corollary it follows that:

THEOREM 2. *Let $\mathfrak{F} = \{F\}$ with F of odd order. Then $G/Z(G)$ is a Frobenius group of odd order.*

The restriction in Theorems 1 and 2 that F have odd order is necessary. For example if $\mathfrak{F} = \{SL_2(3)\}$ then $U_3(3)$ possesses an $\mathfrak{F}$-set. The following theorem is however true:

THEOREM 3. *Let $\mathfrak{F}$ be the collection of Frobenius groups whose kernel is an elementary 2-group. Then $G \in \mathfrak{F}$.*

The analogous theorem for $\mathfrak{F}$ the collection of groups F of order pm with $(m, 2p) = 1$, probably holds. Some progress is made in this paper toward such a result.

The proof of Theorem 3 is combinatorial. The proof of Theorem 1 is more complicated, and uses signalizer arguments. A contradiction is arrived at by showing a minimal counterexample has 2-rank at most 2, or possesses a proper 2-generated core.

Certain specialized notation and terminology is used. A *D-subgroup* of G is a subgroup H with $\langle H \cap D\rangle = H$. Given $X \leq G$, $\theta(X) = \langle X \cap D\rangle$. $\mathcal{N}(X)$ is the set of proper D-subgroups of G normalized by X, and $\mathcal{N}^*(X)$ the set of maximal elements of $\mathcal{N}(X)$. $\mathcal{N} = \mathcal{N}(1)$ and $\mathcal{N}^* = \mathcal{N}^*(1)$. $m(G)$ is the 2-rank of G. $O_\infty(G)$ is the largest normal solvable subgroup of G. $F(X)$ is the set of fixed points of X under its action by conjugation on D.

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2. $\mathfrak{F}$-sets

Throughout this section p is a fixed odd prime, and D is an $\mathfrak{F}$-set of a non-abelian finite group G, such that the members of D have order p. $\mathfrak{F}$ will be one of the following collections of groups:

2.1. *The collection of groups F with $F/Z(F)$ Frobenius.*
2.2. *The collection of groups in* 2.1 *of odd order.*
2.3. *The collection of groups of order mp where* $(2p, m) = 1$.

LEMMA 2.4. *Let $\mathfrak{F}$ be as in* 2.1. *Then:*
(1) *If H is a D-subgroup, $H \cap D$ is an $\mathfrak{F}$-set of H.*
(2) *If α is a homomorphism of G then $D\alpha$ is an $\mathfrak{F}$-set of $G\alpha$.*
(3) *If A and B are in D then A is conjugate to B in $\langle A, B\rangle$.*

Proof. (1) is trivial. Let G be a minimal counterexample to (2) and (3). Then $G = \langle A, B\rangle$ for some A and B in D and $Z(G) = 1$. Let H and K be the Frobenius compliment of G containing A, and the Frobenius kernel of G, respectively. Then $C_G(A) \leq H$, so K is a p'-group and thus $B \cap K = 1$. So $B^k \leq H$ for some $k \in K$. (3) now follows from minimality of G. In (2), $G\alpha$ is not Frobenius, so $K \leq \ker(\alpha)$. Thus $G\alpha \cong \langle A, B^k\rangle\alpha \in \mathfrak{F}$ by minimality of G.

LEMMA 2.5. *Let $\mathfrak{F}$ be as in* 2.1, *let $G \in \mathfrak{F}$, $A \in D$ and $\bar{G} = G/Z(G)$. Then either*

(1) *$\bar{G}$ has Frobenius kernel $\bar{G}'$ and compliment A, or*
(2) *$\bar{G}$ has a Frobenius compliment isomorphic to $SL_2(3)$ and $p = 3$.*

Proof. Let G be a minimal counterexample. Then $Z(G) = 1$. Let H be the Frobenius compliment containing A. By 2.4, $D \cap H$ is an $\mathfrak{F}$-set of the Frobenius compliment H of G, and as $H \neq A$ there exists some B in $H \cap D$ distinct from A. Minimality of G implies either $H = \langle A, B\rangle$ or $\langle A, B\rangle \cong SL_2(3)$. Assume $H = \langle A, B\rangle$, and let $K/Z(H)$ be the Frobenius kernel of $H/Z(H)$. Then K is a nilpotent Frobenius compliment, so $O(K) = J$ is cyclic. It follows that $J \cap C(A) = 1$, as $AJ \in \mathfrak{F}$. But AJ is a Frobenius compliment so $[A, j] = 1$ for any $j \in J$ of prime order. Thus $J = 1$. Similarly it follows that K is a quaternion group. As $[A, K] \neq 1$, minimality of G implies $H = A \quad \cong SL_2(3)$.

So for every choice of distinct A and B in D, $H \neq \langle A, B\rangle \cong SL_2(3)$. It follows from [3] that $H \cong U_3(3)$. But $U_3(3)$ is not a Frobenius compliment.

LEMMA 2.6. *Let $\mathfrak{F}$ be as in* 2.2 *with $G \in \mathfrak{F}$. Then the center of G is generated by the centers of* 2-*generator D-subgroups of G.*

Proof. Set $Z = Z(G)$, let $\langle a\rangle = A \in D$ and set $E = a^G$. Let G be a minimal counterexample. As $G \in \mathfrak{F}$, the centers of all 2-generator D-subgroups of G lie in $Z(G)$. Thus minimality of G implies all such centers are trivial.

Let $b, c \in E$, and $H = \langle a, b\rangle$. Then $ab \equiv a^2 \bmod H'$, so as H is Frobenius with kernel H' and $p > 2$, $ab = d^2$ for some $d \in a^H$. Similarly considering

$\langle d, c\rangle$, $d^2c^{-1} \in E$. Therefore $ab\bar{c}^1 \in E$ and thus $E\bar{c}^1 = \bar{a}^1E$ for all $a, c \in E$. So $\bar{a}^1E = E\bar{c}^1 = \bar{c}^1E$ and therefore $\bar{a}^1E$ is normalized by G.

Now let M/Z be a minimal normal subgroup of G/Z. Then $M = Z \times [a, M]$ and $[a, M] = a^{-1}E \cap M$ is normalized by G. Thus minimality of G implies $G/[a, M]$ is a Frobenius group, whereas $1 \neq M/[a, M]$ centralizes a, a contradiction.

LEMMA 2.7. *Let $\mathfrak{F}$ be as in 2.2, and assume $G' = Q$ is a q-group for some prime q. Then $G \in \mathfrak{F}$.*

Proof. Let G be a minimal counterexample, let $A \in D$ and set $Z = Z(Q)$. Clearly $Z(G) = 1$, so $C(A) \cap Z = 1$. Set $\bar{G} = G/Z$. Minimality of G implies $\bar{G} \in \mathfrak{F}$, so as $C(A) \neq A$, 2.5 and 2.6 imply there exists B in D distinct from A such that $\bar{H} = \langle \bar{A}, \bar{B}\rangle$ has a nontrivial center. Thus the center of H contains an element u not in the center of G. Let Γ be the collection of 2-generator D-subgroups X of G such that $Zu \cap Z(X)$ is nonempty. By 2.4, G^D is transitive, and minimality of G implies $\bar{u}$ is in the center of $\bar{G}$, so $Q = \langle X \cap Q : X \in \Gamma\rangle$. But as $Z = Z(Q)$, $Zu = Z(Zu \cap Z(X))$ is centralized by $X \cap Q$, so $\langle Z, u\rangle \leq Z(Q) = Z$, a contradiction.

LEMMA 2.8. *Let $\mathfrak{F}$ be as in 2.2 and assume G is solvable. Then $G \in \mathfrak{F}$.*

Proof. Let G be a minimal counterexample and let $A \in D$. Clearly $Z(G) = 1$. Let M be a minimal normal subgroup of G. Then M is an elementary abelian q-subgroup for some prime q and minimality of G implies $G/M \in \mathfrak{F}$. Set $K = G'$. Suppose $\bar{K}$ is nilpotent. Then minimality of G implies K is a q-group and 2.7 yields a contradiction. So $\bar{K}$ is not nilpotent and there exists a prime $r \neq q$ dividing the order of K. Let R be an A invariant Sylow r-subgroup of $\bar{K}$. As $\bar{K}$ is not nilpotent, minimality of G implies $K = MR$, AR is generated by any two members of $AR \cap D$, and AR acts irreducibly on M.

Suppose $H = \langle A, B\rangle$ is a 2-generator D-subgroup. Then either H is conjugate to AR or $H' \leq M$. Let $m = |M|$, $n = |M : C_M(A)|$, and

$$k = |R : C_R(A)|.$$

Then D has order nk, so there are $nk - 1$ members B of D distinct from A. There are m/n D-subgroups H conjugate to AR containing A, and $|H \cap D| = k$; there are $n - 1$ members B of D distinct from A with $\langle A, B\rangle' \leq M$. Therefore

$$nk - 1 = m(k - 1)/n + n - 1.$$

It follows that $m = n^2$. Thus letting $AR = \langle A, B\rangle$, $M = C_M(A) \times C_M(B)$.

Extend $GF(p)$ to a splitting field F for AR and M to a vector space V over F. Then $\dim_F V = 2\dim_F(C_V(A)) = 2r$. Let V_i be the absolutely irreducible components of V, and set $r_i = \dim_F(C_{V_i}(A))$. Then $r = \sum r_i$, and as $C_V(A) \cap C_V(B) = 1$, $2r \geq \sum 2r_i$. So $\dim_F V_i = 2r_i$ is even, impossible as $|AR|$ is odd.

LEMMA 2.9. *Let $\mathfrak{F}$ be as in 2.3, and let S be a Sylow 2-subgroup of G. Then*

(1) *for any $X \leq S$, $F(X) = C_D(X)$, and*
(2) *$F(S)$ is nonempty.*

Proof. Let $X \leq S$ centralize $A \in D$ and fix $B \in D$. Then X acts on $H = \langle A, B\rangle$ of odd order, so all X invariant Sylow p-subgroups of H are conjugate in $C_H(X)$ to A. In particular as X centralizes A and normalizes B, X centralizes B.

Next let T be a maximal subgroup of S fixing a point of D. Suppose $T \neq S$. Then T is of index 2 in some $R \leq S$ and R acts on $F(S)$. Thus maximality of T implies R has a cycle (A, B) of length 2 in D. Then R acts on $H = \langle A, B\rangle$, and as $H \cap D = A^H$ has odd order, $F(R)$ is nonempty, a contradiction. This yields (2).

Finally assume (1) is false. Then by the first paragraph, $C_D(S)$ is empty. Let $A \in F(S)$ and $T = C_S(A)$. Then $S/T \leq \mathrm{Aut}(A)$ is cyclic and T is the set of elements x of S with $C_D(x)$ nonempty. Thus $N(T)$ controls fusion in S and considering the transfer of G to S/T, G has a subgroup of index two. But this is impossible as $G = \langle D\rangle$.

3. A signalizer theorem

In this section the following hypothesis is assumed:

HYPOTHESIS 3.1. *$\mathfrak{F}$ is the collection of groups F of odd order with $F/Z(F)$ Frobenius. p is a fixed odd prime and D is an $\mathfrak{F}$-set of G such that each member of D has order p. $O_\infty(G) = 1$ and each member of $\mathcal{N}$ is solvable.*

LEMMA 3.2. *Let E be an elementary 2-group of rank at least two, and $H \in \mathcal{N}(E)$. Then $H = \langle\theta(C_H(U)) : |E : U| = 2\rangle$.*

Proof. $H/Z(H) = \langle C_{H/Z(H)}(U) : |E : U| = 2\rangle$. By 2.8, $H/Z(H)$ is Frobenius, while by 2.9, there exists $A \in C(E) \cap H \cap D$. Thus

$$C_{H/Z(H)}(U) = \theta(C_H(U))Z(H)/Z(H).$$

So setting $K = \langle\theta(C_H(U)) : |E : U| = 2\rangle, H = KZ(H)$. Thus as $|H : H'| = p$, $Z(H) \leq K$, so $H = K$.

THEOREM 3.3. *Let E be an elementary 2-group of rank 3. Then $\mathcal{N}^*(E)$ contains a unique member.*

For the remainder of this section let M_1 and M_2 be distinct members of $\mathcal{N}^*(E)$ with $M_1 \cap M_2$ maximal. By 2.9, E centralizes a member of $M_i \cap D$, so maximality of $M_1 \cap M_2$ implies there exists $A \in M_1 \cap M_2 \cap C_D(E)$.

$$Z(M_1) \cap M_2 = Z(M_1) \cap Z(M_2) = 1.$$

Thus either $M_1 \cap M_2$ is Frobenius or $A = M_1 \cap M_2$. As $m(E) = 3$, there exists $e \in E^{\#}$ with $\theta(C(e)) \cap M_i > A, i = 1, 2$. Thus maximality of $M_1 \cap M_2$

implies $M_1 \cap M_2$ is Frobenius. Let q be a prime distinct from p and $1 \neq Q$ the Sylow q-subgroup of $M_1 \cap M_2$. Let Z_i be the Sylow q-subgroup of $Z(M_i)$.

As each member of $\mathcal{M}$ is nilpotent, if Q is Sylow in M_1, then maximality of M_i implies $M_1 = \theta(N(Q))$ and Q is not Sylow in M_2. As $m(E) = 3$, with 3.2 there exists $e \in E$ such that $\theta(C_{M_1}(e))$ has a nontrivial Hall q'- group R, and a Sylow q-group Q_2 of $\theta(C_{M_2}(e))$ is not contained in Q. Let $\theta(N(R)) \leq M_3 \in \mathcal{M}^*(E)$. Then $\langle Q, Q_2\rangle \leq Q_3 \in \mathrm{Syl}_q(M_2 \cap M_3)$, and as above AQ_3 is Frobenius. So $AQ < \theta(N_{AQ_3}(Q)) \leq M_1$ contradicting Q Sylow in M_1.

So Q is not Sylow in M_i, $i = 1, 2$. But maximality of Q implies $M_1 \cap M_2 = \theta(N_{M_i}(Q))$ for $i = 1$ or 2, say the former. Thus M_1 is a q-group and $N_{M_1}(Q) = Z_1(M_1 \cap M_2)$. In particular $Z_1 \neq 1$ and thus $M_1 \in \mathcal{M}^*$.

Lemma 3.4. *If Z_1 acts on a D-subgroup H with $A < H \leq M \in \mathcal{M}^*(E)$, H' a q-group with $Z(H) \neq 1$, then $A \neq M_1 \cap M \neq M_1 \cap M_2$.*

Proof. Choose M_2 so that either Q is maximal or $Q = 1$. Let

$$U_2 = N_{Z_2}(QZ_1), \quad X = QZ_1U_2 \quad \text{and} \quad Y = \theta(N_{M_1}(X)),$$

Then $M_1 \cap M_2 < Y$. If $M_1 \cap M_2 = \theta(N_{M_1}(Q))$ then $Z_1 \cap Y \neq 1$ while if $Q = 1$ then as $[Y, U_2] \neq 1$, the same holds. So

$$\{M_1\} = \theta(N(X)) \quad \text{and} \quad N(X) \cap Z_2 = N(X \cap M_1) \cap Z_2 = N(Z_1Q) \cap Z_2 = U_2.$$

Thus $U_2 = Z_2$. But then arguing as above on M_2, $\{M_2\} = \mathcal{M}^*(\theta(N(Z_1 Z_2 Q))) = \{M_1\}$, a contradiction.

Lemma 3.5. *Q is abelian.*

Proof. If not then $1 \neq Q' = (QZ_i)' \trianglelefteq N_{M_i}(QZ_i)$, so $Q < \theta(N_{M_i}(Q'))$, contradicting the maximality of Q.

Let P be the Sylow q-subgroup of M_2 and U a 4-group contained in E with $C_Q(U) \neq 1$. For some $u \in U^{\#}$, $\theta(C_{AP}(u)) \nleq M_1 \cap M_2$. Let $Y = QZ_2 \cap \theta(C_{M_2}(u))$. $Y < \theta(N(Y)) \cap P$ and as Q is abelian, $Q \leq \theta(N(Y))$. Thus maximality of $M_1 \cap M_2$ implies M_2 is the unique member of $\mathcal{M}^*(E)$ containing $\theta(N(Y))$.

Maximality of $M_1 \cap M_2$ implies $\theta(N(Z_1Q)) \leq M_1$ and either $M_1 \cap M_2 = \theta(N_{M_2}(Q))$ or $\theta(N(Q)) \leq M_2$. As Z_1 acts on $\theta(N(Q))$, with 3.4 and our initial remark, it is the former. So there is symmetry between M_1 and M_2.

Suppose $Y \cap Z_2 = 1$. Then $[Y, Z_1] = 1$, so Z_1 acts on $\theta(N(Y))$. By 3.4, Q is Sylow in $\theta(N(Y)) \cap \theta(N(Q))$, and $Q < P \cap \theta(N(Y))$, so $\theta(N(Y)) \cap Z_2 \neq 1$. Therefore 3.4 yields a contradiction. It follows that:

Lemma 3.6. *$Z_2 \cap \theta(C(u)) \neq 1$ and $M_2 = \theta(C(Z_2)) \in \mathcal{M}^*$.*

By symmetry there exists $v \in U^{\#}$ with $Z_1 \cap \theta(C(v)) \neq 1$.

Lemma 3.7. *Let $\theta(C(uv)) \leq M_3 \in \mathcal{M}^*(E)$. Then either $M_3 = M_1$ or M_2, or $Z_3 \cap \theta(C(uv)) \neq 1$ and $M_1 \cap M_2 = M_1 \cap M_3 = M_2 \cap M_3$.*

Proof. Assume $M_3 \neq M_1$ or M_2 and choose $M_3 \neq M \in \mathcal{M}^*(E)$ with $\theta(C(U)) \leq M \cap M_3$ maximal. Then by 3.6, $\theta(C(w)) \cap M_3 \neq 1$ for some $w \in U^\#$ and as $M_1 \neq M_3 \neq M_2$, $w = uv$. Further $Z(M) \cap \theta(C(x)) \neq 1$ for some $x \in U^\#$, say $x = v$, so $M = M_1$. Let $AX = M_1 \cap M_2 \cap M_3$. $1 \neq C_Q(U) \leq X$ and as $M_i \cap M_j$ is abelian,

$$\langle M_i \cap M_j : 1 \leq i < j \leq 3\rangle \leq \theta(N(X)) = M_1 \cap M_2.$$

So $M_1 \cap M_2 = M_1 \cap M_3$ by maximality of $M_1 \cap M_3$.

LEMMA 3.8. $|\mathcal{M}^*(E)| > 3$.

Proof. There exists $e \in E^\#$ with $\theta(C_{M_i}(e)) \nleq M_j$, $i \neq j$. Thus $|\mathcal{M}^*(E)| \geq 3$. Assume equality. Then $\theta(C(e)) \leq M_3$, and $M_1 \cap M_3 \nleq M_1 \cap M_2$. So arguing as in 3.7, $A = M_1 \cap M_2 \cap M_3$. Thus for $a \in E^\#$ with $\theta(C(a)) \leq M_1$, a inverts $M_2 \cap M_3$. Now for some M_i, say M_1, there exists $e_i \in E^\#$, $1 \leq i \leq 3$, with $\theta(C(e_i)) \leq M_1$. Further $\theta(C(e_i e_j)) \nleq M_1$, so we may choose $\theta(C(e_1 e_i)) \leq M_2$, $i = 2, 3$, and $\theta(C(e_2 e_3)) \leq M_3$. Now e_i inverts $M_2 \cap M_3$ and thus $b = e_2 e_3$ centralizes $M_2 \cap M_3$. Also

$$M_3 = \langle \theta(C_{M_3}(e_i e_j) : i \neq j\rangle = \theta(C(b))(M_2 \cap M_3) \leq \theta(C(b)),$$

so b centralizes M_3.

Suppose $C_{Z_1}(b) = W \neq 1$. Then W acts on Z_3 and centralizes a nontrivial subgroup of Z_3, which acts on Z_1. Thus $X = N_{Z_3}(Z_1) \neq 1$ acts on $[Z_1, b] = V_1 \neq 1$ by 3.4. So $V = C_{V_1}(X) \neq 1$ acts on Z_3 and $V = [VZ_3, b] \trianglelefteq VZ_3$, and therefore Z_3 acts on $M_1 = \theta(C(V))$, contradicting 3.4.

Thus $W = 1$. So $Z_1 = C_{Z_1}(e_1 e_2)C_{Z_1}(e_1 e_3)$ acts on $\langle\theta(C(e_1 e_2)), \theta(C(e_1 e_3))\rangle = M_2$, contradicting 3.4.

LEMMA 3.9. *$Z_3 \cap \theta(C(uv)) \neq 1$ and for $M \neq M_i$, $1 \leq i \leq 3$, $M \cap M_3$ is maximal and v inverts $(M \cap M_3)'$.*

Proof. Let $M_i \in \mathcal{M}^*(E)$, $1 \leq i \leq 4$, choosing the groups with $Z_3 \cap \theta(C(uv)) \neq 1$ if possible. If $M_1 \cap M_3 \cap M_4 \neq 1$, then by 3.6, 3.7, and choice of M_i, $M_i \cap M_j = M_1 \cap M_2$, so for each i there is $x \in U^\#$ with $Z_i \cap \theta(C(x)) \neq 1$, a contradiction. Thus u and v invert $(M_3 \cap M_4)' = Y$, so uv centralizes Y. By 3.6 there exists $e \in E^\#$ with $Z_3 \cap \theta(C(e)) \neq 1$. Suppose $\theta(C_{M_4}(e)) = A$, Then e inverts $M_4'/C_{Z_4}(e)$ which is therefore abelian. So as

$$M_4'/C_{Z_4}(e) = [A, M_4/C_{Z_4}(e)],$$

$Z(M_4/C_{Z_4}(e)) = 1$ and thus $[Z_4, e] = 1$. So Z_4 acts on M_3, contradicting 3.4.

Therefore $1 \neq \theta(C_{M_4}(e))' \leq Y$, so as $[Y, uv] = 1$, arguing as above $\theta(C(uv)) \nleq M_1$ or M_2. So by 3.7 we may choose $Z_3 \cap \theta(C(uv)) \neq 1$.

As $M_3 \cap M_4 \neq A$, it is maximal by 3.6 and 3.7.

We now complete the proof of Theorem 3.3. Let $Y = (M_3 \cap M_4)'$ and $uv \in W$ a 4-group in E with $C_Y(W) \neq 1$. $Y = (M_3 \cap M_5)'$, some M_5. $[uv, Y] = 1$, so

$$M_3 = \langle\theta(C_{M_3}(w): w \in W^\#\rangle \leq YC(uv) \leq C(uv).$$

W acts on Z_1, so we may assume $Z = C_{Z_1}(uv) \neq 1$. By 3.4, $[Z_1, uv] \neq 1$, so

$$1 \neq [Z_1, uv] \cap C(C_{Z_3}(Z)) \leq C(Z_3),$$

against 3.4.

4. The case $m(G) \geq 3$

LEMMA 4.1. *Assume $m(G) \geq 3$, G has no subgroup of index 2, and let u be an involution in G. Then there exists an elementary 2-subgroup E of rank 3 containing u. Let S be a Sylow 2-subgroup of G containing E. Then there exists a 4-group $W \trianglelefteq S$ and an elementary subgroup V of S containing W with $m(V) \geq 3$ and $|E \cap V| \geq 4$.*

Proof. Let S be a Sylow 2-subgroup of G. As $m(G) \geq 3$, there exists a 4-group $W \trianglelefteq S$. Let $T = C_S(W)$. If E is an elementary subgroup of order 8 in S, then choose $V = (E \cap T)W$. Let u be an involution in S and suppose $m(C_S(u)) < 3$. Then $u \in S - T$. But as G has no subgroup of index 2, $u^G \cap T$ is nonempty. So $m(C_G(u)) \geq 3$.

LEMMA 4.2. *Assume Hypothesis 3.1 and let $m(G) \geq 3$. Then G has a proper 2-generated core.*

Proof. By 3.3, if E is an elementary 2-subgroup of rank 3, then $\mathcal{M}^*(E)$ contains a unique member M. Choose E with M of maximal order. Let S be a Sylow 2-subgroup of G containing E and W a 4-group normal in S. By 4.1 there exists an elementary subgroup V of S containing W, of rank at least 3, such that $|E \cap V| \geq 4$. Now by 3.2,

$$M = \langle \theta(C(x)) : x \in (E \cap V)^{\#} \rangle,$$

so $\{M\} = \mathcal{M}^*(V)$. Therefore $M = \langle \theta(C(w)) : w \in W^{\#} \rangle$, and thus S normalizes M. Set $T = C_S(W)$. Then $m(C_S(u)) \geq 3$ for any involution in T, so by 3.3, $\theta(C(u)) \leq M$. Suppose $\theta(C(s)) \nleq M$ for some involution s in $S - T$. Then $m(C_S(s)) = 2$, so $Z(S)$ contains a unique involution z. Let R be a Sylow 2-subgroup of $C(s)$ containing z. By 4.1, $m(R) \geq 3$. Further if $m(C_R(z)) \geq 3$ then $\mathcal{M}^*(C_R(z))$ contains a unique member K and

$$M = \langle \theta(C_M(x)) : x \in \langle s, z \rangle^{\#} \rangle \leq K.$$

So maximality of M implies $M = K$, contradicting the choice of s. Therefore s is the unique involution in the center of R, so s is conjugate to z. As $s \in S - T$ and T has index 2 in S, s is not rooted in S, a Sylow 2-subgroup of $C(z)$. Therefore z is not rooted in $C(s)$, so $C_S(s)$ is a 4-group. It follows from a result of Suzuki [6] that S is dihedral or semidihedral, and thus in particular $m(S) = 2$, a contradiction.

Set $H = N_G(M)$ and let $X \leq S$ with $m(X) \geq 2$. We have shown that $\theta(C(x)) : x \in X^{\#} \rangle = M$, so $N(X) \leq H$. Thus H contains a 2-generated core of G.

5. The proof of Theorem 1

Let G be a minimal counterexample to Theorem 1. By 2.8, G is not solvable, so minimality of G implies $O_\infty(G) = 1$. Let M be a minimal normal subgroup of G, and let $A \in D$. M is not in the center of G, so $[A, M] \neq 1$. Thus as $[A, M] \trianglelefteq M$, $[A, M]$ is semisimple. Then $A[A, M]$ is a nonsolvable D-subgroup of G, so $G = AM$ and $M = [A, M]$. M is the direct product of simple subgroups M_i permuted transitively by A. Let S be a Sylow 2-subgroup of M_1. Then $A[A, S]$ is a solvable D-subgroup and $[A, S]$ is a 2-group, so $[A, S] = 1$. Therefore $G' = M = M_1$ is simple.

Now by 4.2 either $m(G) \leq 2$ or $m(G) \geq 3$ and G has a proper 2-generated core. In the first case [1] implies $M \cong L_2(q)$, $L_3(q)$, $U_3(q)$, A_7, or M_{11}, q odd. In the second case [2] implies $M \cong L_2(q)$, $Sz(q)$, $U_3(q)$, q even, or J_1, the Janko group of order 175,560.

Let $A = \langle a \rangle$. By 2.9, a induces an automorphism of M centralizing a Sylow 2-subgroup of M. $L_2(q)$, $Sz(q)$, $U_3(q)$, q even, J_1, A_7, and M_{11} do not admit such an automorphism of odd order. Then G does not contain a strongly embedded subgroup, so for an involution $u \in G$, $\theta(C(u))$ is not cyclic. But if $M = L_3(q)$ or $U_3(q)$, q odd, $L = \text{Aut}(M)$, and u is an involution in M, then $O(C_L(u))$ is cyclic. So $M \cong L_2(q^p)$, q odd, and a induces a field automorphism on M.

Now if p divides the order of M, then q^2 is congruent to 0 or 1 modulo p, and therefore $|M : C_M(a)| = q^{p-1}(q^{2p} - 1)/(q^2 - 1) \equiv 0 \bmod p$. So a is not in the center of a Sylow p-subgroup of G, a contradiction. Therefore p does not divide the order of M, so a normalizes a subgroup Q of order q^p in M. Then $\theta(N(Q)) \notin \mathfrak{F}$, a contradiction.

This completes the proof of Theorem 1.

6. The proof of Theorem 3

Let $\mathfrak{F}$ be the class of Frobenius groups whose kernel is an elementary 2-group. Let G be a minimal counterexample to Theorem 3. Let $A \in D$ and a a generator of A. For $\langle b \rangle \in D$ write $a \sim b$ if b is conjugate to a in $\langle a, b \rangle$.

Suppose $p = 3$ and let $A \neq B \in D$, and $Q = \langle A, B \rangle'$. Then $B = A^x$ for some $x \in Q^\#$, so $\langle A, B \rangle = \langle A, x \rangle$ and thus A acts irreducibly on Q. So $|Q| = 4$ and $\langle A, B \rangle$ is isomorphic to the alternating group on 4 letters. Therefore [3] yields a contradiction. So $p > 3$.

Suppose $O_\infty(G) \neq 1$. Then minimality of G implies $G = AG'$ and $\sim$ is an equivalence relation. Further for $b, c \in a^G$, ab^{-1}, bc^{-1} and ac^{-1} have order 1 or 2, so as $(ab^{-1})(bc^{-1}) = ac^{-1}$, ab^{-1} commutes with ac^{-1}. But arguing as in 2.6, $a^{-1}a^G$ is normalized by G, so $G' = \langle a^{-1}a^G \rangle$ is an elementary 2-group. So $O_\infty(G) = 1$.

Let H be a proper D-subgroup of maximal order; we may assume $A \leq H$. Minimality of G implies H' is an elementary 2-subgroup. Let $\langle b \rangle = B \in D - H$ with $a \sim b$. Define

$$\Delta = \{ac^{-1} : c \in a^H \quad \text{and} \quad b \sim c\}$$

As $|H \cap D| > p - 1$, $\langle \Delta \rangle \neq 1$. But for $ac^{-1} = x \in \Delta$, x, cb^{-1} and ab^{-1} all have order 1 or 2, and $x(cb^{-1}) = ab^{-1}$, so x commutes with ab^{-1}. Thus $x^b = x^a \in H'$, so if $\langle \Delta \rangle = H'$, then $G = \langle H, B \rangle$ normalizes H', impossible as $O_\infty(G) = 1$. So $\langle \Delta \rangle < H'$. Therefore

$$\Gamma = \{bd^{-1} : d \in a^H \quad \text{and} \quad b \not\sim d\}$$

has order at least $|H \cap D|/2$. Let $x \in \Delta$ and $d \in a^H$. Then $x^{bd^{-1}} = x^{ad^{-1}} = x$. So $\Gamma \subseteq C(x)$. Therefore $K = \langle \Gamma \rangle \neq G$. Also for each $y \in \Gamma$, $\langle y \rangle \in D$, so K is a D-subgroup. Finally if bd^{-1} and bc^{-1} are in Γ with $c \neq d$, then $(bc^{-1})^{-1}bd^{-1} = cd^{-1}$ is an involution, so $\langle bd^{-1} \rangle \neq \langle bc^{-1} \rangle$. Therefore $|K \cap D| \geq |H \cap D|/2$. But $|H \cap D| - 2^{n+1}$ and $|K \cap D| = 2^{n+r}$ with $2^{n+1} \equiv 2^{n+r} \equiv 1 \bmod p$, so $r \geq 1$, and $|K \cap D| \geq |H \cap D|$. Thus maximality of H implies $|K \cap D| = |H \cap D|$.

Now if $|\Gamma| > |H \cap D|/2$, then $Q = \langle \bar{u}v : u, v \in \Gamma \rangle = H'$, so $x \in K \leq C(x)$, a contradiction as minimality of G implies $Z(K) = 1$. Therefore $|\Gamma| = |H \cap D|/2 = |Q|$. So $P = \langle \Delta \rangle$ also has order $|H \cap D|/2$. But as $p > 3$, $|H \cap D| > 4$, so $Q \cap P \neq 1$. Thus we may assume $x \in Q \cap P \leq K \leq C(x)$, a contradiction.

This completes the proof of Theorem 2.

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California Institute of Technology
Pasadena, California