ON FINITE SIMPLE GROUPS OF LENGTH 8

BY

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1. Introduction

Let G be a finite group. Let $H_0 > H_1 > \cdots > H_n$ be a chain of subgroups of G, where H_i is a proper subgroup of H_{i-1} $(1 \leq i \leq n)$. Then we say that the chain has length n. For a fixed group G we denote by $l(G)$ the maximum of chain lengths, where the chain ranges over all possible ones. We call $l(G)$ the length of G.

The purpose of this paper is to prove the following

THEOREM. *Let G be a finite simple group of length* 8 *with S_2-subgroups of order 2^7. Then G is isomorphic to $PSL(2,127)$.*

Using this statement together with the recent results[2] of D. Gorenstein and K. Harada on finite simple groups with S_2-subgroups of sectional rank 4 and the characterization of finite groups with abelian S_2-subgroups of J. H. Walter [15], one can obtain a classification of all finite simple groups of length 8. More precisely, we have the

COROLLARY. *Let G be a finite simple group of length* 8. *Then G has either abelian S_2-subgroups or S_2-subgroups of sectional rank* 4. *Especially, G is a known group.*

Proof of the corollary. Let T be an S_2-subgroup of G. Suppose that T is neither of sectional rank 4 nor abelian. Then $|T| = 2^6$, $T' \neq 1$ and either $|T/D(T)| = 2^5$ or there exists an elementary abelian subgroup E of T of order 32. Assume that $|T/D(T)| = 2^5$. Now $T' = D(T) = \langle z \rangle$, $z^2 = 1$. By a theorem of Glauberman [3], there exists an element z_1, $z_1 \sim_G z$, $z_1 \in T \setminus \langle z \rangle$. It is $|C_T(z_1)| \geq 2^5$, as $|T'| = 2$. Since $\langle z \rangle = \mho^1(T)$, for all S_2-subgroups T of $C(z)$, it follows that $z_1 \notin Z(T)$ and $C_T(z_1)$ is elementary abelian of order 32.

Thus, in any case, T contains an elementary abelian subgroup E of order 32 and $|Z(T)| \geq 2^3$. If $|Z(T)| = 2^3$, then $|T/Z(T)| = 2^3$ with $|T,E| = 2$, $Z(T) \leq E$, and we get a contradiction by a result of Harada [5a]. Therefore $|Z(T)| = 2^4$. Let t be an element of order 4 in T and z_1 an inovlution in $E \setminus Z(T)$. Then $z_1 \in \langle Z(T), t \rangle$ and by a result of Thompson [13, Lemma 5.38] $z_1 \sim_G z$ for some $z \in Z(T)$. Since $E \lhd T_1 \leq C(z_1)$, for some S_2-subgroup

Received September 28, 1972

[1] This work was done at the University of Mainz under the direction of Professor D. Held, while the author was a fellow of Alexander von Humboldt foundation. It constitutes the main part of the author's doctorial dissertation.

[2] Professor D. Gorenstein has informed me in a letter about these results.

T_1 of $C(z_1)$, and $T_1 \neq T$, it follows that $N(E) = \langle T, T_1\rangle = TP$, where P is a subgroup of odd prime order.

Suppose that $N(T) > T$. Then $N(T) = TQ$, Q a subgroup of odd prime order. It is not $Q \leq N(E)$, because $T \ntriangleleft N(E)$. But there are at most two elementary abelian subgroups of order 32 in T, a contradiction.

Therefore $N(T) = T$. By a theorem of Burnside, it follows that two distinct elements of $Z(T)$ are never conjugate in G. Let v be an involution of

$$Z(T) \cap Z(T_1) \leq Z(N(E)) \cap E.$$

Since $N(E)$ is maximal in G, we have $C(v) = N(E)$. Also $|\, Z(T) \cap Z(T_1)\,| \geq 2^3$. By a result of Glauberman [3], there is an element $v_1 \in T\backslash Z(T)$, $v_1 \sim_G v$. Now $C(v) \neq C(v_1)$ and $C(v) \cap C(v_1)$ contains $M = \langle Z(T), v_1\rangle$ with $|\, M\,| = 2^5$. Since M contains the S_2-subgroups of $Z(C(v))$ and $Z(C(v_1))$, we have

$$V = Z(C(v)) \cap Z(C(v_1)) \neq 1.$$

It follows that $C(V) \geq \langle C(v), C(v_1)\rangle = G$, a contradiction. The corollary is proved.

In the whole paper G will denote a finite simple group satisfying the assumptions of our theorem. Moreover, if a group is given by its generators, then the generators whose order is not stated are involutions, and the pairs of elements whose interaction is not stated commute, e.g.,

$$E = \langle a, b, c \mid\rangle = \langle a, b, c \mid a^2 = b^2 = c^2 = 1, a^b = a^c = a, b^c = b\rangle.$$

The other notation is standard (cf. Thompson [13]).

2. Some known and auxiliary results

Finite simple groups with short chains of subgroups were investigated by Z. Janko in his papers [8] and [9], and by K. Harada in his paper [5]. They have proved the following theorems, which we shall use frequently:

THEOREM OF JANKO [8]. *Let G be a finite non-abelian simple group whose length $l(G)$ is at most four. Then G is isomorphic to $PSL(2, p)$, where $p = 5$ or p is such prime that $p - 1$ and $p + 1$ are products of at most three primes and $p \equiv \pm 3$ or ± 13* (mod 40).

THEOREM OF JANKO [9]. *Let G be a finite non-abelian simple group whose length $l(G)$ is at most five. Then G is isomorphic to $PSL(2, q)$ for some prime power q.*

THEOREM OF HARADA [5]. *Let G be a finite non-abelian simple group whose length $l(G)$ is at most seven. Then G is isomorphic to one of the following groups: $PSL(2, q)$ for a suitable prime power q; $PSU(3, 3^2)$; $PSU(3, 5^2)$; A_7, the alternating group of degree seven; M_{11}, the Mathieu group of degree eleven; J_1, the first Janko group of order* 175560.

We note that $l(J_1) = l(A_7) = 6$ and $l(M_{11}) = l(PSU(3,3^2)) = l(PSU(3,5^2)) = 7$.

We shall often need the following

CRITERION OF NONSIMPLICITY. *Let G be a finite group of even order with the properties:*

(i) *An S_2-subgroup T of G is maximal in G.*

(ii) *There is an involution t in T, such that $|T|:|C(t)|_2$ is greater than 2 and $C(t)$ contains precisely two conjugate classes of involutions under G.*

Then G is not simple.

Proof. Suppose that G is simple. By a theorem of Burnside it follows from (i) and (ii) that $Z(T)$ is cyclic. Let z be the involution of $Z(T)$. If u is an other involution of T then the following relations are obviously equivalent:

(a) $u \in Z_2(T)$,

(b) $\langle u, z\rangle \lhd T$,

(c) $|T:C_T(u)| = 2$.

Assume at first, there is an involution u in $Z_2(T)\backslash Z(T)$. Then by (b), $\langle u, z\rangle \lhd T$ and so $\langle u, z\rangle \in \mathsf{U}(2)$. Hence by Lemma 5.38 of Thompson [13] the group $C(t)$ contains a conjugate u_1 of u. By a result of Harada [5, p. 663] and by (c) it follows $u_1 \nsim_G z$ and $|T|:|C(u_1)|_2 = 2$. Thus by the condition (ii) of our criterion we have $z \nsim_G u_1 \nsim_G t$, which contradicts the same condition.

Therefore, z is the unique involution in $Z_2(T)$. Now, by (c), the group T contains no involution v with $|T:C_T(v)| = 2$, and by (b) no elementary abelian normal subgroup of order 8. Since also T is maximal in G, all the conditions of the main theorem of Janko-Thompson [11] hold. But none of the groups in the statement of this theorem satisfies the conditions of our criterion.

Hence G is not simple. The criterion is proved.

In view of Lemma 5.38 of Thompson [13], our criterion has the following

COROLLARY. *Let G be a finite simple group of even order with the properties:*

(i) *An S_2-subgroup T of G is maximal in G.*

(ii) *There is an involution t in T, such that $C(t)$ contains precisely two conjugate classes of involutions under G.*

Then G has precisely two classes of involutions.

3. Solvability of 2-local subgroups

We shall prove at first the

LEMMA. *Let G be a finite simple group of length 8 with S_2-subgroups of order 2^7. Then all 2-local subgroups of G are solvable.*

Proof of the lemma. Suppose a 2-local subgroup H of G is not solvable. We show in several steps, that this assumption is contradictory. The following propositions are all proved under the above assumption.

PROPOSITION 1. *In the group G there is an involution t such that the centralizer C of t in G is not solvable.*

Proof. We can suppose that $H = N(E)$, where E is elementary abelian. By Huppert [6], $l(H/E) \geq 4$ and so $l(E) \leq 3$. If $l(E) \leq 2$, then $C(E)$ is not solvable because $A(E)$ is solvable. Let be $|E| = 2^3$. Now $H/C(E)$ is isomorphic to a subgroup of $PSL(2, 7)$ and H/E is a simple group of length 4. Thus $H = C(E)$. The assertion holds for some $t \in E^{\#}$ in both cases.

PROPOSITION 2. *If $C/\langle t\rangle$ is simple, then $C = \langle t\rangle \times F$, where $F \cong PSL(2, q)$ for some prime power q.*

Proof. Since $l(C/\langle t\rangle)$ is at most 6 it follows by Harada [5] that $C/\langle t\rangle$ is isomorphic to $PSL(2, q)$ for some prime power q, or to the first Janko group J_1 or to the alternating group A_7.

Let S be an S_2-subgroup of C and T an S_2-subgroup of G containing S. Then $\langle t\rangle \times Z(T) \leq Z(S)$. Suppose first $C/\langle t\rangle \cong PSL(2, q)$, with $8 \neq q \neq 9$. Then the Schur multiplier of $C/\langle t\rangle$ equals 2, as $l(PSL(2, p^f)) \geq 7$ for $f \geq 4$. Since S is not generalized quaternion we get $C = \langle t\rangle \times F$ with $F \cong PSL(2, q)$. If $C/\langle t\rangle \cong PSL(2, 8)$ or $PSL(2, 9)$, S is elementary or a direct product of $\langle t\rangle$ and a dihedral group of order 8 respectively, because all involutions in $C/\langle t\rangle$ are conjugate, and thus C splits over $\langle t\rangle$ by Gaschütz [2], as there are involutions in $C\backslash\langle t\rangle$.

Suppose now that $C/\langle t\rangle \cong J_1$ or A_7. As in the both cases just considered we have $C = \langle t\rangle \times F$, with $F \cong J_1$ or A_7 respectively, for the same reasons. But every involution in F and therefore also in C contains a 3-element in its centralizer. However, for a central involution z of G in C the group $C(z)$ is an S_2-subgroup of G, a contradiction.

PROPOSITION 3. *The group $C/\langle t\rangle$ is not simple.*

Proof. Assume $C/\langle t\rangle$ is simple. By Proposition 2, $C = \langle t\rangle \times F$, $F \cong PSL(2, q)$, $q = p^f$, p a prime. Moreover $f \leq 3$ for $p = 2$. Therefore an S_2-subgroup V of F is elementary of order 8 or dihedral of order at most 32. The group $S = \langle t\rangle \times V$ is an S_2-subgroup of C. Since the involutions of F are all conjugate under F and $S = \langle t\rangle \times V$ is not an S_2-subgroup of G there are precisely two classes of involutions in S under G. Applying our criterion we get a contradiction if $|V|$ is smaller than 32. Thus we can suppose that $V \cong D_{32}$, the dihedral group of order 32.

Let T be an S_2-subgroup of G containing S. Since $C(z) = T$ for any involution z in $Z(T)$ it follows that $F \cong PSL(2, 31)$. We can write

$$V = \langle a, b \mid a^{16} = 1, a^b = a^{-1}\rangle$$

for some $a, b \in V$. Obviously $\langle a^8 \rangle = \Omega_1(\mho^1(S)) \lhd T$ and therefore $C(a^8) = T$.

Choose $v \in T \backslash S$ to be an element of smallest possible order. By a result of Harada [5] it must be $v^x = v_1$ for some $v_1 \in S$ and some $x \in G$. It is clear that $x \notin T$.

If $|v| > 2$ it follows $v_1^{|v|/2} = a^8 = x^{-1}v^{|v|/2}x$. Since $a^8 \not\sim_G t$, $t \sim ta^8$, $t \sim ta^\delta b$ for all δ and $x \notin T$, we get $v^2 = a^\delta b$ for some δ. It is $\langle a^2 \rangle = \mho^1(S) \lhd T$ and thus $(a^4)^v = a^{4\zeta}$, $\zeta \in \{1, 3\}$. Now $(a^4)^{v^2} = (a^4)^{\zeta^2} = a^4 = (a^4)^{a^\delta b} = a^{-4}$, a contradiction.

We conclude that $|v| = 2$. Since

$$\langle a^8, t \rangle = Z(S) \lhd T \quad \text{and} \quad v \in C(a^8) \backslash C(t)$$

it must be $t^v = ta^8$. Since $ta^\delta b \sim ta^8 \sim t \not\sim_G a^8 \sim a^\delta b$ we have $b^v = a^\vartheta b$ and $a^v = a^\eta$. Using $v^2 = 1$ we get

$$\eta^2 \equiv 1 \pmod{16}; \qquad \vartheta(1 + \eta) \equiv 0 \pmod{16}.$$

Replacing b by $a^\lambda b$ for a suitable λ, we can assume, that

$$(\eta, \vartheta) \in \{(1, 0), (1, 8), (7, 0), (9, 0), (15, 0), (15, 1)\}.$$

Consider $S_1 = \langle t, a^8, b \rangle$. We have $N_F(\langle a^8, b \rangle) \cong \Sigma_4$ because $F \cong PSL(2, 31)$. Also $N_1 = N(S_1) \cap C = \langle t \rangle \times N_F(\langle a^8, b \rangle)$, $|N_1| = 2^4 \cdot 3$ and $C(S_1) = S \cap C(b) = S_1$.

Suppose $N_1 < N(S_1)$. Then $|N(S_1):N_1| = 4$ because t is under $N(S_1)$ conjugated into $\{tz, tb, tzb\}$ and this set is a conjugate set under an 3-element of $N_F(\langle a^8, b \rangle)$. Therefore $|N(S_1)| = 2^6 \cdot 3$ and $N(S_1)/S_1 \cong \Sigma_4$, the symmetric group of degree four, because $A(S_1) \cong PSL(2, 7)$. It follows that $|O_2(N(S_1))| = 2^5$. Now a^8 and b have under $N(S_1)$ precisely three conjugates and so $C(b) \cap N(S_1)$ and $C(a^8) \cap N(S_1)$ are S_2-subgroups of $N(S_1)$. It follows

$$O_2(N(S_1)) \leq C(b) \cap C(a^8) \quad \text{and} \quad |C(b) \cap S| \geq 2^4$$

as $|T:S| = 2$. But $C(b) \cap S = S_1$ with $|S_1| = 2^3$, a contradiction.

Thus $N(S_1) = N_1$. It follows $|N_T(S_1)| = 2^4$ and so $N_T(S_1) \leq S$. But $v \in T \backslash S$ and $\langle t, a^8 \rangle^v = \langle t, a^8 \rangle$, therefore $b^v \in T \backslash S_1$. We conclude that $(\eta, \vartheta) \equiv (15, 1)$, and

$$T = \langle t, a, b, v \mid a^{16} = 1, t^v = ta^8, a^b = a^{-1}, a^v = a^{-1}, b^v = ab \rangle.$$

Now $|a^\alpha v| = 2$, $|ta^\alpha v| = 4$, $|t^\tau a^\alpha bv| = 32$ for all α, τ.

Suppose tv is fused with an element of S, i.e. $x^{-1}tvx = s$, with $s \in S$, $x \in G$. Since $s^2 = (tv^2) = a^8$, it would follow $x \in C(a^8) = T$ which is not possible. Hence tv is not fused with any element of S. By a simplicity criterion of Harada [5] must $(tv)^2 = a^8$ be fused now with an element in $T \backslash S$. We can obviously suppose that $v \sim a^8$. But $ta^4 \in C_T(v)$ and $(ta^4)^2 \neq v$, while the square of each element of order 4 in $T = C(a^8)$ equals a^8, in contradiction with $v \sim a^8$. Proposition 3 is completely proved.

PROPOSITION 4. *Let N be the maximal solvable normal subgroup of C and K/N a minimal normal subgroup of C/N. Then K/N is simple and uniquely determined. Moreover $C_{C/N}(K/N) = N/N$. Thus C/N is isomorphic to a* subgroup *of $A(K/N)$.*

Proof. The simplicity of K/N is obvious. If K_1/N is also a minimal normal subgroup of C/N, we have $K = K_1$ or $K \cap K_1 = N$. But from $K \cap K_1 = N$ we get $|K/N|\,|K_1/N| = |KK_1/N| \,\big|\, |C/K|\,|K/N|$, and so $|K_1/N| \,\big|\, |C/K|$ a contradiction to $l(C/K) \leq 2$, and to $l(K_1/N) \geq 4$ by Huppert [6]. Therefore $K = K_1$ and K is unique.

Set $C_{C/N}(K/N) = L/N$. We have $K \cap L = N$, as K/N is non-abelian. It follows that $|L/N| \,\big|\, |C/K|$. But $l(C/K) \leq 2$, $L \trianglelefteq C$ and so L is solvable. Therefore $L = N$.

PROPOSITION 5. *We have $N > \langle t \rangle$.*

Proof. Assume $N = \langle t \rangle$. Then by Proposition 3, $K \neq C$. Thus $l(K/N) \leq 5$ and by Janko [9], $K/N \cong PSL(2, q)$, q a prime power.

Assume that $2 \nmid |C/K|$. By Proposition 4 and by Dieudonné [1] it must be $K/N \cong PSL(2, p^3)$ for some prime p. As in Proposition 2 we conclude that $K = \langle t \rangle \times F$ with $F \cong PSL(2, p^3)$. Since $l(F) \leq 5$, we have $p = 2$ or $p = 3$ and we get a contradiction in both cases applying our criterion.

Therefore $2 \,\big|\, |C/K|$. Moreover $|C/K| = 2$. For otherwise we would have $l(K/N) = 4$ and so, by Janko [8], $K/N \cong PSL(2, p)$, $p \geq 5$. Hence by Proposition 4 and by Dieudonné [1], it follows that $C/N \cong PGL(2, p)$ and so $|C/K| = 2$, a contradiction.

Assume now that $p = 2$. Since $l(PSL(2, 2^f)) \geq 6$ for $f \geq 4$ and $PSL(2, 4) \cong PSL(2, 5)$ we can suppose that $K/N \cong PSL(2, 8)$. But now $|A(K/N)| = 3 \cdot |K/N|$, a contradiction to $|C/K| = 2$.

Thus we can assume that $p \neq 2$. Let S be an S_2-subgroup of C and $S_1 = S \cap K$.

We prove next that for each S_2-subgroup T of G containing S, we have $Z(T) \cap S_1 \neq 1$.

Assume conversely that there is an S_2-subgroup T of G containing S such that $Z(T) \cap S_1 = 1$. Because $Z(T) \leq S$ and $|S : S_1| = 2$ we must have $Z(T) = \langle z \rangle$ with $|z| = 2$, and $S = S_1 \times \langle z \rangle$. As $K/N \cong PSL(2, p^f)$, $p \neq 2$, $l(K/N) \leq 5$, we have $f \leq 3$. If $f = 1$ or $f = 3$, then by Proposition 4 and by Dieudonné [1], we conclude that $C/N \cong PGL(2, p^f)$. But an S_2-subgroup of $PGL(2, p^f)$ is dihedral of order of least 8, a contradiction to $S/N = S_1/N \times \langle z \rangle N/N$. Thus $f = 2$, $K/N = PSL(2, p^2)$. Since $l(K/N) \leq 5$, we must have $p^f = 3^2$. So we can assume that

$$K/N \cong PSL(2, 9) \cong A_6 .$$

We have $A(K/N) \cong P\Gamma L(2, 9)$ and this group contains two subgroups isomorphic to $PGL(2, 9)$ and to Σ_6, the symmetric group of degree six, respectively, whose intersection is $PSL(2, 9)$. Moreover

$$P\Gamma L(2, 9)/PSL(2, 9) \cong Z_2 \times Z_2 .$$

Thus by Proposition 4, C/N is isomorphic to one of the three subgroups of index 2 in $P\Gamma L(2, 9)$, which contain $PSL(2, 9)$. The corresponding S_2-subgroups can be easily computed and one gets

$$B_1 \cong \langle a, b \mid a^8 = 1, a^b = a^{-1}\rangle,$$

$$B_2 \cong \langle c, d, f \mid c^4 = 1, c^d = c^{-1}\rangle,$$

$$B_3 \cong \langle af, b \mid (af)^8 = 1, (af)^b = (af)^\gamma\rangle \quad \text{with} \quad \gamma \in \{3, 7\},$$

where B_1, B_2, B_3 correspond to $PGL(2, 9)$, Σ_6 and to the third of the mentioned subgroups respectively. It holds $S/N \cong B_i$ for some i, $i = 1, 2, 3$. But $\mid Z(B_1) \mid = \mid Z(B_3) \mid = 2$ and therefore $S/N \cong B_2$, as $\mid Z(S/N) \mid \geq 2^2$. We get $C/N \cong \Sigma_6$. Hence $\langle z, t\rangle/\langle t\rangle$ has an element $x\langle t\rangle/\langle t\rangle$ of order 3 in its centralizer. We can assume that $\mid x \mid = 3$ and we have $\langle z, t\rangle^x = \langle z, t\rangle$. But x acts nontrivially on $\langle z, t\rangle$ as $z \notin C(z) = T$, which is a contradiction to $z \not\sim_G t$. Our assertion is proved.

Thus for each S_2-subgroup T of G containing S, $Z(T) \cap S_1 \neq 1$. In the following let z be an involution of $Z(T) \cap S_1$ for some T.

The group S_1/N is dihedral of order at most 16. Assume first that $S_1/N \cong D_4$, D_n denoting the dihedral group of order n. We know that all involutions of S_1/N are conjugate under K/N. This fact and $z \in S_1$ imply that $S_1 = \langle t\rangle \times S_0$, and hence it follows that $K = N \times F$, with $F \cong PSL(2, p^f)$, by a result of Gaschütz [2]. Obviously F char K and so $F \lhd C$. We have $z = t^\tau f$, $\tau \in \{0, 1\}$, $f \in F$ and $C_C(f)$ is a 2-group because $C(z) = T$. Therefore $p^f = 5$, $F \cong PSL(2, 5) \cong A_5$. By Proposition 4 and Dieudonné [1] it follows that $C/N \cong PGL(2, 5) \cong \Sigma_5$ and so $S/N \cong D_8$. Thus $Z(S)$ is of order at most 4 and we get $Z(T) = \langle z\rangle$. Let $V = \langle v_1, v_2 \mid\rangle = S_1 \cap F$. If $c \in S\backslash S_1$, then $c \in N(S_1) \cap N(F) \leq N(V)$. Moreover, by our criterion we can suppose that $c^2 = 1$. Thus we can write

$$S = \langle t, v_1, v_2^c, c \mid v_2^c = v_1 v_2\rangle.$$

Now, $\mho^1(S) = \langle v_1\rangle$, and $Z(S) = \langle t, v_1\rangle = \langle t, z\rangle$. We conclude that $v_1 = z$ and $t \sim_T tz$. As known, there is a subgroup R of F, $\mid R \mid = 3$, with $VR \cong A_4$. Therefore $t \sim tv_1 \sim_R tv_2 \sim_R tv_3$ and $z = v_1 \sim_R v_2 \sim_R v_1 v_2$. Hence $\mid N(S_1) \mid = 2^4 \cdot 3$ or $\mid N(S_1) \mid = 2^6 \cdot 3$, since $N(S_1) \cap C = SR$.

Assume first that $\mid N(S_1) \mid = 2^4 \cdot 3$, i.e. $N(S_1) = SR$. Let $s \in T\backslash S$ with $s^2 \in S$ and $\langle S, s\rangle = T_1$. Then

$$T_1 = \langle t, v_1, v_2, c, s \mid t^s = tv_1, v_2^c = v_1 v_2, v_2^s = t^\alpha v_1^\beta v_2^\gamma c^\delta, c^s \in S, s^2 \in S\rangle.$$

Now $\delta = 1$, because $s \notin N_{T_1}(S_1) = S$. Since $\mid \iota_2 c \mid = 4$ it must be $\gamma = 0$, i.e. $v_2^s = t^\alpha v_1^\beta c$. Hence $(v_1 v_2)^s = t^\alpha v_1^{\beta+1} c$ and we see that S has precisely two classes of involutions, in contradiction with our criterion.

Therefore $|N(S_1)| = 2^6 \cdot 3$ and $t \sim tv_1 \sim tv_2 \sim tv_1 v_2$ under $N(S_1)$. We easily check that $C(S_1) = S_1$ and therefore $N(S_1)/S_1 \cong \Sigma_4$, $N(S_1)/S_1$ being isomorphic to a subgroup of $PSL(2, 7)$. Hence $O_2(N(S_1)) = M$ is of order 2^5 and a S_3-subgroup R of $C(t) \cap N(S_1)$ acts faithfully on M/S_1. It follows that $C(t) \cap M = S_1$ and so $t \sim tv_1 \sim ti_2 \sim tv_1 i_2$ under M. We can now apply the theorem of Janko-Thompson [11]. As none of the groups of its statement satisfies our condition it follows that there is an involution s in T with $|C_T(s)| = 2^6$. But now, $s \not\sim_G z$, because otherwise $N(C_T(s)) \geq \langle T, C(s)\rangle$ would be too great. We have $\langle z, s\rangle \lhd T$ and there is a conjugate $\langle u_1, u_2 \,|\rangle$ of $\langle z, s\rangle$ in S, with $\langle u_1, u_2\rangle \lhd C(u_1 u_2)$, and $|C(u_1 u_2)| = 2^7$, by Lemma 5.38 of Thompson [13]. The elements u_1, u_2 are conjugate neither with t nor with z. Therefore u_1, u_2 belong to the set $\{c, v_1 c, tc, tv_1 c\}$. As $u_1 u_2$ is a central involution it follows

$$(u_1, u_2) \in \{(c, v_1 c), (tc, tv_1 c)\} \quad \text{and} \quad \langle u_1, u_2\rangle \lhd C(u_1 u_2) = C(v_1) = T.$$

There is a $u \in MS$ with $t^u = tv_2$. Also $v_1^u = v_1$, since $Z(S) \geq Z(T)$ and $z = v_1$ is the only central involution in $Z(S)$, and thus T is unique, $MS \leq T$. Hence with $u_1 = t^\varepsilon c$, $\varepsilon \in \{0, 1\}$, we get $(t^\varepsilon c)^u = t^\varepsilon v_1^\zeta c$, $\zeta \in \{0, 1\}$. But on the other hand $(t^\varepsilon c)^u = t^\varepsilon v_2^\varepsilon c^u$, hence $c^u = v_2^\varepsilon v_1^\zeta c$ and we get $(t^u)^{c^u} = (t^c)^u$, i.e. $tv_1 v_2 = tv_2$, a contradiction.

Thus $S_1/N \not\cong D_4$.

Suppose next that $S_1/N \cong D_8$. By Janko [8], [9], we have $K/N \cong PSL(2, p^f)$ and $l(K/N) = 5$. It is $|Z(S_1)| = 4$ and thus $Z(S_1) = \langle t, z \,|\rangle$. Since all involutions in K/N are conjugate we see that S_1 possesses 11 involutions and 4 elements of order 4. Let $\langle t, z, a \,|\rangle$ be an elementary abelian normal subgroup of S_1. Then we can write

$$S_1 = \langle t, z, a, b \mid a^b = t^\alpha z a\rangle = \langle t\rangle \times \langle a, b\rangle.$$

Thus, by a result of Gaschütz [2],

$$K = \langle t\rangle \times F, \quad \text{with} \quad F \cong PSL(2, p^f).$$

We have F char K char C and so F char C. Also K contains the element z with $C(z) = T$. From this we conclude that $F \cong PSL(2, 7)$ or $F \cong PSL(2, 9)$. Let be $c \in S \backslash S_1$, $S_1 \cap F = \langle h, k \mid h^4 = 1, h^k = h^{-1}\rangle$. We have

$$S = \langle t, h, k, c \mid h^4 = 1, c^2 = t^\iota h^\mu k^\nu, h^k = h^{-1}, h^c = h^\sigma, k^c = h^\beta k\rangle.$$

One easily sees that $S' \leq \langle h\rangle$ and especially $\Omega_1(S') = \langle h^2\rangle$. Assume first that $F \cong PSL(2, 7)$. Then by Proposition 4 and by Dieudonné [1], $C/N \cong PGL(2, 7)$ and $S/\langle t\rangle \cong D_{16}$. Thus $|Z(S)| = 4$, and hence $Z(S) = \langle z, t\rangle = \langle t, h^2\rangle$, $Z(T) = \langle z\rangle$. Since $\langle h^2\rangle$ char S we get $z = h^2$ and $t \sim_T zt$. All the involutions in F are conjugate in F. If there are no involutions in $S\backslash S_1$, then there are exactly two classes of involutions in S under G, by a result of Glauberman [3], and by our criterion we get a contradiction.

Thus we can suppose that $c^2 = 1$ and we can write

$$S = \langle t, h, k, c \mid h^4 = 1, h^k = h^{-1}, h^c = h^\sigma, k^c = h^\beta k\rangle.$$

Now $S = \langle t\rangle \times \langle h, k, c\rangle$ and by a result of Gaschütz [2] we have

$$C = \langle t\rangle \times F_1 \quad \text{with} \quad F \cong PGL(2, 7) \quad \text{and} \quad F < F_1 .$$

Since an S_2-subgroup of F_1 is isomorphic with D_{16}, we now have

$$S = \langle t, d, k \mid d^8 = 1, d^k = d^{-1}\rangle \quad \text{with} \quad d^2 = h,$$

for some $d \in S\backslash S_1$. We have under S seven classes of involutions in S with the representatives $z = d^4$, t, zt, k, tk, dk, tdk. But $z \sim k$, $tz \sim tk$ under F and $t \sim tz$ under T, and so $z \sim k$, $t \sim tz \sim tk$. Thus there are at most four classes of involutions in S under G with the representatives z, t, dk, tdk, all involutions in S_1 being fused either with z or with t. Applying the theorem of Janko-Thompson [11] we conclude that there is an involution s in T with $|C_T(s)| = 2^6$ and $s \not\sim_G z$, $s \not\sim_G t$. By Lemma 5.38 of Thompson [13], there is a conjugate $\langle u_1 u_2 \mid\rangle$ of $\langle z, s\rangle$ in S with $\langle u_1, u_2\rangle \lhd (u_1 u_2)$ and $u_1 u_2 \sim z$. Now

$$u_1, u_2 \in \{dk, d^3k, d^5k, d^7k, t, tdk, td^3k, td^5k, td^7k\} \quad \text{and} \quad [u_1, u_2] = 1.$$

Since $t \sim td^4 \not\sim d^4 = z$ and $u_1 u_2 \sim z$ it must be that $u_1 u_2 = z$ and $|C_T(u_i)| = 2^6$, for $i = 1, 2$. But then $|C_S(u_i)| \geq 2^4$, a contradiction to $C_S(u_i) = \langle u_i\rangle \times Z(S)$ for $i = 1, 2$, as one can directly see.

Thus we can suppose that $F \cong PSL(2, 9) \cong A_6$. We conclude as before that C/N is isomorphic to one of the three subgroups of index 2 in $P\Gamma L(2, 9)$ which contain $PSL(2, 9)$, and that S/N is isomorphic to one of the following groups: D_{16}, S_{16} the semidihedral group of order 16 or $Z_2 \times D_8$.

The case $S/N \cong D_{16}$ yields to a contradiction in the same way as in the case $F \cong PSL(2, 7)$. Suppose next that

$$S/N \cong S_{16} = \langle a, b \mid a^8 = 1, a^b = a^3\rangle.$$

Here $S_1/N \cong \langle a^2, b\rangle$ and therefore all involutions of S are in S_1. Also $|Z(S/N)| = 2$ and thus $Z(S) = \langle t, z\rangle$, $Z(T) = \langle z\rangle$. But $S_1 = \langle t\rangle \times (S_1 \cap F)$ and there are 3 classes of involutions in S_1 under K, with the representatives t, tz, z. By the theorem of Glauberman [3] we conclude that S_1 contains precisely two classes of involutions under G. Applying our criterion we get a contradiction.

Thus we may assume that $S/N \cong D_8 \times Z_2$,

$$S/N = \langle \bar{c}, \bar{d}, \bar{f} \mid \bar{c}^4 = \bar{1}, \bar{c}^{\bar{d}} = \bar{c}^{-1}\rangle$$

with $S_1/N = \langle \bar{c}, \bar{d}\rangle$, c, d, $f \in S$, where $\bar{x} = xN$ for all $x \in C$. In this case $C/N \cong \Sigma_6$.

We had $S_1 = \langle t, h, k \mid h^4 = 1, h^k = h^{-1}\rangle$. Without loss of generality we may suppose that $c = h$, $d = k$ and we get

$$S = \langle t, h, k, f \mid h^4 = 1, f^2 = t^\varphi, h^k = h^{-1}, h^f = h, k^f = k\rangle, \qquad \varphi \in \{0, 1\},$$

as $\langle h, k\rangle \lhd S$. If $f^2 = t$, then $\langle t\rangle = \mho^1(Z(S))$ char S, which contradicts $|S| = |C(t)|_2$. Thus

$$S = \langle t, h, k, f \mid h^4 = 1, h^k = h^{-1}\rangle = \langle t\rangle \times \langle h, k\rangle \times \langle f\rangle.$$

By a result of Gaschütz [2] we get

$$C = \langle t\rangle \times F_1 \quad \text{with} \quad F_1 \cong \Sigma_6$$

and $F < F_1$, $F_1 = \langle F, f\rangle$, $F \cong A_6$.

All the involutions of $F_1\backslash F$ have a 3-element in its centralizer and hence there is no central involution in $C\backslash\langle t\rangle F$. Thus

$$\Omega_1(Z(T)) \leq Z(S_1) = \langle t, h^2\rangle$$

and $\Omega_1(Z(T)) = \langle z\rangle$ is of order 2. Thus $Z(T)$ is cyclic and $z = h^2$, as $\langle h^2\rangle = S'$. But $Z(S) = \langle t, h^2, f\rangle$ is elementary abelian and therefore $Z(T) = \langle z\rangle$. Hence T containing S is unique and $z = h^2$ is the unique central involution of G, which is in $Z(S)$. Thus

$$S_1 = N(Z(S)) \leq T = C(h^2).$$

Since $S_1 > S$ the element t has some conjugate $t_1 \neq t$ in $Z(S)$.

Let $S < T_1 < T$. Then $|Z(T_1) \cap Z(S)| \geq 4$, and $|\Omega_1(Z(T_1))| \leq 4$, as $Z(T) = \langle z\rangle$. Hence $\Omega_1(Z(T_1)) = V \leq Z(S)$ is of order 4 and $V \lhd T$. With $V = \langle z, s \mid\rangle$ we have $t \not\sim_G s \sim_T sz \not\sim_G z$.

In view of conjugance of all the involutions of F we get the following conjugate classes in S: t, $h^2 \sim k \sim hk \sim h^2k \sim h^3k$, $th^2 \sim tk \sim thk \sim th^2k \sim th^3k$, $f \sim hkf \sim h^3kf$, $kf \sim h^2f \sim h^2kf$, $tf \sim thkf \sim th^3kf$, $tkf \sim th^2f \sim th^2kf$. Since z is the unique central involution in $Z(S_1)$ it follows that $h^2 \sim k \sim hk \sim h^2k \sim h^3k$ are all the central involutions of G in S. Also we easily see that

$$E_1 = \langle t, h^2, f, k\rangle \quad \text{and} \quad E_2 = \langle t, h^2, f, hk\rangle$$

are the unique elementary abelian subgroups of order 16 in S. Identifying F_1 with Σ_6 we can take $h \equiv (1234)(56)$, $k \equiv (12)(34)$ and $f \equiv (56)$. The element $r_1 \equiv (123)$ normalizes E_1 and the element $r_2 \equiv (125)(346)$ normalizes E_2. We have

$$N(E_1) \geq \langle E_1, r_1, hk\rangle = W_1 \quad \text{and} \quad N(E_2) \geq \langle E_2, r_2, k\rangle = W_2.$$

Also $W_1/E_1 \cong W_2/E_2 \cong \Sigma_3$. Since E_1 and E_2 contain each precisely 3 central involutions h^2, k, h^2k and h^2, hk, h^3k respectively, and $|N(E_i)| = 2^5 \cdot 3p_i$, with $p_i = 1$ or p_i a prime, it follows $|N(E_i)| \in \{2^5 3, 2^6 3\}$, for $i = 1, 2$. But $\langle E_i, r_i\rangle \trianglelefteq N(E_i)$ and so all S_3-subgroups of $N(E_i)$ are contained in $C(t)$. As mentioned, t has some conjugate $t_1 \neq t$ in $Z(S)$ under T. Now $\langle r_1, r_2\rangle \leq C(t) \cap C(t_1)$ because of symmetry and because of uniqueness of E_1 and E_2 in S. But $C_{Z(S)}(r_1) = \langle t, f\rangle$ and $C_{Z(S)}(r_2) = \langle t, h^2f\rangle$ and so $C_{Z(S)}(\langle r_1, r_2\rangle) = \langle t\rangle$, a contradiction to $t \neq t_1$.

Thus $S_1/N \not\cong D_8$.

We can therefore assume that $S_1/N \cong D_{16}$. We have $K/N \cong PSL(2, p^f)$. Since the Schur multiplier of K/N is 2 and $\langle t, z\rangle \leq Z(S)$ contradicting $K \cong SL(2, p^f)$, it follows that $K = N \times F$, with $F \cong PSL(2, p^f)$. We have $F \cong PSL(2, 17)$ because all the involutions of F are conjugate and $z \in K$, $C(z) = T$. By Proposition 4 and Dieudonné [1] we have $C/N \cong PGL(2, 17)$ and an S_2-subgroup S/N of C/N is dihedral of order 32. Now

$$S_1 = \langle t, h, k \mid h^8 = 1, h^k = h^{-1}\rangle,$$

with $F \cap S_1 = \langle h, k\rangle$ and

$$S/N = \langle \bar{c}, \bar{d} \mid \bar{c}^{16} = \bar{d}^2 = \bar{1}, \bar{c}^{\bar{d}} = \bar{c}^{-1}\rangle \quad \text{with } c, d \in S,$$

where $\bar{x} = xN$, for $x \in S$. We can set $d = k$. We also have $|c| = 16$, $c^2 = h^{\sigma}t^{\iota}$, $c^k = c^{15}t^{\rho}$. Replacing if necessary h by h^{σ}, we get $c^2 = t^{\iota}h$. Obviously $F \lhd C$ and thus $F \cap S \lhd S$. Hence $\langle h, k\rangle \lhd S$ and so $k^c \in \langle h, k\rangle$, which yields $\iota = \rho$. We get

$$S = \langle t, c, k \mid c^{16} = 1 (c^2 = t^{\iota}h), c^k = t^{\iota}c^{15}\rangle.$$

We have $Z(S) = \langle t, c^8\rangle$ and $S' = \langle h\rangle$. We conclude that $Z(T) = \langle z\rangle$, where $z = c^8 = h^4$. Also $t \sim_T tc^8$.

Since all the involutions of F are conjugate in F and $t \in C(F)$ it follows that $t \sim_T th^4 \sim th^{\mu}k$ for all $\mu \in \{0, 1, 2, \cdots, 7\}$. Set $B_{\mu} = \langle t, h^4, h^{\mu}k\rangle$. We have $C(B_{\mu}) = C(t) \cap T \cap C(h^{\mu}k) = B$. By Huppert [7, II.8.16], the normalizer of $\langle h^4, h^{\mu}k\rangle$ in F is isomorphic to Σ_4, because $p = 17$. Thus $|N_K(B_{\mu})| = 2^4 3$.

In B_{μ} there are two conjugate classes under G:

$$t \sim_T th^4 \sim_L th^{\mu}k \sim_L th^{4+\mu}k \quad \text{and} \quad h^4 \sim_L h^{\mu}k \sim_L h^{4+\mu}k,$$

where $L = N_K(B_{\mu})$.

Hence $\omega = |N(B_{\mu}):N_C(B_{\mu})| \in \{1, 4\}$ and $|N(B_{\mu}):N_T(B_{\mu})| = 3$. Assume first, that $\omega = 4$. Since $|N_C(B_{\mu})| \geq |N_K(B_{\mu})| = 2^4\cdot 3$, it follows that $N_C(B_{\mu}) = N_K(B_{\mu})$ and $|N(B_{\mu})| = 2^6\cdot 3$. This yields $|N_T(B_{\mu})| = 2^6$ and therefore $|N_S(B_{\mu})| \geq 2^5$, a contradiction to $N_S(B_{\mu}) \leq N_C(B_{\mu}) = N_K(B_{\mu})$.

Thus $\omega = 1$, i.e. $N(B_{\mu}) \leq C(t)$. Since $N_T(B_{\mu})$ is an S_2-subgroup of $N(B_{\mu})$, we have $N_T(B_{\mu}) = N_S(B_{\mu}) = \langle t, h^2, h^{\mu}k\rangle$ and so $|N(B_{\mu})| = 2^4\cdot 3$ for every μ.

Suppose $\iota = 1$, i.e. $c^2 = th$ and $c^k = tc^{15}$. Now $\Omega_1(S) = S_1 = \langle t, h, k\rangle$. We have $C(t) \cap C(th^4) = S$. Since $t \sim_T th^4$, S is an S_2-subgroup of $C(th^4)$ also. Also $S_1 \lhd T$, as $S \lhd T$. Therefore $(h^{\mu}k)^v = t^{\xi}h^{\eta}k$, as $(h^4)^v = h^4$, $t^v = th^4$, for every $v \in T\backslash S$. Thus $B_{\mu}^v = B_{\eta}$ is in $C(t)$ of the same type as in $C(th^4)$. But then $N(B_{\eta}) \leq C(t) \cap C(th^4) = S$, a contradiction. It follows that $\iota = 0$ and we get

$$S = \langle t, c, k \mid c^{16} = 1, c^k = c^{-1}\rangle = \langle t\rangle \times \langle c, k\rangle \quad \text{with} \quad c^2 = h.$$

One can easily compute that $S' = \langle c^2\rangle$, $Z(S) = \langle t, c^8\rangle$, $Z_2(S) = \langle t, c^4\rangle$, $Z_3(S) = \langle t, c^2\rangle$ and $\Omega_4(S) = \langle t, c\rangle$.

Let now $v \in T \backslash S$ be an element of smallest possible order. Because of the above relations we can write

$$T = \langle t, c, k, v \mid c^{16} = 1, v^2 = t^\rho c^\sigma k^\tau, t^v = tc^8,$$
$$c^k = c^{-1}, c^v = t^\vartheta c^\iota, k^v = t^\delta c^\varepsilon k\rangle \quad \text{with} \quad 2 \nmid \iota.$$

In S we now have the following conjugate classes of involutions under S: $t, tc^8, c^8, k \sim c^\alpha k, ck \sim c^{\alpha+1}k, tk \sim tc^\alpha k, tck \sim tc^{\alpha+1}k$, for all α with $2 \mid \alpha$. But also $c^8 \sim_C k, t \sim_T tc^8 \sim_C tk$.

For $B = \langle c^8, t, k_0\rangle$ with k_0 an arbitrary involution of $S_1 \backslash \langle c^8, t\rangle$ we have, as shown, $N(B) \leq C(t)$ and $3 \,|\, |N(B)|$. Assume $S_1^w = S_1$, for a $w \in T \backslash S$. Then $B \leq S_1 \cap S_1^w$ and so $N(B) \leq C(t) \cap C(tc^8)$, as $t^w = tc^8$, which is a contradiction to $3 \,|\, |N(B)|$. Thus $S_1 \cap S_1^w = \langle t, c^2\rangle = Z_3(S)$. But $S/Z_3(S) \cong E_4$ and $S_1^w = \langle t, c^2, ck\rangle$, as $\langle t, c\rangle$ is of exponent 16. We get $k^v = t^\delta c^{2\zeta} ck \sim_S t^\delta ck \sim c^8$, $(tk)^v = tc^8 t^\delta c^{2\zeta} ck \sim t^{\delta+1} ck \sim tk \sim t$, for some integers δ, ζ. We see that there are precisely 2 classes of involutions in S, which contain t and c^8 respectively. Moreover $k^v = t^\delta c^\varepsilon k$ with $2 \nmid \varepsilon$.

For $x \in S \backslash \langle t, c^8\rangle$, $|x| = 2$, the group $\langle t, c^8, x\rangle$ is of considered type B_μ either in S_1 or in S_1^w and so

$$N(\langle t, c^8, x\rangle) \leq C(t) \quad \text{or} \quad N(\langle t, c^8, x\rangle) \leq C(tc^8).$$

Since $N_T(\langle t, h^4, h^\mu k\rangle) = \langle t, h^2, h^\mu k\rangle$, we have

$$N_T(\langle t, h^4, (h^\mu k)^w\rangle) = N_{T^w}(\langle t, h^4, h^\mu k\rangle^w) = \langle t, h^2, h^\mu k\rangle^w = \langle t, h^2, (h^\mu k)^w\rangle.$$

Hence $N_T(\langle t, c^8, x\rangle) = \langle t, c^4, x\rangle$ and so $x^w \notin \langle t, c^8, x\rangle$, for all x and all w chosen as above.

Let $y = t^\alpha c^\beta k^\gamma \in S$. Then $y^2 = c^{\beta+(-1)^\gamma \beta}$. Thus $x = t^\alpha c^\beta k$ with arbitrary α, β. We have

$$c^{v^2} = (t^\vartheta c^\iota)^v = c^{8\vartheta+\iota^2} = c^{t^\rho c^\sigma k^\tau} = c^{(-1)^\tau},$$

as $2 \nmid \iota$. Hence $8\vartheta + \iota^2 \equiv (-1)^\tau \pmod{16}$. If $\vartheta = 0$, then $\iota^2 \equiv (-1)^\tau \pmod{16}$ and it follows that $\tau = 0$, $\iota \in \{1, 7, 9, 15\}$. If $\vartheta = 1$, then $8 + \iota^2 \equiv (-1)^\tau \pmod{16}$ and we get $\tau = 0$, $\iota \in \{3, 5, 11, 13\}$.

Now

$$k^{v^2} = (t^\delta c^\varepsilon k)^v = t^{\vartheta\varepsilon} c^{8\delta+(\iota+1)\varepsilon} k = k^{t^\rho c^\sigma} = c^{-2\sigma} k$$

and hence $\vartheta\varepsilon \equiv 0 \pmod 2$, $8\delta + (\iota + 1)\varepsilon \equiv -2\sigma \pmod{16}$. As $2 \nmid \varepsilon$ we get $\vartheta = 0$, and we can write

$$T = \langle t, c, k, v \mid c^{16} = 1, v^2 = t^\rho c^\sigma, t^v = tc^8, c^k = c^{-1}, c^v = c^\iota, k^v = t^\delta c^\varepsilon k\rangle,$$

with $\rho, \delta \in \{0, 1\}$, $\iota \in \{1, 7, 9, 15\}$, $2 \nmid \varepsilon$.

Also we have the equation

$$(*) \qquad 2\sigma \equiv 8\delta - (\iota + 1)\varepsilon \pmod{16}.$$

We consider in the following the particular cases for ι.

Case (a) $\iota = 1$. By $(*)$, $\sigma + \varepsilon \equiv 4\delta \pmod 8$. Now $(kv)^8 = 1$, but $v^8 = c^{4\sigma} \neq 1$, because $2 \nmid \varepsilon$, $2 \mid \sigma + \varepsilon$. Thus we have a contradiction to the minimality of order of v.

Case (b) $\iota = 7$. From $(*)$ and $2 \nmid \varepsilon$ it follows that $\sigma \epsilon \{0, 8\}$ for $\delta = 1$, and $\sigma \epsilon \{4, 12\}$ for $\delta = 0$.

Suppose first that $\delta = 0$. Since the order of v is minimal, v must be conjugated under G with an element s ϵ S, by a criterion of Harada [5]. It is $|v| = |s| = 8$. But $x^{-1}vx = s$ gives $x^{-1}v^4x = s^4$, *i.e.* $x^{-1}c^8x = c^8$, which implies $x \epsilon T = C(c^8)$, a contradiction.

Therefore $\delta = 1$ and $\sigma \epsilon \{0, 8\}$. If $\rho = 1$ we would have $(v^2)^v = v^2 = (tc^\sigma)^v = t^v c^\sigma = tc^\sigma$, as $c^\sigma \epsilon Z(T)$. But $t^v = tc^8$ and we get a contradiction. Thus $\rho = 0$ and we have

$$T = \langle t, c, k, v \mid c^{16} = 1, v^2 = c^\sigma, t^v = tc^8, c^k = c^{-1}, c^v = c^7, k^v = tc^\varepsilon k\rangle,$$

with $\sigma \epsilon \{0, 8\}$, $2 \nmid \varepsilon$.

If $\sigma = 8$, then $|v| = 4$ and by the criterion of Harada [5] we get the same contradiction as above. Therefore $\sigma = 0$, $v^2 = 1$.

One computes easily that $|t^\alpha c^\beta kv| = 32$ for all α, β; $|t^\alpha c^\beta v| = 4$ for $\alpha + \beta \equiv 1 \pmod 2$; $|t^\alpha c^\beta v| = 2$ for $\alpha + \beta \equiv 0 \pmod 2$.

Consider the element tv. From $tv = x^{-1}sx$, $s \epsilon S$, $x \epsilon G$ it follows that $x \epsilon T$, a contradiction. Therefore by Harada [5], $(tv)^2 = c^8$ is conjugate with some involution in $T \backslash S \cdot$ of the form $w = t^\alpha c^\beta v$, $\alpha + \beta \equiv 0 \pmod 2$. Since $w^c = xc^{-6}$ and $v^k = tc^{-\varepsilon}v$, with $2 \nmid \varepsilon$, all the involutions of $T\backslash S$ are conjugate with another and therefore with c^8. In particular $v \sim c^8$. But $tc^4 \epsilon C(v)$, $(tc^4)^2 = c^8 \neq v$, while in $C(c^8) = T$ all the elements of order 4 have c^8 as its square, a contradiction.

Case (c) $\iota = 9$. By $(*)$ $\sigma + 5\varepsilon \equiv 4\delta \pmod 8$. It is now $(kv)^8 = 1 \neq v^8$, a contradiction to the minimality of the order of v.

Case (d) $\iota = 15$. The relation $(*)$ gives $\sigma \equiv 4\delta \pmod 8$, and so $\sigma \epsilon \{0, 4, 8, 12\}$. From $(v^2)^v = v^2$ it follows $\sigma \equiv 4\rho \pmod 8$. Thus $\rho = \delta$. If $v^2 \neq 1$ we get the same contradiction as in Case (b). Therefore $v^2 = 1$ and we can write

$$T = \langle t, c, k, v \mid c^{16} = 1, \quad t^v = tc^8, \quad c^k = c^{-1}, \quad c^v = c^{-1}, \quad k^v = c^\varepsilon k\rangle,$$

with $2 \nmid \varepsilon$. Now $|c^\beta v| = 2$, $|tc^\beta v| = 4$ and $|t^\alpha c^\beta kv| = 32$, for all α, β. We have $v^c = c^{-2}v$, $v^k = c^{-\varepsilon}v$ and therefore all the involutions in $T\backslash S$ are conjugate. Considering the element tv we get the contradiction in the same way as in the Case (b).

Proposition 5 is completely proved.

PROPOSITION 6. *Let $C \neq K$. Then $|C:K| = 2$ and $C/N \cong PGL(2, p)$ with $K/N \cong PSL(2, p)$, p a prime, $p \geq 5$. In particular, an S_2-subgroup of C/N is dihedral of order* 8 *in the considered case.*

Proof. By Proposition 5 we have $l(N) \geq 2$. If $C \neq K$, then $l(K/N) = 4$, as $l(G) = 8$, by Huppert [6]. Proposition 4 and the results of Janko [8] and Dieudonné [1] give now the assertion.

PROPOSITION 7. *We have* $|N| \neq 4$.

Proof. Assume $|N| = 4$. We shall prove that this assumption yields to a contradiction.

Since K/N is simple, we conclude that $C_C(N) \geq K$ and by Janko [9] we have $K/N \cong PSL(2, p^f)$, p a prime. If $p = 2$, then $p^f \in \{4, 8\}$. If $p \neq 2$ then $S/N \cong D_k$, the dihedral group of order k, $k \in \{4, 8, 16\}$, where S is an S_2-subgroup of C. Let T be an S_2-subgroup of G containing S, the chosen S_2-subgroup of C, and $z \in S$ a central involution of T. Since $C_C(N) \geq K$ we have $z \in S \backslash N$.

Assume now that N is cyclic. We consider first the case $C = K$. If $p^f = 8$, then $C/N \cong PSL(2, 8)$, S/N is elementary abelian and all the involutions of S/N are conjugate in $N_{C/N}(S/N)$. Since $\langle z\rangle \times N \leq Z(S)$ it follows $S' = \langle 1\rangle$ and thus $S = N \cap S_1$, with S_1 elementary abelian. Therefore $\langle t\rangle = \mho^1(S)$, which contradicts $|C(t)|_2 = |S|$. Hence $p^f \neq 8$.

If $S/N \cong D_k$, $k \in \{4, 8, 16\}$ we again have $\langle z\rangle \times N \leq Z(S)$, and $\langle t\rangle = \mho^1(Z(S))$ in each case, which yields to the same contradiction as above.

Thus $C \neq K$ if N is cyclic, and by Proposition 6 we have

$$|C:K| = 2, \qquad K/N \cong PSL(2, p), \qquad C/N \cong PGL(2, p).$$

Suppose that $C(N) = C$. Since S/N is dihedral of order 8, we would have $Z(S) = \langle z\rangle \times N$ and so $\langle t\rangle = \mho^1(Z(S))$, a contradiction. Therefore $C(N) = K$. Now $z \in K$ and $S_1 = S \cap K = \langle z\rangle \times N$ is abelian. Since all involutions of S_1/N are conjugate in K/N, we have $S_1 = N \times L$, for a subgroup L of S_1. By a result of Gaschütz [2], it follows that $K = N \times F$, with $F \cong PSL(2, p)$. Let $S_1 \cap F \geq \langle v_1, v_2 \mid \rangle$; hence $S_1 = \langle n, v_1, v_2 \mid n^4 = 1\rangle$, with $\langle n\rangle = N$. We have $F \lhd C$, $S_1 \cap F \lhd S$. Since $Z(S) \leq \langle z\rangle \times N$, it follows that $Z(S) = \langle z, t\rangle$ with $n^2 = t$ and one of the v_i, $i = 1, 2$, say v_1 belongs to $Z(S)$. Thus $Z(S) = \langle n^2, v_1\rangle$. We have $\langle t\rangle = \mho^1(S_1)$ and hence there is an automorphism φ of S with $S_1^\varphi \neq S_1$. Obviously $S_1S_1^\varphi = S$, $S_1 \cap S_1^\varphi \leq Z(S)$ and $S_1 \cap S_1^\varphi$ is of order 8, a contradiction to $|Z(S)| = 4$.

We have proved that N is not cyclic.

Assume next that N is a four-group, $N = \langle t, n \mid \rangle$.

We consider first the case $C = K$. Here $C(N) = C$. Suppose at first that $C/N \cong PSL(2, 8)$. Because all involutions of S/N are conjugate in C/N and $\langle z\rangle \times N \leq Z(S)$, we conclude by a result of Gaschütz [2] that $C = N \times F$ with $F \cong PSL(2, 8)$. We know that there is an element r in F, $|r| = 7$, such that $S \lhd SR$, where $R = \langle r\rangle$. Denote $T_1 = N_T(S)$. Then $N(S) = T_1 R$, where $|T_1| = 2^6$, and $N(S)/S \cong D_{14}$. Let $s \in T \backslash T_1$. Then $[T_1 = S \cdot S^s$, $Z(T_1) \geq S \cap S^s \lhd T$ and $|S \cap S^s| = 2^4$. By Suzuki [12 there is a complement L of S in $N(S)$ with

$$L = \langle r, w \mid r^7 = 1, r^w = r^{-1}\rangle,$$

as there are involutions in $S^s\backslash S$ and $S = O_2(N(S))$. By Gorenstein [4, 5.2.3] we have $S = C_S(R) \times [S, R]$. Obviously $S \cap Z(T_2) = C_S(w)$, with $T_2 = S\langle w\rangle \sim_{N(S)} T_1$. Denote $C_S(R) = U$, $[S, R] = V$. Then $|U| = 4$, $|V| = 8$ and $|C_S(w)| = 16$. We have $U^w = U$, $V^w = V$ and hence $C_S(w) = C_U(w) \times C_V(w)$. It follows $C_U(w) = U$ or $C_V(w) = V$. But $U = N$ and $U = C_U(w) \leq Z(T_2)$ would imply $T_2 \leq C(N) = C$, a contradiction.

Hence $V = C_V(w)$. Let $v_1 \in V^{\#}$. Now $v_1^{rw} = v_1^{rw} = v_1^r = v_1^{r^{-1}}$, in contradiction with the faithful action of r on $[R, S]$.

Suppose next, that $S/N \cong D_4$, $C/N \cong PSL(2, p^f)$. Since $PSL(2, 4) \cong PSL(2, 5)$, we can assume without loss that $p \neq 2$. Similarly as in the previous cases we conclude that $C = N \times F$, with $F \cong PSL(2, p^f)$. Since $N \leq Z(C)$, $C(z) = T$ and $z \in C\backslash N$, one easily sees that

$$F \cong PSL(2, 5) \cong A_5.$$

In F there is an r, $|r| = 3$, such that $R = \langle r\rangle$ normalizes the S_2-subgroup S of C. Therefore $|N(S)| = 2^5\cdot 3\cdot k$, k a prime or $k = 1$, where $3k \leq 12$, as the elements of N are not central involutions. Hence $k \in \{1, 2, 3\}$.

Assume first that $k = 3$. Let T_1 be an S_2-subgroup of $N(S)$ and $T_1 < T_2 < T$. If $s \in T_2\backslash T_1$, then $T_1 = SS^s$, $S \cap S^s \leq Z(T_1)$ and $|S \cap S^s| = 2^3$. Since $S = C_{T_1}(S)$, the group T_1 is not abelian and hence $S \cap S^s = Z(T_1)$. It follows that $N \cap Z(T_1) \neq 1$ and we may assume that $n \in Z(T_1)$. Thus $n \not\sim_G t$. But now $z \in Z(T) \cap S$ would have 9 conjugates in S under $N(S)$ and t would have the remaining 6 involutions as conjugates, because $|C_{N(S)}(t)| = 2^4\cdot 3$. This, however, contradicts $t \not\sim n \not\sim z$. It follows that $k \neq 3$.

Assume now $k = 2$, i.e. $N(S) = T_2 R$, with $|T_2| = 2^6$. We have $T_2 \ntriangleleft N(S)$ as $l(G) = 8$. Let t, t_1, t_2, t_3 be the conjugates of t in S under $N(S)$. Since R has no fixpoints on $S\backslash N$ it follows that $t_i \in S\backslash N$, for $i = 1, 2, 3$. Consider $C(t_1) \cap C(t)$. Because of $C = N \times F$, $F \cong A_5$, t_1 has no element of odd order in its centralizer in C. Therefore $C(t_1) \cap C(t) = S$. If $O_2(N(S)) = S$, it would be $O_{2,2'}(N(S)) = SR \leq C(t) \cap C(t_1)$, a contradiction. It follows that $O_2(N(S)) = T_1$ is of order 2^5, and $N(S)/T_1 \cong D_6$, $N_T(S) = N_T(T_1) = T_2$.

Let $v \in T\backslash T_2$. Then $S\cdot S^v \leq T_2$, $\bar{S} = SS^v \triangleleft T$.

Suppose that $|\bar{S}| = 2^5$. Then $\bar{S}_0 = S \cap S^v = Z(\bar{S})$ is of order 8. We have $\bar{S}_0^r \neq \bar{S}_0$, as $T \neq T^r$. Hence $|\bar{S}_0^r \cup \bar{S}_0| = 12$ and t, t_1, t_2, t_3 are the only elements of S, which are conjugate with t under G. Similarly, there is an element of $\bar{S}_0^{r^2}$, which is not contained in $\bar{S}_0^r \cup \bar{S}_0$. Since this is also not conjugate with t, there are at least 13 elements of S, which are not conjugate with t, a contradiction to $t \sim t_1 \sim t_2 \sim t_3$.

Therefore $\bar{S} = T_2$ and $|S \cap S^v| = 4$. Now $T_2 = S\langle a_1, a_2 \mid \rangle$, $S \cap \langle a_1, a_2\rangle = 1$,

$\langle a_1, a_2\rangle \leq S^v$, and by Gaschütz [2], S has a complement B in $N(S)$. Since $B \cong N(S)/S$ and $S \lhd T_1 \lhd T_1 R \lhd T_2 R = N(S)$ is the corresponding $\{2, 2'\}$-series, we have $T_2/S \cong D_4$ and

$$B = \langle b \mid \rangle \times \langle r, a \mid r^3 = 1, \quad r^a = r^{-1}\rangle$$

for some involutions $a, b \in N(S)\backslash S$, assuming $B \geq R$.

By Gorenstein [4, 5.2.3], we get $S = C_S(R) \times [S, R]$, with $C_S(R) = N$. Since $S^a = S^b = S$ and $R^a = R^b = R$, we now have $N = N^a = N^b = N^r$, and so $N \lhd N(S)$, a contradiction to $t \in N$, $t \sim_{N(S)} t_1 \in S\backslash N$.

It remains to consider the case $k = 1$. Now t has 2 conjugates under $N(S)$. Since R acts fixpointfree on $S\backslash N$, we may suppose that $t \sim_{N(S)} n$.

Let T_1 be an S_2-subgroup of $N(S)$ and $T_1 < T_2 < T$. For $s \in T_2\backslash T_1$ we have $SS^s = T_1$ and $Z(T_1) = S \cap S^s$ is of order 8, as $S = C_T(S)$. Since S has a complement in T_1, it is $N(S) = SB$, with $B \cong D_6$ or $B \cong Z_6$. We may assume that

$$B = \langle r, v \mid r^3 = 1, r^v = r^\alpha\rangle, \qquad \alpha \in \{1, -1\}.$$

We have $T_1 \sim S\langle v\rangle$ and thus $\mid Z(S\langle v\rangle) \mid = 8$. We have again $S = U \times V$, with $U = C_S(R)$, $V = [S, R]$. Since $U^r = U$, $V^r = V$ it follows that $C_S(v) = C_U(v) \times C_V(v) = Z(S\langle v\rangle)$. Because of $\mid U \mid = \mid V \mid = 4$, we must have $C_U(v) = U$, or $C_V(v) = V$. But $t \in C_S(R)$ and $t^v \neq t$. Thus $V = C_V(v)$. If $\alpha = -1$ we get a contradiction to the faithful action of $\langle r\rangle$ on V. Therefore $T_1 \lhd N(S)$ and we can write

$$N(S) = \langle t, n, h_1, h_2, r, v \mid r^3 = 1, t^v = n, n^v = t, h_1^r = h_2, h_2^r = h_1 h_2\rangle,$$

where $\langle t, n, h_1, h_2\rangle = S$.

Now $Z(N(S)) = \langle tn\rangle$ and $N(S) \lhd N(T_1) = T_2 R$. Thus

$$C(tn) \geq \langle N(T_1), C(t)\rangle > N(T_1) \quad \text{and} \quad l(N(T_1)) = 7,$$

a contradiction.

Suppose now that $S/N \cong D_8$ or D_{16}. Then $K/N \cong PSL(2, p^f)$ with $l(C/N) = 5$ and thus $l(C/N) = 7$. Since $Z(S/N) \cong Z_2$ it follows that $Z(S) = N \times Z(T)$ is elementary abelian of order 8. Let $S < T_1 < T$ with $\mid T{:}T_1 \mid = 2$. Then $\mid Z(T_1) \cap Z(S) \mid \geq 4$ and hence $Z(T_1) \cap N \neq 1$, a contradiction to $C(n) = C(t) = C(tn)$, because of maximality of $C(t)$. Thus $S/N \ncong D_8, D_{16}$ also.

We have proved that $C \neq K$. Hence by Proposition 6,

$$\mid C{:}K \mid = 2, \qquad K/N \cong PSL(2, p), \qquad C/N \cong PGL(2, p),$$

where p is a prime, $p \geq 5$, and an S_2-subgroup S/N of C/N is dihedral of order 8.

Suppose first that $C(N) = C(t) = C$. Now $\mid S \mid = 32$ and we have $Z(S) = \langle z\rangle \times N$. If $S < T_1 < T$, then $\mid Z(T_1) \cap Z(S) \mid \geq 4$, and so $Z(T_1) \cap N \neq 1$, a contradiction to $C(N) = C$, because of maximality of C.

Thus we have $C(N) = K$. Let $S_1 = S \cap K$. Now $Z(T) \leq S_1$, because

$K = C(N)$. Since $\langle z \rangle \times N \leq Z(S_1)$ and $|S_1| = 16$, S_1 is abelian. Moreover S_1 is elementary abelian, as all involutions of S_1/N are conjugate under K/N. Thus $K = N \times F$, with $F \cong PSL(2, p)$. It is $z \in K$, $C(z) = T$ and all involutions of F are conjugate in F. Therefore we conclude, that $F \cong PSL(2, 5) \cong A_5$.

Let $c \in S \backslash S_1$. Then $C = (N \times F)\langle c \rangle$. Since $S/N \cong D_8$ the element c does not act trivially on $S \cap F = V \lhd S$. Let $V = \langle v_1, v_2 \mid \rangle$. Then we can write

$$S = \langle t, n, v_1, v_2, c \mid c^2 = t^\iota n^\nu v_1^\alpha v_2^\beta, n^c = tn, v_1^c = v_2, v_2^c = v_1 \rangle,$$

$\iota, \nu, \alpha, \beta \in \{0, 1\}$.

It is $Z(S) = \langle t, z \rangle = \langle t, v_1 v_2 \rangle$, and $Z(T) = \langle z \rangle$, because $Z(S/N)$ is of order 2. For $S < T_1 < T$ we get therefore $t \sim_{T_1} zt = t^\varepsilon v_1 v_2$, where either $\varepsilon = 0$ or $\varepsilon = 1$. Since $(c^2)^c = c^2 \in Z(S)$, we have $c^2 = t^\iota v_1^\alpha v_2^\alpha$, with $\alpha, \iota \in \{0, 1\}$. But now $(n^\alpha v_1^\beta c)^2 = 1$, and replacing c by $n^\alpha v_1^\beta c$, we can write

$$S = \langle t, n, v_1, v_2, c \mid n^c = tn, v_1^c = v_2, v_2^c = v_1 \rangle.$$

Consider now $N(S_1)$. Obviously $|N(S_1)| = 2^5 \cdot 3k$, $k = 1$ or k a prime, as V has a normalizer $VR \cong A_4$ in F, $R \cong Z_3$. Since $z \in S_1$ and z is not conjugated with any element of N, we have $3k \leq 12$, and hence $k \in \{1, 2, 3\}$.

Let $k \neq 2$. Then $S = N_T(S_1)$. For $S_1 < S < T_1 < T$ and $s \in T_1 \backslash S$ it follows that $S_1 S_1^s = S$ and $S_1 \cap S_1^s \leq Z(S)$ is of order 8, a contradiction to $|Z(S)| = 4$.

Therefore $k = 2$, $N(S_1) = T_2 R$, with $|T_2| = 2^6$. If $T_2 \lhd N(S_1)$, T_2 would have a too large normalizer. Thus $T_2 \ntriangleleft N(S_1)$.

We have $t \sim_{T_2} t^\alpha v_1 v_2 \sim_R t^\alpha v_1$, with $\alpha = 0$ or $\alpha = 1$. Consider

$$C_0 = C(t) \cap C(t^\alpha v_1).$$

We have $C_0 \cap K = C_K(v_1) = S_1$, as $C_F(v_1) = V$. Since $C_0/K \cong C_0/(C_0 \cap K)$ and $|C_0 K/K| \mid 2$, C_0 is a 2-group.

If $O_2(N(S_1)) = S_1$, then

$$S_1 R \trianglelefteq N(S_1) \quad \text{and} \quad S_1 R \leq C(t).$$

But $t \sim_{N(S_1)} t^\alpha v$, and therefore it must be also $S_1 R \leq C(t^\alpha v)$, a contradiction to $3 \nmid |C_0|$.

Hence $O_2(N(S_1)) = T_3$ with $|T_3| = 2^5$. Obviously we can suppose that $T_2 > S$. But then $Z(T_2) \leq Z(S)$ and $Z(S)$ is of order 4. Let $v \in T \backslash T_2$. Then $S_1 S_1^v \leq T_2$ and $S_1 S_1^v \lhd T$.

Suppose that $|S_1 S_1^v| = 2^5$. Then $S_1 \cap S_1^v \leq Z(S_1 S_1^v)$ and thus

$$|\Omega_1(Z(S_1 S_1^v))| \geq 2^3.$$

It follows that $|Z(T_2)| \geq 4$ and therefore $Z(T_2) = Z(S) = \langle t, v_1 v_2 \rangle$, a contradiction to $2^6 \nmid |C(t)|$.

Therefore $|S_1 S_1^v| = 2^6$, i.e. $S_1 S_1^v = T_2$. But now $S_1 \cap S_1^v \leq Z(T_2)$ and hence $|Z(T_1)| = 4$, which yields the same contradiction as in the preceding case.

We have shown that the assumption $|N| = 4$ yields a contradiction in all cases. The Proposition is completely proved.

PROPOSITION 8. *The group N is not a 2-group.*

Proof. Since $l(C/N) \geq 4$ and therefore $l(N) \leq 3$, it remains by Propositions 5 and 7 to show, that $|N| \neq 2^3$.

Assume the contrary, i.e. $|N| = 8$. By Huppert [6] and Janko [8], the group C/N is simple and $C/N \cong PSL(2, p)$, p a prime, $p \geq 5$, with $l(C/N) = 4$.

We have shown in the proof of the Proposition 1, that $N \leq Z(C)$ if N is elementary abelian. One can easily see that the same holds also in the other cases, where N is abelian, because of the simplicity of C/N. In all cases an S_2-subgroup of C/N is dihedral of order 4, by Janko [8], and C/N contains alternating groups A_4.

Suppose at first that $N \cong E_8$, the elementary abelian group of order 8. In C there is a subgroup M, with $|M| = 2^5 \cdot 3$, $M/N \cong A_4$, and $|O_2(M)| = 2^5$. Denote $O_2(M) = A$. If $N = Z(A)$, then $N \lhd A_1$ for $A < A_1$ with $|A_1 : A| = 2$, which contradicts $C = N_G(N)$ and $|C|_2 = |A|$. Therefore $Z(A) > N$ and A is abelian. We have $M = AR$, with $|R| = 3$. Since $M/N \cong A_4$ the group R acts faithfully on A and therefore on $\Omega_1(A)$. Since $N \leq \Omega_1(A) \cap C(R)$, it follows $\Omega_1(A) = A$ and A is elementary abelian. By Gaschütz [2] N has now a complement F in C, i.e. $C = N \times F$, with $F \cong PSL(2, p)$. Consider now $N(A)$. We see that $N(A)/A \cong D_6$. Let A_1 be an S_2-subgroup of $N(A)$, $A < A_1 < T$, where T is an S_2-subgroup of G. For $v \in T \backslash A_1$ we have $AA^v = A_1$ and $Z(A_1) \geq A \cap A^v$, because $A \ntriangleleft T$. Thus $|Z(A_1)| = 2^4$. With $a \in A^v \backslash A$ we have $A_1 = A\langle a \rangle$, and hence A has a complement L in $N(A) = A_1 R$. We can write $N(A) = AL$, with

$$L = \langle r, m \mid r^3 = 1, r^m = r^{-1} \rangle.$$

Now, by Gorenstein [4, 5.2.3], we have $A = U \times V$, where $U = C_A(R)$, $V = [A, R]$. Obviously $U^r = U^m = U$, $V^r = V^m = V$. Therefore $C_A(m) = C_U(m) \times C_V(m)$. Since $A\langle m \rangle \sim_{N(A_1)} A_1$, we have $|Z(A\langle m \rangle)| \geq 2^4$. If $A\langle m \rangle = Z(A\langle m \rangle)$, then $C_A(m) = A$, otherwise $C_A(m) = Z(A\langle m \rangle)$. We have $U = N$ and $|V| = 4$ because of the faithful action of R on A. It follows that $U = C_U(m)$ or $V = C_V(m)$. But $U = C_U(m) = N$ implies $Z(T_1) \cap N \neq 1$ for the S_2-subgroup T_1 of G containing $A\langle m \rangle$. But this is impossible, because $C(N) = C$. Hence $V = C_V(m)$, which contradicts again the faithful action of R on V.

Therefore $N \ncong E_8$.

Suppose now that $N \cong Z_4 \times Z_2$ or $N \cong Z_8$. We know that $N \leq Z(C)$. Let S be an S_2-subgroup of C. Since $N = Z(C)$, we have $N \cap Z(T) = 1$ for each S_2-subgroup T of G containing S. It follows that there is a central involution $z \in Z(T)$ in $S \backslash N$. But $N_C(S) = SR$, with $R = \langle r \rangle$ of order 3 and $SR/N \cong A_4$. It follows that $S = N \times \langle z \rangle \times \langle z^r \rangle$. Now $\mho^1(S) \leq N$ and $N(\Omega_1(S)) = C$, which contradicts $2 \nmid |C:S|$.

Assume at last that $N \cong D_8$ or $N \cong Q_8$, the quaternion group of order 8. Since C/N is simple and $A(N)$ is solvable it must be again $C = C(N)$. Now, the contradiction follows as in the preceding case.

The proposition 8 is proved.

PROPOSITION 9. *We have* $l(N) > 2$.

Proof. In view of Propositions 5 and 7 it remains to show that $|N| \neq 2 \cdot q$, where q is an odd prime.

Assume the contrary, i.e. $|N| = 2 \cdot q$, q an odd prime. Then

$$N = \langle t \rangle \times \langle m \rangle,$$

with $|m| = q$. Since $A(N)$ is solvable it must be $C(N) = K$ or $C(N) = C$. But there are central involutions in C and thus $C(N) = K \neq C$, as S_2-subgroups of G are maximal. By Proposition 6 we now have

$$C/N \cong PGL(2, p), \qquad K/N \cong PSL(2, p)$$

and an S_2-subgroup S/N of C/N is dihedral of order 8. Let $S_1 = S \cap K$ and $z \in Z(T)$, $|z| = 2$, where T is an S_2-subgroup of G containing S. But then $S/N = S_1/N \times \langle z \rangle N/N$, a contradiction to $S/N \cong D_8$.

PROPOSITION 10. *If t is an involution of G, then $C = C(t)$ is solvable.* (This contradicts Proposition 1, proving our lemma.)

Proof. By Propositions 8 and 9 it remains to consider the case, where $l(N) = 3$, $|N| = 2q_1 q_2$, with q_1, q_2 primes, which are not both even. By Janko [8] we have $C/N \cong PSL(2, p)$, p a prime, $p \geq 5$, as $l(C/N) = 4$.

Suppose at first that q_1 and q_2 are both odd.

If $q_1 \neq q_2$, let be $q_1 > q_2$ and Q_1 the S_{q_1}-subgroup of N. Then $Q_1 \lhd C$, $C/C_C(Q_1)$ is cyclic and it must be that $C \neq C_C(Q_1)$, as C contains some central involution of G, which cannot be in $C_C(Q_1)$. Now $C > C_C(Q_1) > Q_1\langle t \rangle > Q_1 > 1$ is a normal series and $C > N > Q_1\langle t \rangle > Q > 1$ is already a chief series. It follows that $C/C_C(Q_1) \cong N/Q_1\langle t \rangle$ is of odd order and so $C_C(Q_1)$ contains nevertheless an S_2-subgroup of C, a contradiction.

Thus we can assume that $q_1 = q_2 = q$ is an odd prime. Now $N = \langle t \rangle \times Q$, where $|Q| = q^2$. Since $C_C(Q)$ contains no central involutions of G and C does contain such involutions, it must be that $C_C(Q) = N$, $N_C(Q) = C$. Therefore C/N is isomorphic to a subgroup of $A(Q)$ and Q must be elementary abelian. Let S be an S_2-subgroup of C and T an S_2-subgroup of G containing T. Since $SN/N \cong S/S \cap N \cong D_4$ and $Z(T)^{\#} \leq S \backslash S \cap N$, S is abelian. If S is not elementary abelian, then $\langle t \rangle = \mho^1(S)$ char S, which is a contradiction. Hence $C = \langle t \rangle \times F$, with $Q \leq F$, $F/Q \cong PSL(2, p)$ and $Q = C_L(Q)$. Let V be an S_2-subgroup of L. Then $V = \langle v_1, v_2 \mid \rangle$ and $v_1 Q \sim_L v_2 Q \sim_L v_3 Q$, where $v_3 = v_1 v_2$. It follows $v_1 \sim_L v_2 \sim_L v_3$. By a result of Brauer and Wielandt [16], we get

$$| Q |\cdot| C_Q(V) |^2 = | C_Q(v_1) |\cdot| C_Q(v_2) |\cdot| C_Q(v_3) | = | C_Q(v_1) |^3.$$

It follows that $| C_Q(v_1) | = q^2$, a contradiction to $Q = C_L(Q)$.

Suppose now that $| N | = 2^2 q$, q an odd prime. Now $N = QM$, $| Q | = q$, $| M | = 4$. Obviously $Q \lhd C$ and also $C = N \cdot N_C(M)$, by the Frattini argument. Assume that $M \leq C(Q)$. Since $C/C_C(Q)$ is cyclic and $C_C(Q) \geq N$, it follows $C_C(Q) = C$, a contradiction, because C contains central involutions of C. Now, $C/N = N \cdot N_C(M)/N \cong N_C(M)/M \cong PSL(2, p)$ and so $| C:N_C(M) | = q$. Since $| M | = 4$, it follows that $C_C(M) = N_C(M)$ and hence $C_C(M)/M \cong PSL(2, p)$. In particular M contains no central involution of G.

Let S be an S_2-subgroup of C contained in $C_C(M)$. Since

$$C_C(M)/M \cong PSL(2, p), \qquad p \geq 5,$$

there is a subgroup U/M in $C_C(M)$ isomorphic to A_4, with $S/M \lhd U/M$. Now $U = SR$, $R = \langle r \rangle$, $| r | = 3$. Let T be an S_2-subgroup of G containing S. Then $Z(T) \leq Z(S)$ and $Z(T) \cap M = 1$. Thus there is an involution $z \in Z(T)$ in $S \backslash M$ and so $S = M \times \langle z \rangle \times \langle z^r \rangle$. Consequently $C_C(M) = M \times F$, with $F \cong PSL(2, p)$. It must be now $M \cong D_4$, because otherwise $\langle t \rangle = \mho^1(S)$ char S, a contradiction. Hence from it follows $C = \langle t \rangle \times L$.

Obviously $F \leq L$, as $L \cap F \lhd F$. Also $Q \leq L$. Let $M \cap L = \langle m \rangle$. Then

$$Q\langle m \rangle = N \cap L \lhd L \quad \text{and} \quad L/N \cap L \cong PSL(2, p).$$

Consider $C_L(Q)$. Since $m \notin C_L(Q)$, $L/C_L(Q) \cong W \leq A(Q)$ and $C_L(Q) \geq Q$, it follows that $| L:C_L(Q) | = 2$, $L = C_L(Q)\langle m \rangle$, with $C_L(Q)/Q \cong PSL(2, p)$. One can easily see that $F \leq C_L(Q)$, $Q \cap F = 1$ and thus

$$C_L(Q) = Q \times F, \qquad L = (Q \times F)\langle m \rangle = F \times Q\langle m \rangle.$$

From $C = \langle t \rangle \times L$, it follows now that

$$C = \langle t \rangle \times Q\ m \rangle \times F,$$

with $F \cong PSL(2, p)$ and $Q\langle m \rangle \cong D_{2q}$.

The group S contains a central involution z of G and z is of the form $t^\alpha m^\beta h$, $h \in F \cap S$. Since $C(z) = T$ is an S_2-subgroup of G, we have $\beta = 1$ and $h \neq 1$. In particular $| h | = 2$. It follows now, that $p = 5$, $F \cong PSL(2, 5)$.

Let $V = S \cap F = \langle v_1, v_2 \mid \rangle$. Then

$$S = \langle t, m, v_1, v_2 \mid \rangle, \quad N_F(V) = VR = \langle v_1, v_2, r \mid r^3 = 1, v_1^r = v_2, v_2^r = v_1 v_2 \rangle$$

with $R = \langle r \rangle$.

Therefore $| N(S) | = 2^5 \cdot 3k$, with $k = 1$ or k a prime. Since the central involutions of G in S have the form mentioned above, one easily sees, that z has at most 6 conjugates in S. Thus $k = 1$ or $k = 2$.

Assume first that $k = 2$, i.e. $N(S) = T_1 R$, where $| T_1 | = 2^6$. Obviously $T_1 \not\triangleleft N(S)$. Since t has 4 conjugates under $N(S)$, t, t_1, t_2, t_3 say, and R acts fixed-point-free on $S\backslash M$, it follows that $t_1, t_2, t_3 \in S\backslash M$. Moreover we can suppose that $t_1^r = t_2$, $t_2^r = t_3$.

If $O_2(N(S)) = S$, then $O_{2,2'}(N(S)) = SR$ char $N(S)$. Consider $C(t) \cap C(t_1)$. Since $t \sim_{N(S)} t_1$, it follows $SR \leq C(t) \cap C(t_1)$, a contradiction to $C_S(R) = M$.

Thus $O_2(N(S)) = T_2$, with $| T_2 | = 2^5$, $N(S)/T_2 \cong D_6$. We can assume that $T > T_1 > T_2 > S$. Let $v \in T\backslash T_1$. It is $S_1 = SS^v \leq T_1$ and $S_1 \triangleleft T$. Also $S_2 = S \cap S^v \triangleleft T$, because $T_1 = N_T(S)$.

Suppose at first that $| S_1 | = 2^5$. Then $S_2 = Z(S_1)$ is of order 8, as $S = C_T(S)$. If $S_2^{r^i} = S_2^{r^j}$, for $r^i \neq r^j$, $S_2^{r^i}$ would have a too great normalizer. Thus S_2, S_2^r, $S_2^{r^2}$ are all different and their union contains at least 13 elements, as their pairwise intersections contain 4 elements. But t is not conjugate with any element of arbitrary group $S_2^{r^i}$, because $S_2^{r^i} \leq Z(S_1^{r^i})$, but $2^5 \nmid | C(t) |$. Therefore t would have at most 3 conjugates, a contradiction.

Thus we can suppose that S_1 has order 2^6, $S_1 = T_1$, and S_2 is of order 4. Now S has a complement in T_1 and by Gaschütz [2] also in $N(S)$, i.e. $N(S) = SB$, $S \cap B = 1$, where B is a subgroup of $N(S)$.

We have $S \triangleleft T_2 \triangleleft T_2 R \triangleleft T_1 R = N(S)$ as a $\{2, 2'\}$-series of $N(S)$ and $T_1/S \cong D_4$. We may suppose that $r \in B$, and we can write

$$B = \langle b \mid \rangle \times \langle r, a \mid r^3 = 1, r^a = r^{-1} \rangle.$$

Let us denote $U = C_S(R) = \langle t, m \rangle$. Now $U = U^r = U^a = U^b = \langle t, m \rangle$ and therefore $\langle t, m \rangle \triangleleft N(S)$, a contradiction to $t \sim_{N(S)} t_1 \in S\backslash\langle t, m \rangle$.

Therefore we must have $k = 1$ and so $N(S) = T_1 R$, with $| T_1 | = 2^5$. Since $\langle t, m \rangle = C_S(R)$ it must be $t \sim_{N(S)} t^\tau m = m_1$, where either $\tau = 0$ or $\tau = 1$, because t has precisely 2 conjugates in $N(S)$.

Let $T_1 < T_2 < T$, where T is an S_2-subgroup of G. If $s \in T_2 \backslash T_1$, we have $SS^s = T_1$, $Z(T_1) = S \cap S^s$ is of order 8, and S has a complement in T_1. Therefore also $N(S) = SB$, with $B \cong D_6$ or $B \cong Z_6$.

Let $B = \langle r, a \mid r^3 = 1, r^a = r^\varepsilon \rangle$, $\varepsilon \in \{-1, 1\}$. Then $\langle S, a \rangle \sim T_1$, hence $| Z(\langle S, a \rangle) | = 8$. By Gorenstein [4, 5.2.3], we get $S = U \times V$, where $U = C_S(R)$, $V = [S, R]$, and $U^a = U^r = U$, $V^a = V^r = V$. Moreover,

$$Z(S\langle a \rangle) = C_S(a) = C_U(a) \times C_V(a).$$

Since U and V have order 4 and $C_S(a)$ has order 8, it must be $U \leq C(a)$ or $V \leq C(a)$. But $t \in U$, $[t, a] \neq 1$. Thus $V \leq C(a)$. Let $V = \langle v_1, v_2 \mid \rangle$ and $v_1^r = v_2$, $v_2^r = v_1 v_2$. If $\varepsilon = -1$, we get $v_1^{ra} = v_1^{r^{-1}} = v_1^{ara} = v_1^{ra} = v_1^r$, a contradiction to the faithful action of r on V. Therefore $\varepsilon = 1$. We have now

$$N(S) = \langle t, m_1, v_1, v_2, a, r \mid r^3 = 1, t^a = m_1, v_1^r = v_2, v_2^r = v_1 v_2 \rangle.$$

For $T_1 = \langle t, m_1, v_1, v_2, a \rangle$ we have now $Z(T_1) = \langle v_1, v_2, tm_1 \rangle$. Consider

$C(tm_1)$. It is $T_1 \leq C(tm_1)$, $|T_1| = 2^5$ and $F \leq C(tm_1)$, $F \cong PSL(2, 5)$. Thus $C(tm_1)$ is not solvable. Let N_1 be the maximal solvable normal subgroup of $C(tm_1)$. By Proposition 9, $l(N_1) = 3$ and therefore $C(tm_1)/N_1 \cong PSL(2, p)$ by Janko [8] and an S_2-subgroup of $C(tm_1)/N_1$ is of order 4. But 2^5 is a divisor of the order of $C(tm_1)$ and therefore $|N_1| = 2^3$, which contradicts the Proposition 8.

Thus $l(N) = 3$ also yields to a contradiction and Proposition 10 is proved. We conclude that all the centralizers of the involutions in G are solvable. This however contradicts Proposition 1 and so completes the proof of our lemma.

4. Proof of the theorem

By our lemma all the 2-local subgroups of G are solvable. In the following T will always denote an S_2-subgroup of G and z an involution in $Z(T)$. We shall prove the theorem in several steps.

PROPOSITION 11. *Two different elements of $Z(T)$ are never conjugate in G. Each element of $Z(T)$ is conjugate under G with an element of $T\backslash Z(T)$. If $Z(T) \cap T^g \neq 1$, with $g \in G$, it follows that $\langle Z(T), Z(T^g)\rangle \leq T \cap T^g$. It is $|Z(T)| = 2$ or $|Z(T)| = 4$.*

Proof. By a theorem of Burnside and by Glauberman [3], the first and the second assertion follow. Suppose $v \in Z(T) \cap T^g$. Then $Z(T^g) \leq C(v) = T$ and thus $Z(T^g) \cap T \neq 1$. Hence also

$$Z(T) \leq T^g \quad \text{and} \quad \langle Z(T), Z(T^g)\rangle \leq T \cap T^g.$$

By the second assertion, for every $s \in Z(T)^{\#}$ there is some $h \in G$, such that $s^h \in T\backslash Z(T)$. Thus $s^h \in Z(T^h) \cap T$, with $T^h \neq T$ and so

$$\langle Z(T), Z(T^h)\rangle \leq T \cap T^h.$$

Because of maximality of T we have $Z(T) \cap Z(T^h) = 1$ and also $|T \cap T^h| \mid 2^5$. Since

$$|\langle Z(T), Z(T^h)\rangle| = |Z(T) \times Z(T^h)| = |Z(T)|^2 \mid |T \cap T^h|$$

it follows that $|Z(T)| = 2$ or $|Z(T)| = 4$. The proposition is proved.

In Propositions 12–20 we shall suppose that $SCN_3(2) \neq 0$, the case $SCN_3(2) = 0$ remaining to be considered in the following.

PROPOSITION 12. *Let U be an element of $SCN_3(T)$. Then the set $\mathcal{V}_G(U; 2')$ is trivial.*

Proof. By Gorenstein [4, 8.5.6] and by our lemma, the assertion follows, because of maximality of T.

PROPOSITION 13. *If A belongs to $U(2)$, then the set $\mathcal{V}_G(A:2')$ is trivial.*

Proof. Let $U^*(2)$ be the set of B such that
(i) $B \leq G$ and B is of type $(2, 2)$,
(ii) $N(B) = T^g$, for some $g \in G$.

Since we have supposed that $SCN_3(2) \neq 0$, we have $U^*(2) \neq 0$.

Let $N(B) = T, B \in U^*(2)$. Since $| A(B) | = 6, | T:C(B) | \mid 2$. Suppose that $C(B)$ centralizes some non-trivial $2'$-subgroup Q of G. Then $C(B) \neq T$ because of maximality of T and so $| C(B) | = 2^6$. Since $l(G) = 8$, it must be that $| Q | = q$ a prime. If $U \in SCN_3(T)$, then by Proposition 12, $\mathscr{V}_G(U; 2')$ is trivial. Thus $U \nleq C(B)$ and therefore $C(B)U = T$. But now $| \Omega_1(U) \cap C(B) | \geq 2^2$. Also $B \nleq U$, since otherwise $U \leq C(B)$. It follows that $C = B(\Omega_1(U) \cap C(B))$ is elementary abelian of order at least 2^3. Thus there exists a subgroup Y with $C \leq Y \in SCN_3(T)$. As above we have $C(B)Y = T$. But now

$$B \leq Z(C(B)) \cap Z(Y) \leq Z(C(B)Y) = Z(T)$$

and thus $C(B) = T$, a contradiction. Thus for a $B \in U^*(2)$, $C(B)$ centralizes no nontrivial $2'$-subgroup of G.

Therefore Hypothesis 7.1 of Thompson [14] holds, and by Lemma 7.1 of Thompson [14], any subgroup A of G belonging to $U(2)$ centralizes all the elements of $\mathscr{V}_G(A; 2')$.

If $1 \neq H \in \mathscr{V}_G(A; 2')$ then $HA = H \times A$ and thus $H \leq N(A) = T$, a contradiction to $2 \nmid | H |$. Hence $\mathscr{V}_G(A; 2')$ is trivial.

Proposition 14. *Let $A \in U(2)$ and $A < H < G$, H a solvable group. Then $O_{2'}(H) = 1$, $H/O_2(H)$ is faithfully represented on $O_2(H)/D(O_2(H))$ and $C_H(O_2(H)) \leq O_2(H)$. Moreover $\Pi(H) \in \{2, 3, 5, 7, 31\}$.*

Proof. By Proposition 13, $\mathscr{V}_G(A; 2')$ is trivial and therefore $O_{2'}(H) = 1$. By Gorenstein [4, 6.3.4], the first assertion holds. Since

$$| O_2(H)/D(O_2(H)) | \mid 2^6,$$

we have $| H/O_2(H) | \mid | GL(6, 2) | = 2^{15} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31$. Hence the second assertion follows.

Proposition 15. *Let $A \in U(2)$ and let B be a p-subgroup of G satisfying one of the following conditions:*

(a) $p = 3$ *and* $B \cong Z_9$ *or* $3^3 \mid | B |$;

(b) $p \in \{5, 7, 31\}$ *and* $p^2 \mid | B |$;

(c) $p \notin \{2, 3, 5, 7, 31\}$.

Then the group $\langle A, B \rangle$ is not solvable.

Proof. Suppose the contrary holds. If $\langle A, B \rangle = H$ is a solvable group, we can apply the Proposition 14. Denote $E = O_2(H)/D(O_2(H))$. Since $| E | \mid 2^6$, $l(H) \leq 7$, $H/O_2(H) \cong L \leq A(E)$ and as $GL(4, 2)$ and $GL(5, 2)$ have elementary abelian S_3-subgroups of order 3^2, the assertion follows.

Proposition 16. *$Z(T)$ contains no elementary abelian group of order* 4.

Proof. Let $H = N(U)$, U a nontrivial 2-subgroup of G and $O_2(H) \leq T$. Then $U \leq T$ and so $Z(T) \leq H$. Since $Z(T) \in U(2)$ we have, by Proposition 14, that $H/O_2(H)$ is faithfully represented on $O_2(H)/D(O_2(H))$. Suppose there is in H a non-cyclic S_p-subgroup P, for an odd prime p. Then $9 \leq |P|$. Since $l(H) \leq 7$ and $O_2(H)P \leq H$, it follows that $|O_2(H)| \mid 2^5$. Any element $s \in Z(T)$ has $|P|$ different conjugates under P. Since $Z(T) \leq Z(O_2(H))$, $O_2(H)$ has, by Proposition 11, at least $|Z(T)^{\#}||P| \geq 3 \cdot 9 = 27$ involutions in its center. Consequently, $O_2(H)$ is elementary abelian of order 2^5 and $|H/O_2(H)| = 3^2$. Since $O_2(H) \lhd H$ and $O_2(H) < T$, it follows that $2^6 \cdot 3^2 \mid |N(O_2(H))|$, a contradiction to $l(G) = 8$. Thus all S_p-subgroups in H, for odd primes p, are cyclic. Therefore we can apply the theorem of Janko [10]. But none of the groups in the list of the theorem satisfies our conditions. The assertion follows.

PROPOSITION 17. *Suppose that $Z(T)$ contains a cyclic group of order* 4. *Let t be an involution of G and $C = C(t)$. Then $|C| = 2^{\alpha}3^{\beta}$ with $\alpha \in \{3, 4, 5, 6, 7\}$, $\beta \in \{0, 1\}$.*

Proof. Let $T \cap C$ be an S_2-subgroup of C. By Thompson [13, Lemma 5.38], $C(t)$ contains a subgroup $A \in U(2)$. By Proposition 14, $O_{2'}(C) = 1$ and $C/O_2(C)$ is faithfully represented on $O_2(C)/D(O_2(C))$. It follows that $\langle t\rangle Z(T) \leq O_2(C)$ and so $\langle t\rangle Z(T) \leq Z(O_2(C)) = X$. Obviously, we may assume that t is non-central. Let P be an S_p-subgroup of C, p an odd prime. Then $X = C_X(P) \times [P, X]$ and P is represented faithfully on $\Omega_1(X)$ as $Z(T) \leq X$. Assume $|P| > 3$. Then $|\Omega_1(X)| \geq 2^4$ and hence $|X/\mho^1(X)| \geq 2^4$. But $|\mho^1(X)| \geq 2^3$ as $z \in \mho^1(X)$. This is however a contradiction to $l(C) \leq 7$. The proposition is proved.

PROPOSITION 18. *$Z(T)$ contains no cyclic group of order* 4.

Proof. Let $N = N(U)$, where U is a nontrivial 2-subgroup of G. Suppose, there is an odd prime p such that $p^2 \mid |N|$ and let P be an S_p-subgroup of N.

Obviously, we can suppose, that U is elementary abelian and also that $N \cap T$ is an S_2-subgroup of N. Denote $M = C(U)$.

Clearly $|U| < 2^6$. If $|U| = 2^5$, it would be $|Z(T)U| \geq 2^6$, a contradiction to $l(N) \leq 7$, because $Z(T)U \leq M \leq N$.

Suppose $|U| = 2^4$. Since $l(N) \leq 7$ it must be $U \cap Z(T) = \langle z\rangle$. Now $N = (UZ(T))P$, with $|P| = p^2$, and $UZ(T)$ is abelian. It is $M = C(U) = UZ(T)$, because none element of $P^{\#}$ centralizes $z \in U$. Now

$$\langle z\rangle = \mho^1(M) \text{ char } N,$$

which contradicts $C(z) = T$.

Suppose next, that $|U| = 2^3$ or 2^2. If $U \cap Z(T) = \langle z\rangle$, then $z \in U$ would have $p^2 \geq 9$ conjugates in U, which is impossible. Therefore $U \cap Z(T) = 1$, $U \times Z(T) \leq M$. Assume first, that $O_{2'}(N) \neq 1$. Then $M \geq UZ(T)O_{2'}(N)$ and $O_{2'}(N) \cong Z_3$ by Proposition 17. But Z_3 has not an automorphism of order 4, which contradicts $C(Z(T)) = T$, $Z(T) \leq N$. Thus $O_{2'}(N) = 1$. By Gorenstein [4, 6.3.4], $N/O_2(N)$ is faithfully represented on

$O_2(N)/D(O_2(N))$. Therefore

$$Z(T) \leq O_2(N) \quad \text{and} \quad |O_2(N)/D(O_2(N))| \geq 2^4,$$

as $p^2 \mid |N/O_2(N)|$, $p \neq 2$. It follows that $|O_2(N)| = 2^5$ and $\langle z \rangle = D(O_2(N))$ char N, a contradiction to $C(z) = T$.

Suppose at last that $|U| = 2$. Then Proposition 17 contradicts the assumption $p^2 \mid |N(U)|$.

We have proved that $p^2 \nmid |N(U)|$ for all nontrivial 2-subgroups U, if p is an odd prime. Hence all S_p-subgroups of $N(U)$ are cyclic and we can apply the theorem of Janko [10]. This yields a contradiction again as in the proof of Proposition 16.

PROPOSITION 19. *Let $Z(T)$ be cyclic of order 2 and t an involution of G, $C = C(t)$. Then $|C| = 2^\alpha p^\beta$, $p \in \{3, 7\}$, $\alpha \in \{2, 3, 4, 5, 6, 7\}$, $\beta \in \{0, 1\}$. If $O_2(C)$ is not abelian, then $p = 3$.*

Proof. Let $T \cap C$ be an S_2-subgroup of C. By Thompson [13, Lemma 5.38], C contains a subgroup $A \in U(T)$. By Proposition 14, we get $O_{2'}(C) = 1$ and $C/O_2(C)$ is faithfully represented on $O_2(C)/D(O_2(C))$. Therefore $Z(T) = \langle z \rangle \leq O_2(C)$. If C is a 2-group the assertion holds. Thus we can suppose that C is not a 2-group and so $t \not\sim_G z$. We have $\langle z \rangle \times \langle t \rangle \leq Z(O_2(C))$. Also $|O_2(C)| \mid 2^5$ because otherwise $O_2(C)$ would have a too large normalizer.

Assume first that $O_2(C)$ is not abelian. Then $|Z(O_2(C))| \mid 8$ and since C is solvable, there exists a Hall $2'$-subgroup B of C. Because of $C(z) = T$ and $z \not\sim_G t$, it must be that $|B| \leq 6$ and so $|B| = 5$ or 3. But if $|B| = 5$, then $|Z(O_2(C))| = 8$ and B acts faithfully on $Z(O_2(C))$, which is impossible. Therefore $|B| = 3$, if $O_2(C)$ is nonabelian.

Assume now that $O_2(C)$ is abelian. Let B be again a Hall $2'$-subgroup of the solvable group C. Then $|B| + 2 \leq |\Omega_1(O_2(C))|$. But

$$|\Omega_1(O_2(C))| \leq 2^4,$$

because T contains no elementary abelian subgruop of order 2^5, as $|Z(T)| = 2\cdot$ It follows that $|B| \leq 14$ and thus $|B| \in \{3, 5, 7, 11, 13\}$ or $|B| = 9$. Let $\Omega_1(O_2(C)) = K$. Then by Gorenstein [4, 5.2.3], $K = C_K(B) \times [K, B]$ and B acts faithfully on $[K, B]$ if $|B|$ is a prime.

Therefore $|B| \neq 11, 13$. Suppose $|B| = 9$. Then $|\Omega_1(O_2(C))| = 16$. First let $O_2(C) = \Omega_1(O_2(C))$. Then B acts faithfully on $O_2(C)$ and thus on $[K, B]$. The element z has precisely 9 conjugates under B because $C(z) = T$. Therefore some conjugate of z under B is in $[K, B]$ and thus all nine are in $[K, B]$. It follows that $[K, B] = K$, a contradiction to $t \in C_K(B)$. Hence $O_2(C) > \Omega_1(O_2(C))$ and thus $|O_2(C)| = 2^5$. But now $N(O_2(C))$ would be too large, a contradiction. Thus $|B| \neq 9$.

If $|B| = 5$, then $|K| \in \{8, 16\}$ as $C(z) = T, z \in K$. Since B acts faithfully on K, it must be that $|K| = 16$ and $C_K(B) = 1$, which contradicts $t \in C_K(B)$. The assertion of the proposition is completely proved.

PROPOSITION 20. *$Z(T)$ is not cyclic of order 2. The assumption $SCN_3(2) \neq 0$ is contradictory.*

Proof. It is clear that the first assertion, together with the Propositions 11, 16, and 18, implies the second.

Suppose that $Z(T) = \langle z \rangle$. By the theorem of Janko [10], we can suppose, that there is an elementary abelian 2-subgroup A of G, such that $N(A) = N$ has a non-cyclic S_p-subgroup for some odd prime p.

Let $M = C(A)$ and $a \in A$. Then $M \leq C(a)$. Let $C(a) \cap T$ be an S_2-subgroup of $C(a)$ containing A. By Thompson [13, Lemma 5.38], $C(a)$ contains a subgroup $B \in U(2)$. By Proposition 14, $O_{2'}(C(a)) = 1$ and $C(a)/O_2(C(a))$ is faithfully represented on $O_2(C(a))/D(O_2(C(a)))$. Thus $Z(T) = \langle z \rangle \leq O_2(C(a)) \cap M \lhd M$. Obviously $z \in O_2(M)$.

Suppose that $O_{2'}(N) \neq 1$. Then $[O_{2'}(N), O_2(M)] = 1$. because $O_2(M)$ char $M \lhd N$. But this is impossible, as $z \in O_2(M)$, $C(z) = T$.

It follows that $O_{2'}(N) = 1$. Now $N/O_2(N)$ is faithfully represented on $O_2(N)/D(O_2(N))$. Since $l(N) \leq 7$ and $p^2 \mid | N/O_2(N) |$, p an odd prime, it must be that

$$| O_2(N) | \mid 2^5 \quad \text{and} \quad | O_2(N)/D(O_2(N)) | \geq 2^4.$$

If $| O_2(N) | = 2^5$, then $| O_2(N)/D(O_2(N)) | = 2^4$, because G contains no elementary abelian subgroup of order 2^5, as $| Z(T) | = 2$. Hence

$$D(O_2(N)) = \langle d \rangle \text{ char } N, \quad \text{with} \quad | d | = 2.$$

It follows $C(d) \geq N$, a contradiction, as $p^2 \nmid | C(d) |$, by Proposition 19.

Thus $O_2(N)$ is elementary abelian of order 2^4. We can obviously replace A by $O_2(N)$ and set $A = O_2(N)$. Since $p^2 \mid | GL(4, 2) |$ it must be that $p = 3$ and $| N | = 2^5 3^2$. Moreover, an S_3-subgroup P of N is elementary abelian.

Let $R = N \cap T$. Then R is an S_2-subgroup of N. Let $R_1 \leq T$, $R_1 > R$ and $| R_1 : R | = 2$. Then $A \ntriangleleft R_1$, $R \lhd R_1$ and thus there exists some $r \in R_1$, with $AA^r = R$. We see that A has a complement in B and therefore also in N. We get $N = AK$, $A \cap K = 1$, with $K \leq N$. We can suppose that $P \leq K$. Then $P \lhd K$, $K = P\langle s \rangle$, with $| s | = 2$. Since $z \in O_2(M) \leq O_2(N)$, it follows that $z \in A$ and z has 9 conjugates in A under N. We denote the set of these conjugates with $\mathfrak{Z}$.

Let $x_1 \in A^{\#} \backslash \mathfrak{Z}$. Then $C_P(x_1) = \langle m_1 \rangle$, with $| m_1 | = 3$. If $m \in P \backslash \langle m_1 \rangle$, for $x_2 = x_1^m$, $x_3 = x_2^m$ it holds $C_P(x_1) = C_P(x_2) = C_P(x_3)$, $x_1 x_2 = x_3$, because x_1 has not more than three conjugates in A under T and because of Proposition 19. Similarly, there is an element

$$y_1 \in A \backslash \mathfrak{Z} \backslash \{x_1, x_2, x_3\}$$

with $C_P(y_1) = m_2$, $| m_2 | = 3$, and for $y_2 = y_1^{m_2}$, $y_3 = y_2^{m_2}$, we have

$$C_P(y_1) = C_P(y_2) = C_P(y_3) \quad \text{and} \quad y_1 y_2 = y_3 .$$

One easily checks, that $P = \langle m_1 \rangle \times \langle m_2 \rangle$, and we may assume that $m = m_2$.

Let $A_1 = \langle x_1, x_2\rangle$, $A_2 = \langle y_1, y_2\rangle$. Then $A = A_1 \times A_2$.

Consider now the action of s on $O_2(N)P$. Let $A\langle s\rangle < S < T$ and $v \in S\backslash A\langle s\rangle$. Then $A\langle s\rangle = AA^v$ and $Z(A\langle s\rangle) \geq A \cap A^v$. Since N/A acts faithfully on $A = O_2(N)$, it follows that $C(A) = A$ and thus $A\langle s\rangle$ is not abelian. Hence $Z(A\langle s\rangle) = C_A(s) = A \cap A^v$ is of order 8 and thus s fixes precisely 8 elements of A.

The element s fixes the set $A_1 \cup A_2$. If $A_1^s \neq A_1$, then we can suppose $x_1^s = y_1$. We now have $x_2^s \neq x_3$ because $x_2 x_3 = x_1$. For the same reason it is not true that $x_2^s = x_2$, $x_3^s = x_3$. Thus $x_i^s = y_j$, with $i, j \in \{2, 3\}$, implying that s fixes only 4 elements in A, a contradiction.

Therefore $A_1^s = A_1$ and $A_2^s = A_2$. Now we can suppose without loss, that $y_1^s = y_1$, $y_2^s = y_3 = y_1 y_2$, $x_1^s = x_1$, $x_2^s = x_2$, because $|C_A(s)| = 8$ and $A = A_1 \times A_2$. One easily sees that this implies $m_1^s = m_1^{-1}$ and $m_2^s = m_2$. Thus we can write

$$N = \langle x_1, x_2, y_1, y_2, m_1, m_2, s \mid m_1^3 = m_2^3 = 1, x_1^{m_2} = x_2, x_2^{m_2} = x_1 x_2,$$
$$y_1^{m_1} = y_2, y_2^{m_1} = y_1 y_2, y_2^s = y_1 y_2, m_1^s = m_1^{-1}\rangle.$$

Consider again S and v chosen as before. We have

$$Z(A\langle s\rangle) = \langle x_1, x_2, y_1\rangle \lhd S.$$

Since $Z(T) = \langle z\rangle$ is of order 2, it must be that $|\Omega_1(Z(S))| \leq 4$. But $\Omega_1(Z(S))$ contains the group $C_{Z(A\langle s\rangle)}(v)$ and therefore is of order at least 4. Consequently, $\Omega_1(Z(S)) = Z(S) = C_{Z(A\langle s\rangle)}(v)$ and this group is of order 4, because $C(A) = A$ and $A \leq S$.

Let $Z(S)^{\#} = \{s_1, s_2, z\}$. Now $\langle s_1, s_2\rangle \lhd T$ and $2^6 \mid |C(s_1)| = |C(s_2)|$, as $s_1 \sim_T s_2$. If $|C(s_1)| = |C(s_2)| = 2^7$, then $N(C_T(Z(S))) \geq \langle C(s_1), C(s_2)\rangle$ and thus $C(s_1) = C(s_2)$. Since $Z(T)$ has order 2, this implies $s_1 = s_2$, which is a contradiction. Therefore $C(s_i)$ are not S_2-subgroups of G and especially $s_i \not\sim_G z$, for $i = 1, 2$.

It is $Z(S) \leq Z(A\langle s\rangle) = \langle x_1, x_2, y_1\rangle$. But s_1, s_2 are not central involutions and hence $s_1, s_2 \in \{x_1, x_2, x_3, y_1\}$. Also $s_1 s_2 \in \langle x_1, x_2\rangle y_1$, as $s_1 s_2 = z$. Thus we can suppose $s_1 = y_1$, $s_2 = x_i$, for some $i \in \{1, 2, 3\}$. It follows that $x_1 \sim x_2 \sim x_3 \sim y_1 \sim y_2 \sim y_3$ and $|C(x_i)| = |C(y_i)| = 2^6 \cdot 3$, for $i = 1, 2, 3$, as $s_1 \sim_T s_2$ and $\langle m_1\rangle \leq C(x_1)$.

Consider again $A\langle s\rangle = AA^v$. Since $|Z(A\langle s\rangle)| = 8$ one can easily see, that all the involutions of $A\langle s\rangle$ are contained in $A \cup A^v$. Thus $s \in A^v\backslash A$ and $A\langle s\rangle\backslash A$ contains precisely 8 involutions which belong to $A^v = C_A(S) \times \langle s\rangle$. Here $C_A(s) = \langle x_1, x_2, y_1\rangle$, as one checks directly. Since the groups A and A^v are conjugate and $C_A(s)$ contains 4 non-central and 3 central involutions, $A^v\backslash A$ must contain still 2 non-central and 6 central involutions.

Since $m_2^s = m_2$, s is not central. Also $y_1^{m_2} = y_1$ and thus $(y_1 s)^{m_2} = y_1 s$. Therefore $y_1 s$ and s are the both non-central involutions of $A^v\backslash A$. We see, that $x_1 \sim x_2 \sim x_3 \sim y_1 \sim y_2 \sim y_3 \sim s \sim y_1 s$ form a class of involutions in $A\langle s\rangle$ under G, all the others involutions of $A\langle s\rangle$ being central in G.

We have proved that $|C(y_1)| = 2^6 \cdot 3$ and that $A\langle s, m_2\rangle \leq C(y_1)$. We can suppose that $A\langle s\rangle < S < C(y_1)$ with $T = N(S)$, S being an S_2-subgroup of $C(y_1)$. Now $\langle S, m_2\rangle \leq N(A\langle s\rangle)$ and hence $N(A\langle s\rangle) = C(y_1)$. In $S\backslash A\langle s\rangle$ there exists an involution v. Otherwise $A\langle s\rangle = \Omega_1(S) \lhd T$, a contradiction. We get $S = \langle x_1, x_2, y_1, y_2, s, v\rangle$. We have

$$\langle x_1, x_2, y_1\rangle = Z(A\langle s\rangle) \lhd S.$$

Since $y_1 \in Z(S)$, we have $\langle x_1, x_2\rangle^v = \langle x_1, x_2\rangle$ and so $\langle x_1, x_2\rangle \lhd S$. We can suppose that $x_i^v = x_i$, $x_{i+1}^v = x_{i+2}$, for some i, where the indices are taken mod 3. Here we have used the fact that $Z(T)$ is of order 2. On the other hand $\langle y_1, y_2\rangle \not\!\lhd S$, as $A \not\!\lhd S$. Especially $\langle y_1, y_2\rangle^v \cap (A\langle s\rangle\backslash A) \neq 0$. Since $y_1^v = y_1$, it follows that $y_2^v \in \{s, y_1 s\}$. Now

$$S = \langle x_1, x_2, y_1, y_2, s, v \mid x_i^v = x_i, x_{i+1}^v = x_{i+1} x_i,$$
$$y_2^s = y_1 y_2, y_2^v = y_1^\alpha s, s^v = y_1^\alpha y_2\rangle$$

where $i \in \{1, 2, 3\}$, $\alpha \in \{0, 1\}$, and $x_1 x_2 = x_3$.

Obviously $Z(S) = \langle x_i, y_1\rangle$. From here we get

$$S/Z(S) = \bar{S} = \langle \bar{x}_{i+1}, \bar{y}_2, \bar{s}, \bar{v} \mid \bar{y}_2^{\bar{s}} = \bar{s}, \bar{s}^{\bar{v}} = \bar{y}_2\rangle$$

where $\bar{m} = mZ(S)$ for $m \in S$. One can easily compute that $Z(\bar{S}) = \langle \bar{x}_{i+1}, \bar{y}_2 \bar{s}\rangle$ and therefore

$$Z_2(S) = \langle x_i, y_1, x_{i+1}, y_2 s\rangle = \langle x_1, z_2, y_2 s\rangle = \langle x_1, x_2\rangle \times \langle y_2 s\rangle.$$

But now $(y_2 s)^2 = y_2(y_1 y_2) = y_1$. Thus $\langle y_1\rangle = \mho^1(Z_2(S))$ char $S \lhd T$. It follows that $C(y_1) = T$, a contradiction, because s_1 is not a central involution in G.

Thus, $Z(T)$ is not of order 2, and the Proposition 20 holds, as noted at the beginning of the proof.

Now, we are in the position to prove our theorem.

From Proposition 20 it follows that $SCN_3(2) = 0$. Thus we can apply the theorem of Janko-Thompson [11]. One can easily see, that among the groups listed in this theorem only $PSL(2, 127)$ satisfies the conditions of our theorem.

The theorem is proved.

Acknowledgment. It is a pleasure to express my gratitude to Professor D. Held, who was my advisor in this research. He has given me a large amount of help at all stages of the work.

I wish to thank Professor Z. Janko for suggesting the problem to me and for his kind help during the work.

I also thank Dr. U. Dempwolff for very profitable conversations.

My thanks go to the Alexander von Humboldt foundation for its generous support.

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