MEASURES WHOSE POISSON INTEGRALS ARE PLURIHARMONIC

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1. Introduction

1.1 Let V be a vector space over $\mathbf{C}$ of complex dimension n with an inner product. If x and y are in V, then we will denote by $\langle x, y\rangle$ the inner product of x and y. We will denote by B the class of all x in V such that $\langle x, x\rangle < 1$, by $\bar{B}$ the class of all x in V such that $\langle x, x\rangle \leq 1$, and by S the class of all x in V such that $\langle x, x\rangle = 1$. We recall that the Poisson kernel of B is the function $\beta : \bar{B} \times B \to (0, \infty)$ defined by

$$\beta(x, y) = [(1 - \langle y, y\rangle)/(1 - \langle x, y\rangle)(1 - \langle y, x\rangle)]^n.$$

With regard to why β is called the Poisson kernel of B we refer to Proposition 2.4. (We remark that β is the Poisson kernel with respect to the Bergman metric on B and not the Euclidean metric.) If Y is a locally compact Hausdorff space, then we will denote by $M_+(Y)$ the class of all Radon measures on Y. Thus if $\mu \in M_+(Y)$ and $E \subset Y$, then $\mu(E) \geq 0$. We will denote by $M(Y, \mathbf{R})$ the real linear span of those μ in $M_+(Y)$ for which $\mu(Y) < \infty$ and we will denote by $M(Y, \mathbf{C})$ the complex linear span of those μ in $M_+(Y)$ for which $\mu(Y) < \infty$. (Thus if Y is compact, then $M(Y, \mathbf{C})$ is the complex linear span of $M_+(Y)$.) We recall that if X and Y are sets, if f is a function defined on the Cartesian product $X \times Y$, and if $(s, t) \in X \times Y$, then f_s and f^t are the functions defined on Y and X respectively by $f_s(y) = f(s, y)$ and $f^t(x) = f(x, t)$. If $\mu \in M(S, \mathbf{C})$, then we define $\mu^\# : B \to \mathbf{C}$ by

$$\mu^\#(y) = \int \beta^y \, d\mu.$$

Thus $\mu^\# \in C^\infty(B)$.

The purpose of this paper is to prove the theorems that follow (1.3, 1.5, 1.7, 1.13, 1.15, 1.17). These theorems with one exception (Theorem 1.5) are on measures whose Poisson integrals are pluriharmonic.

1.2. If $\mu \in M(S, \mathbf{C})$, then we will denote by spt (μ) the support of μ.

1.3. THEOREM. *If $n \geq 2$, if $\mu \in M(S, \mathbf{C})$, if $\mu^\#$ is pluriharmonic, and if $\mu \neq 0$, then* spt $(\mu) = S$.

1.4. If Z is a topological space and if $Y \subset Z$, then we will denote (as is usual) by Y^0 the interior of Y. Furthermore we will denote (as is usual) by $C(Z)$ the class of all continuous functions $f : Z \to \mathbf{C}$. We will denote by $A(B)$ the class of all functions in $C(\bar{B})$ that are holomorphic on B. We will denote (as is usual) by $\mathbf{T}$ the class of all z in $\mathbf{C}$ such that $z\bar{z} = 1$.

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1.5. THEOREM. *Let f and g be in $A(B)$ and let*

$$Y = \{x : x \in S, \mid f(x) \mid = \mid g(x) \mid \}.$$

If $Y^0 \neq \emptyset$ and if $n \geq 2$, then $f = cg$ where $c \in \mathbf{T}$.

1.6. We will denote by σ the Radon measure on S which assigns to each open subset of S its Euclidean volume (for the purpose of defining σ we regard S as the Euclidean sphere of real dimension $2n - 1$).

1.7. THEOREM. *If $\mu \in M(S, \mathbf{C})$, if $\mu^{\#}$ is pluriharmonic, if $E \subset S$, and if $zE = E$ for every z in $\mathbf{T}$, then*

$$\sigma(S)\mu(E) = \mu(S)\sigma(E).$$

1.8. If k is a positive integer, then we will denote by H_k the class of all members of the polynomial ring $\mathbf{C}[\chi : \chi \in V^*]$ that are homogeneous of degree k.

1.9. COROLLARY. *If $n \geq 2$, if $\mu \in M(S, \mathbf{C})$, if $\mu^{\#}$ is pluriharmonic, if $f \in \bigcup_{k=1}^{\infty} H_k$, if $f \neq 0$, and if $t \in [0, \infty)$, then*

$$\mu(\{x : x \in S, \mid f(x) \mid = t\}) = 0.$$

1.10. If $\alpha \in [0, \infty)$, then we will denote by H^{α} the Hausdorff measure on S of dimension α.

1.11. COROLLARY. *If $\mu \in M(S, \mathbf{C})$, if $\mu^{\#}$ is pluriharmonic, if $E \subset S$, and if $H^{2n-2}(E) = 0$, then*

$$\mu(\bigcup_{z \in T} zE) = 0.$$

1.12. Let W be a linear subspace of V of complex dimension m and let P be the orthogonal projection of V onto W. We will denote by $U(V)$ the class of all unitary transformations of V. We remark that if $z \in \mathbf{T}$, then $P + z(I - P) \in U(V)$.

1.13. THEOREM. *Let $m \leq n - 1$, let $\mu \in M(S, \mathbf{C})$, and let $\mu^{\#}$ be pluriharmonic. If we define $f : S - W \to \mathbf{C}$ by $f(x) = \mu^{\#}(Px)$, then $f \in L^1(\sigma)$. If $E \subset S$, if E is σ measurable, and if $(P + z(I - P))E = E$ for every z in $\mathbf{T}$, then*

$$\mu(E) = \frac{1}{\sigma(S)} \int_E f \, d\sigma.$$

If $\mu \in M_+(S)$, if $\mu \neq 0$, and if $x \in S - W$, then $f(x) > 0$.

1.14. We remark that if $m = 0$, then by Theorem 1.7 Theorem 1.13 holds.

1.15. THEOREM. *If $n \geq 2$, if $\mu \in M(S, \mathbf{C})$, if $\mu^{\#}$ is pluriharmonic, if $x \in V$, and if $m \leq n - 1$, then $\mid \mu \mid ((x + W) \cap S) = 0$.*

1.16. Let τ be a skew-Hermitian transformation of V. Thus $i\tau$ is Hermitian. We recall that if $t \in \mathbf{R}$, then $e^{t\tau} \in U(V)$. Furthermore let $i\tau \leq 0$. We will denote by $\mathbf{H}$ the class of all z in $\mathbf{C}$ such that $\operatorname{Im}(z) > 0$ and we will denote by $\mathbf{H}^-$ the class of all z in $\mathbf{C}$ such that $\operatorname{Im}(z) \geq 0$. Thus if $z \in \mathbf{H}^-$ and if $x \in S$, then $e^{z\tau}x \in \bar{B}$. Furthermore if $z \in \mathbf{H}$ and if $\tau x \neq 0$, then $e^{z\tau}x \in B$.

1.17. THEOREM. *Let $\tau \neq 0$, let N be the null space of τ, let $z \in \mathbf{H}$, let $\mu \in M(S, \mathbf{C})$, and let $\mu^\#$ be pluriharmonic. If we define $f : S - N \to \mathbf{C}$ by $f(x) = \mu^\#(e^{z\tau}x)$, then $f \in L^1(\sigma)$. If $E \subset S$, if E is σ measurable, and if $e^{t\tau}E = E$ for every t in $\mathbf{R}$, then*

$$\mu(E) = \frac{1}{\sigma(S)} \int_E f \, d\sigma.$$

If $\mu \in M_+(S)$, if $\mu \neq 0$, and if $x \in S - N$, then $f(x) > 0$.

1.18. COROLLARY. *If $\tau \neq 0$, if $\mu \in M(S, \mathbf{C})$, if $\mu^\#$ is pluriharmonic, if $E \subset S$, if $e^{t\tau}E = E$ for every t in $\mathbf{R}$, and if $\sigma(E) = 0$, then $\mu(E) = 0$.*

1.19. We will denote by $G(B)$ the class of all holomorphic homeomorphisms of B. With regard to $G(B)$ we refer to Section 2.1.

1.20. COROLLARY. *Let $Z \in G(B)$, let $\tau \neq 0$, let N be the null space of τ, let $z \in \mathbf{H}$, let $\mu \in M(S, \mathbf{C})$, and let $\mu^\#$ be pluriharmonic. If we define*

$$f : S - Z(N \cap S) \to \mathbf{C}$$

by $f(x) = \mu^\#((Z \circ e^{z\tau} \circ Z^{-1})(x))$, then $f \in L^1(\sigma)$. If $v \in Z(N \cap B)$, if $E \subset S$, if E is σ measurable, if E is $|\mu|$ measurable, and if $(Z \circ e^{t\tau} \circ Z^{-1})E = E$ for every t in $\mathbf{R}$, then

$$\int_E \beta^v \, d\mu = \frac{1}{\sigma(S)} \int_E f \beta^v \, d\sigma. \tag{1.1}$$

If $\mu \in M_+(S)$, if $\mu \neq 0$, and if $x \in S - Z(N \cap S)$, then $f(x) > 0$.

1.21. COROLLARY. *Let $Z \in G(B)$, let $\tau \neq 0$, let $E \subset S$, let*

$$(Z \circ e^{t\tau} \circ Z^{-1})E = E$$

for every t in $\mathbf{R}$, let $\mu \in M_+(S)$, and let $\mu^\#$ be pluriharmonic. If $\sigma(E) = 0$, then $\mu(E) = 0$. If $\mu \neq 0$ and $\mu(E) = 0$, then $\sigma(E) = 0$.

1.22. We will denote by $H(B)$ the class of all holomorphic functions on B. We recall the following fact of the theory of functions on B.

1.23. PROPOSITION. (a) *If $f \in H(B)$ and* $\operatorname{Re}(f) > 0$, *then* $\operatorname{Re}(f) = \mu^\#$ *where $\mu \in M_+(S)$.* (b) *If $\mu \in M(S, \mathbf{R})$ and $\mu^\#$ is pluriharmonic, then $\mu^\# = \operatorname{Re}(f)$ where $f \in H(B)$.*

1.24. It is because of Proposition 1.23 that theorems such as 1.3, 1.7, 1.13, 1.15, and 1.17 are of interest. If the hypotheses of Theorem 1.3 hold, then we do not know if either $(d\mu/d\sigma)\, d\sigma \neq d\mu$ or $(d\sigma/d\mu)\, d\mu \neq d\sigma$ can hold.

2. The Poisson kernel of B

2.1. We will regard $SL(2, \mathbf{R})$ as the class of all 2×2 matrices M of the form

$$M = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

where a and b are in $\mathbf{C}$ and $a\bar{a} - b\bar{b} = 1$. We define

$$\gamma : SL(2, \mathbf{R}) \times S \times \bar{B} \to \bar{B}$$

by

$$\gamma(M, x, y) = \frac{a\langle y, x\rangle + b}{\bar{b}\langle y, x\rangle + \bar{a}} x + \frac{1}{\bar{b}\langle y, x\rangle + \bar{a}} (y - \langle y, x\rangle x)$$

and we define $\delta : U(V) \times SL(2, \mathbf{R}) \times S \times \bar{B} \to \bar{B}$ by

$$\delta(t, M, x, y) = t[\gamma(M, x, y)] = \gamma(M, t(x), t(y)).$$

With regard to the definition of γ we remark that if $x \in S$ and $y \in V$, then $y - \langle y, x\rangle x$ is the orthogonal projection of y into $V \ominus \mathbf{C}x$. Furthermore we remark that $\delta_{(t,M,x)} \in G(B)$ for every triple (t, M, x) in $U(V) \times SL(2, \mathbf{R}) \times S$. We recall the following fact of the theory of functions on B.

2.2 Proposition. *If $Z \in G(B)$, then there is a triple (t, M, x) in*

$$U(V) \times SL(2, \mathbf{R}) \times S$$

such that

$$Z(y) = \delta(t, M, x, y)$$

for all y in B.

2.3. If Y is a topological space, then we will denote by $F_+(Y)$ the class of all Borel functions $f : Y \to [0, \infty)$. The following proposition (which is well known) follows from Proposition 2.2.

2.4. Proposition. *If $(Z, f) \in G(B) \times F_+(S)$, then*

$$\int f \circ Z \, d\sigma = \int f \beta^{Z(0)} d\sigma.$$

(Proposition 2.4 may be proved by means of the following identity which serves to define σ. If $x \in S$, if

$$T = \{y : y \in V, \operatorname{Re}\langle y, x\rangle = 0\},$$

and if $t : T \to S - \{-x\}$ is defined by

$$t(y) = [(4y + 2x)/(4\langle y, y\rangle + 1)] - x,$$

then

$$\int f \, d\sigma = \int f(t(y))[4/(4\langle y, y\rangle + 1)]^{2n-1} \, dy.)$$

2.5. The following proposition follows from Proposition 2.4.

2.6. PROPOSITION. *If* $y \in B$, *then* $\sigma^{\#}(y) = \sigma(S)$.

2.7. If $f \in C(S)$, then we define $f^{\#} : B \to \mathbf{C}$ by

$$f^{\#}(y) = \frac{1}{\sigma(S)} \int \beta^{y} f d\sigma.$$

We will denote (as is usual) by $\mathbf{D}$ the class of all z in $\mathbf{C}$ such that $z\bar{z} < 1$ and by $\mathbf{D}^{-}$ the class of all z in $\mathbf{C}$ such that $z\bar{z} \leq 1$. We recall the following facts (2.8, 2.9, 2.10) of the theory of Poisson integrals.

2.8. PROPOSITION. *If* $f \in C(S)$ *and if* $g : \bar{B} \to \mathbf{C}$ *is defined by* $g \mid B = f^{\#}$ *and* $g \mid S = f$, *then* $g \in C(\bar{B})$.

2.9. PROPOSITION. *If* $\mu \in M(S, \mathbf{C})$, *if* $f \in C(S)$, *and if* $z \in \mathbf{D}$, *then*

$$\int f^{\#}(zx)\, d\mu(x) = \frac{1}{\sigma(S)} \int f(y)\, \mu^{\#}(\bar{z}y)\, d\sigma(y).$$

2.10. PROPOSITION. *If* $\mu \in M(S, \mathbf{C})$ *and* $z \in \mathbf{D}$, *then*

$$\int \mid \mu^{\#}(zx) \mid d\sigma(x) \leq \sigma(S) \mid \mu \mid (S).$$

2.11. The following proposition follows from Proposition 2.8 and Proposition 2.9.

2.12. PROPOSITION. *If* $\mu \in M(S, \mathbf{C})$ *and if* $\mu^{\#} = 0$, *then* $\mu = 0$.

3. The proofs of Theorems 1.3 and 1.5

3.1. We will denote by Δ the class of all pairs (x, y) in $V \times V$ such that $\langle x, y \rangle = 0$ and $x + y \in S$. If $x \in V$ and $r > 0$, then we will denote by $B(x, r)$ the class of all y in V such that $\mid y - x \mid < r$ (where $\mid x \mid = \sqrt{\langle x, x \rangle}$).

3.2. LEMMA. *Let* $v \in S$ *and let* $r \in (0, 1)$. *If* $(x, y) \in \Delta$, *if* $x \in B(v, r)$, *and if* $z \in \mathbf{D}^{-}$, *then* $x + zy \in B(v, 3\sqrt{r})$.

Proof. We have

$$\begin{aligned}\langle y, y \rangle = 1 - \langle x, x \rangle &= (1 + \mid x \mid)(1 - \mid x \mid) \\ &\leq 2(1 - \mid x \mid) = 2(\mid v \mid - \mid x \mid) \leq 2 \mid v - x \mid < 2r,\end{aligned}$$

hence

$$\mid x + zy - v \mid \leq \mid x - v \mid + \mid zy \mid < r + \sqrt{(2r)} < 3\sqrt{r}.$$

3.3. We will now prove Theorem 1.3. Let G be an open subset of S, let $v \in G$, and let r in $(0, 1)$ be such that $B(v, 3\sqrt{r}) \cap S \subset G$. Furthermore let $x \in B(v, r) \cap B$. Since $n \geq 2$, there is a y in V such that $(x, y) \in \Delta$. If $z \in \mathbf{T}$, then (by Lemma 3.2) $x + zy \in G$.

If $\mid \mu \mid (G) = 0$ and if we define $\mu^{\#}$ on G by $\mu^{\#}(w) = 0$ $(w \in G)$, then it follows from the definition of β that $\mu^{\#} \in C(B \cup G)$. We define $f : \mathbf{D}^{-} \to \mathbf{C}$

by $f(z) = \mu^{\#}(x + zy)$. It follows that $f \in C(\mathbf{D}^-)$, that f is harmonic on $\mathbf{D}$, and that $f = 0$ on $\mathbf{T}$. Hence $f(0) = 0$, that is to say $\mu^{\#}(x) = 0$. Thus $\mu^{\#} = 0$ on $B(v, r) \cap B$. It follows by Proposition 1.23 or by the fact that $\mu^{\#}$ is analytic that $\mu^{\#} = 0$. Thus $\mu = 0$ by Proposition 2.12. This completes the proof of Theorem 1.3.

3.4. We will denote by $A(\mathbf{D})$ the class of all functions in $C(\mathbf{D}^-)$ that are holomorphic on $\mathbf{D}$.

3.5. We will now prove Theorem 1.5. (With regard to the following proof we refer to the proof of Theorem 1.3.) If $g(x) = 0$ for all x in Y^0, then $g = 0$ and $f = 0$. We suppose then that there is a vector v in Y^0 such that $g(v) \neq 0$. Let r in $(0, 1)$ be such that $B(v, 3\sqrt{r}) \cap S \subset Y$ and $f(w)g(w) \neq 0$ if $w \in B(v, 3\sqrt{r}) \cap \bar{B}$. Furthermore let $x \in B(v, r) \cap B$. Since $n \geq 2$, there is a y in V such that $(x, y) \in \Delta$. If $z \in \mathbf{T}$, then (by Lemma 3.2) $x + zy \in Y$.

We define $h : \mathbf{D}^- \to \mathbf{C}$ by

$$h(z) = f(x + zy)/g(x + zy).$$

It follows that $h \in A(\mathbf{D})$, that $h(z) \neq 0$ if $z \in \mathbf{D}$, and that $|h(z)| = 1$ if $z \in \mathbf{T}$. Hence $|h(0)| = 1$, that is to say $|f(x)| = |g(x)|$. Thus $|f| = |g|$ on $B(v, r) \cap B$; hence $f = cg$ where $c \in \mathbf{T}$ which completes the proof of Theorem 1.5.

3.6. Let $n \geq 2$ and let $v \in S$. With regard to the proof of Theorem 1.5 Rudin (unpublished) uses the map $z \to x + zy$ to prove that if f is an inner function on B and if the cluster set of f at v is not all of $\mathbf{D}^-$, then f is constant. In particular if f is continuous at v, then f is constant.

4. The proof of Theorem 1.7

4.1. If Y, Z, and N are sets, if $\phi : Y \to Z$, and if $\mu : 2^Y \to N$, then we define $\phi^*(\mu) : 2^Z \to N$ by

$$\phi^*(\mu)(E) = \mu(\{y : y \in Y, \phi(y) \in E\}).$$

With regard to this definition we recall the following fact of measure theory [Federer, p. 72].

4.2. Proposition. *If Y and Z are compact Hausdorff spaces, if $\phi : Y \to Z$ is continuous, and if $\mu \in M_+(Y)$, then $\phi^*(\mu) \in M_+(Z)$. Thus if $\mu \in M(Y, \mathbf{C})$, then $\phi^*(\mu) \in M(Z, \mathbf{C})$.*

4.3. With regard to Proposition 4.2 we remark that if $f \in C(Z)$, then

$$\int f\, d\phi^*(\mu) = \int f \circ \phi\, d\mu.$$

The following proposition follows from Proposition 4.2.

4.4. Proposition. *If Y and Z are locally compact Hausdorff spaces, if*

$\phi : Y \to Z$ is continuous, if $\mu \in M_+(Y)$, and if $\mu(Y) < \infty$, then $\phi^*(\mu) \in M_+(Z)$. Thus if $\mu \in M(Y, \mathbf{C})$, then $\phi^*(\mu) \in M(Z, \mathbf{C})$.

4.5. Let (G, X, T) be a topological transformation group. (Thus by definition G is a locally compact Hausdorff group, X is a locally compact Hausdorff space, $T : G \times X \to X$, etc.) It is assumed that G and X are metric spaces whose closed balls are compact. We will denote by γ a right Haar measure on G. Thus if $(s, f) \in G \times F_+(G)$, then

$$\int f(ts)\, d\gamma(t) = \int f\, d\gamma.$$

It is assumed that $\gamma(G) = 1$ if G is compact. If G is compact and if $\mu \in M(X, \mathbf{C})$, then we define μ^* in $M(X, \mathbf{C})$ by

$$\mu^* = T^*(\gamma \times \mu).$$

4.6. PROPOSITION. *Let X be compact. If $E \subset G$, if $F \subset X$, if E is compact, and if F is open, then $\bigcap_{t \in E} T_t(F)$ is open.*

Proof. We have

$$[\bigcap_{t \in E} T_t(F)]' = \bigcup_{t \in E} T_t(F') = T(E \times F')$$

which completes the proof of Proposition 4.6.

4.7. PROPOSITION. *Let X be compact. If $\mu \in M_+(X)$, if $E \subset X$, and if $T_t(E) = E$ for every t in G, then there is an $F \subset X$ such that F is a G_δ, $E \subset F$, $T_t(F) = F$ for every t in G, and $\mu(E) = \mu(F)$.*

Proof. If $\varepsilon > 0$, then since μ is a Radon measure there is an open set Q such that $E \subset Q$ and $\mu(Q) \leq \mu(E) + \varepsilon$. If

$$Q_\varepsilon = \bigcap_{t \in G} T_t(Q),$$

then $E \subset Q_\varepsilon$, $\mu(Q_\varepsilon) \leq \mu(E) + \varepsilon$, $T_t(Q_\varepsilon) = Q_\varepsilon$ for every t in G, and (by Proposition 4.6) Q_ε is a G_δ. Thus if

$$F = \bigcap_{k=1}^{\infty} Q_{1/k},$$

then F satisfies the conclusions of Proposition 4.7.

4.8. PROPOSITION. *Let X be compact. If $\lambda \in M(G, \mathbf{C})$, if $\lambda(G) = 1$, if $\mu \in M(X, \mathbf{C})$, if $E \subset X$, and if $T_t(E) = E$ for every t in G, then*

$$\mu(E) = T^*(\lambda \times \mu)(E). \tag{4.1}$$

Proof. By Proposition 4.7 there is an $F \subset X$ such that F is a G_δ, $T_t(F) = F$ for every t in G, $\mu(E) = \mu(F)$, and $T^*(\lambda \times \mu)(E) = T^*(\lambda \times \mu)(F)$. If f is the characteristic function of F, then

$$f = \int f \circ T_t\, d\lambda(t),$$

hence

$$\mu(F) = \int \left[\int f \circ T_t \, d\lambda(t) \right] d\mu = \int f \circ T \, d(\lambda \times \mu) = T^*(\lambda \times \mu)(F),$$

hence the identity (4.1) holds.

4.9. For the purpose of the proof of Theorem 1.7 we let $G = \mathbf{T}$, $X = S$, and we define T by $T(z, x) = \bar{z}x$.

4.10. PROPOSITION. *If* $(\mu, y) \in M(S, \mathbf{C}) \times B$, *then*

$$(\mu^*)^{\#}(y) = \int_0^{2\pi} \mu^{\#}(e^{i\theta} y) d\theta. \tag{4.2}$$

Proof. We have

$$\begin{aligned} (\mu^*)^{\#}(y) &= \int \beta^y \, d\mu^* \\ &= \int \beta^y \circ T \, d(\gamma \times \mu) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left[\int \beta(e^{-i\theta} x, y) d\mu(x) \right] d\theta. \end{aligned} \tag{4.3}$$

Furthermore $\beta(e^{-i\theta}x, y) = \beta(x, e^{i\theta}y)$, hence the last term of the string of identities (4.3) is equal to the right side of (4.2) which completes the proof of Proposition 4.10.

4.11. PROPOSITION. *If* $\mu \in M(S, \mathbf{C})$ *and if* $\mu^{\#}$ *is pluriharmonic, then*

$$\sigma(S)\mu^* = \mu(S)\sigma. \tag{4.4}$$

Proof. If $y \in B$, then since $\mu^{\#}$ is pluriharmonic the right side of the identity (4.2) is equal to $\mu^{\#}(0)$; hence $(\mu^*)^{\#} = \mu(S)$ and hence by Proposition 2.6 and Proposition 2.12 the identity (4.4) holds.

4.12. We will now prove Theorem 1.7. By Proposition 4.8 and Proposition 4.11 we have

$$\sigma(S)\mu(E) = \sigma(S)\mu^*(E) = \mu(S)\sigma(E)$$

which completes the proof of Theorem 1.7.

5. The proof of Corollary 1.9

Let

$$E = \{x : x \in S, |f(x)| = t\}.$$

By Theorem 1.5 the function $(\bar{f}f - t^2) \mid S$ is not identically 0. Furthermore $(\bar{f}f - t^2) \mid S$ is analytic; hence $\sigma(E) = 0$ [Federer, p. 240] and hence by Theorem 1.7 $\mu(E) = 0$ which completes the proof of Corollary 1.9.

6. The proof of Corollary 1.11

6.1. We recall the following fact of the theory of Hausdorff measures.

6.2. PROPOSITION. *If* $\alpha \in (0, \infty)$, *if* $E \subset S$, *and if* $H^\alpha(E) = 0$, *then*

$$H^{\alpha+1}(\bigcup_{z \in T} zE) = 0.$$

6.3. Corollary 1.11 follows from Theorem 1.7, Proposition 6.2, and the fact that

$$H^{2n-1}(S)\sigma = \sigma(S)H^{2n-1}.$$

7. The proof of Theorem 1.13

7.1. Let $m \geq 1$. If w is the general point of W, then we let $dw\, d\bar{w}$ be the $2m$-dimensional Lebesgue measure on W. Furthermore we let $B(W) = B \cap W$ and $S(W) = S \cap W$. We define $\theta : [0, \infty) \to (0, \infty)$ by

$$\theta(t) = \sigma(S)/\int_{B(W)} (1 - \langle w, w \rangle)^t \, dw\, d\bar{w}.$$

The following proof (Proposition 7.2) is due to Rudin. It replaces an unnecessarily complicated proof of ours.

7.2. PROPOSITION. *If* $m \leq n - 1$ *and* $f \in F_+(W)$, *then*

$$\int f \circ P \, d\sigma = \theta(n - 1 - m) \int_{B(W)} f(w)(1 - \langle w, w \rangle)^{n-1-m} \, dw\, d\bar{w}.$$

Proof. We define $F : (0, \infty) \to [0, \infty]$ by

$$F(r) = \int_{B(0,r)} f(Px) \, dx\, d\bar{x}$$

and we define $G : (0, \infty) \to [0, \infty]$ by

$$G(r) = \int_{W \cap B(0,r)} f(w)(r^2 - \langle w, w \rangle)^{n-m} \, dw\, d\bar{w}.$$

By the Fubini theorem we have

$$F(r) = aG(r) \tag{7.1}$$

where the constant a is equal to the $(2n - 2m)$-dimensional Lebesgue measure of $B(V \ominus W)$. If $g \in F_+(V)$, then

$$\int_{B(0,r)} g(x) \, dx\, d\bar{x} = b \int_0^r \left[\int g(tx) \, d\sigma(x) \right] t^{2n-1} \, dt$$

where

$$b = 2n \int_B dx\, d\bar{x}/\sigma(S),$$

hence if f is continuous, then

$$F'(1) = b \int f \circ P \, d\sigma. \tag{7.2}$$

Likewise

$$G(r) = c \int_0^r \left[\int f(tw)(r^2 - \langle tw, tw \rangle)^{n-m} \, d\tau(w) \right] t^{2m-1} \, dt$$

where

$$c = 2m \int_{B(W)} dw \, d\bar{w} / \tau(S(W))$$

and (for the purpose of this proof) τ is the Radon measure on $S(W)$ which assigns to each open subset of $S(W)$ its Euclidean volume; hence if f is continuous, then

$$G'(1) = 2(n-m) \int_{B(W)} f(w)(1 - \langle w, w \rangle)^{n-m-1} \, dw \, d\bar{w}. \tag{7.3}$$

Proposition 7.2 follows from (7.1), (7.2), and (7.3).

7.3. Lemma. *If* $(z, f) \in \mathbf{D} \times F_+(V)$, *then*

$$|z|^{2n} \int_B f(zx) \, dx \, d\bar{x} \leq \int_B f(x) \, dx \, d\bar{x}.$$

Proof. For the purpose of the proof of Lemma 7.3 we define $t : V \to V$ by $t(x) = zx$. If $z \neq 0$, then

$$\begin{aligned} |z|^{2n} \int_B f(zx) \, dx \, d\bar{x} &= \int_B f(t(x)) \, | \det(t'_x) |^2 \, dx \, d\bar{x} \\ &= \int_{t(B)} f(x) \, dx \, d\bar{x} \leq \int_B f(x) \, dx \, d\bar{x}. \end{aligned}$$

7.4. Proposition. *If* $f \in L^1(B)$, *then*

$$\operatorname*{limit}_{z \in \mathbf{D}, z \to 1} \int_B | f(zx) - f(x) | \, dx \, d\bar{x} = 0.$$

Proof. Let $\varepsilon > 0$ and let g in $C(\bar{B})$ be such that

$$\int_B | f(x) - g(x) | \, dx \, d\bar{x} < \varepsilon. \tag{7.4}$$

If $z \in \mathbf{D}$, then by Lemma 7.3 and the inequality (7.4) we have

$$\int_B | f(zx) - f(x) | \, dx \, d\bar{x} \leq (|z|^{-2n} + 1)\varepsilon + \int_B | g(zx) - g(x) | \, dx \, d\bar{x},$$

hence

$$\limsup_{z \to 1} \int_B | f(zx) - f(x) | \, dx \, d\bar{x} \leq 2\varepsilon$$

which completes the proof of Proposition 7.4.

7.5. PROPOSITION. *Let* $m \leq n - 1$, *let* $\mu \in M(S, \mathbf{C})$, *and define*

$$f : S - W \rightarrow \mathbf{C}$$

by $f(x) = \mu^{\#}(Px)$. *If*

$$\mu^{\#}(y) = \mu^{\#}(Py) \tag{7.5}$$

for every y *in* B, *then*

$$d\mu = (1/\sigma(S))f\, d\sigma. \tag{7.6}$$

Furthermore if $\mu \neq 0$, *then* $f(x) \neq 0$ *for* σ *almost all* x *in* S.

Proof. Let Y be a linear subspace of V of complex dimension $n - 1$ that contains W and let Q be the orthogonal projection of V onto Y. We have

$$PQ = P. \tag{7.7}$$

If $y \in B$, then by (7.5) and (7.7), $\mu^{\#}(y) - \mu^{\#}(Py) = \mu^{\#}(PQy) = \mu^{\#}(Qy)$.

Thus if $z \in \mathbf{D}$ and $x \in S$, then

$$\mu^{\#}(zx) = \mu^{\#}(zQx), \tag{7.8}$$

hence by Proposition 2.10,

$$\int | \mu^{\#}(zQx) | \, d\sigma(x) \leq \sigma(S) \, | \mu | \, (S). \tag{7.9}$$

If $z \in \mathbf{D}$, then by Proposition 7.2 (with W replaced by Y),

$$\theta(0) \int_{B(Y)} | \mu^{\#}(zy) | \, dyd\bar{y} = \int | \mu^{\#}(zQx) | \, d\sigma(x);$$

hence by (7.9) and the Fatou-Lebesgue lemma,

$$\theta(0) \int_{B(Y)} | \mu^{\#}(y) | \, dyd\bar{y} \leq \sigma(S) \, | \mu | \, (S). \tag{7.10}$$

If $x \in X - Y$, then by (7.5) and (7.7),

$$f(x) = \mu^{\#}(Px) = \mu^{\#}(PQx) = \mu^{\#}(Qx); \tag{7.11}$$

hence by (7.9) and the Fatou-Lebesgue lemma, $\int | f | \, d\sigma \leq \sigma(S) \, | \mu | \, (S)$, and thus $f \in L^1(\sigma)$.

If $z \in \mathbf{D}$, then by (7.8), (7.11), and Proposition 7.2 (with W replaced by Y),

$$\int | \mu^{\#}(zx) - f(x) | \, d\sigma(x) = \theta(0) \int_{B(Y)} | \mu^{\#}(zy) - \mu^{\#}(y) | \, dyd\bar{y};$$

hence by (7.10) and Proposition 7.4 (with V replaced by Y),

$$\operatorname*{limit}_{z \in \mathbf{D}, z \rightarrow 1} \int | \mu^{\#}(zx) - f(x) | \, d\sigma(x) = 0. \tag{7.12}$$

If $(g, z) \in C(S) \times \mathbf{D}$, then by Proposition 2.9,

$$\frac{1}{\sigma(S)} \int g(y)\mu^{\#}(\bar{z}y)\, d\sigma(y) = \int g^{\#}(zx)\, d\mu(x);$$

hence by (7.12) and Proposition 2.8,

$$\frac{1}{\sigma(S)} \int gf\, d\sigma = \int g\, d\mu,$$

and hence (7.6) holds.

If

$$E = \{x : x \in S - Y, f(x) = 0\} \quad \text{and} \quad F = \{y : y \in B(Y), \mu^{\#}(y) = 0\},$$

then by (7.11) and Proposition 7.2

$$\sigma(E) = \theta(0) \int_F dy d\bar{y}. \tag{7.13}$$

If F is of positive $(2n - 2)$-dimensional Lebesgue measure, then (since $\mu^{\#}$ is analytic) $\mu^{\#} = 0$ on $B(Y)$; hence by (7.5) and Proposition 2.12 $\mu = 0$. Thus if $\mu \neq 0$, then F is of zero $(2n - 2)$-dimensional Lebesgue measure; hence by (7.13), $\sigma(E) = 0$ which completes the proof of Proposition 7.5.

7.6. We refer to Section 4.5 for the meanings of the terms that follow. For the purpose of the proof of Theorem 1.13 we let $G = \mathbf{T}$, $X = S$, and we define T by $T(z, x) = Px + \bar{z}(I - P)x$.

7.7. Proposition. *If* $(\mu, y) \in M(S, \mathbf{C}) \times B$, *then*

$$(\mu^*)^{\#}(y) = \frac{1}{2\pi} \int_0^{2\pi} \mu^{\#}(Py + e^{i\theta}(I - P)y)\, d\theta. \tag{7.14}$$

Proof. If $(z, x) \in \mathbf{T} \times S$, then $\beta(T(z, x), y) = \beta(x, Py + z(I - P)y)$. This fact and the proof of Proposition 4.10 will serve to prove Proposition 7.7.

7.8 Proposition. *If* $\mu \in M(S, \mathbf{C})$ *and* $\mu^{\#}$ *is pluriharmonic, then*

$$(\mu^*)^{\#}(y) = \mu^{\#}(Py) \tag{7.15}$$

for every y *in* B.

Proof. Since $\mu^{\#}$ is pluriharmonic the right side of the identity (7.14) is equal to the right side of (7.15) which completes the proof of Proposition 7.8.

7.9. Proposition. *Let* $m \leq n - 1$, *let* $\mu \in M(S, \mathbf{C})$, *and define*

$$f : S - W \to \mathbf{C}$$

by $f(x) = \mu^{\#}(Px)$. *If* $\mu^{\#}$ *is pluriharmonic, then*

$$d\mu^* = (1/\sigma(S)) f\, d\sigma. \tag{7.16}$$

Furthermore if $\mu^* \neq 0$, *then* $f(x) \neq 0$ *for* σ *almost all* x *in* S.

Proof. Proposition 7.9 follows from Proposition 7.8 and Proposition 7.5.

7.10. The first assertion of Theorem 1.13 follows from Proposition 7.9. The second assertion of Theorem 1.13 follows from Proposition 4.8 and Proposition 7.9. The third assertion of Theorem 1.13 follows from the fact that $\beta > 0$.

8. The proof of Theorem 1.15

8.1. PROPOSITION. *If* $\mu \in M(S, \mathbf{C})$, *if* $\mu^\#$ *is pluriharmonic, and if* $m \leq n - 1$, *then* $|\mu|(S(W)) = 0$.

Proof. If $E \subset S(W)$, then $\sigma(E) = 0$, hence by Theorem 1.13 $\mu(E) = 0$ from which fact Proposition 8.1 follows.

8.2. PROPOSITION. *If* $(Z, x, y) \in G(B) \times \bar{B} \times B$, *then*

$$\beta(Z(x), Z(y)) = \beta(x, y)\beta(Z(x), Z(0)).$$

Proof. This follows (by direct verification) from the definition of β and Proposition 2.2. If $x \in S$, then Proposition 8.2 also follows from Proposition 2.4.

8.3. PROPOSITION. *If* $\mu \in M(S, \mathbf{C})$, *if* $(Z, Y) \in G(B) \times G(B)$, *if* $Z \circ Y = I$, *and if* $y \in B$, *then*

$$[(\beta^{Z(0)} \circ Z)Y^*(\mu)]^\#(y) = \mu^\#(Z(y)).$$

Proof. By Proposition 8.2,

$$\begin{aligned}[(\beta^{Z(0)} \circ Z)Y^*(\mu)]^\#(y) &= \int \beta^y \beta^{Z(0)} \circ Z \, dY^*(\mu) \\ &= \int \beta^{Z(y)} \circ Z \, dY^*(\mu) = \int \beta^{Z(y)} d\mu = \mu^\#(Z(y)).\end{aligned}$$

8.4. We will now prove Theorem 1.15. If $Z \in G(B)$ and $Y = Z^{-1}$, then by Proposition 8.3, $[(\beta^{Z(0)} \circ Z)Y^*(\mu)]^\#$ is pluriharmonic; hence by Proposition 8.1,

$$|\mu|(Z(S(W))) = 0. \tag{8.1}$$

Furthermore if $(M, x, y) \in SL(2, \mathbf{R}) \times S(W^\perp) \times \bar{B}(W)$, then

$$\gamma(M, x, y) = (b/\bar{a})x + (1/\bar{a})y$$

where

$$M = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix};$$

thus if $Z = \gamma_{(M,x)}$, then

$$Z(S(W)) = [(b/\bar{a})x + W] \cap S. \tag{8.2}$$

Theorem 1.15 follows from (8.1) and (8.2).

9. The proof of Theorem 1.17

9.1. For the purpose of the proof of Theorem 1.17 we let $G = \mathbf{R}$, $X = S$, and we define T by $T(t, x) = e^{-t\tau}x$. We recall that the Poisson kernel of $\mathbf{H}$ is the function $\alpha:\mathbf{R} \times \mathbf{H} \to (0, \infty)$ defined by

$$\alpha(t, w) = \operatorname{Im}(1/(t - w));$$

we let $z \in \mathbf{H}$, and we let $d\lambda = (1/\pi)\alpha^z d\gamma$. (It is assumed that $\gamma((0, 1)) = 1$.)

9.2. Proposition. *If* $(\mu, y) \in M(S, \mathbf{C}) \times B$, *then*

$$(T^*(\lambda \times \mu))^{\#}(y) = \int \mu^{\#}(e^{t\tau}y)\, d\lambda(t). \tag{9.1}$$

Proof. If $(t, x) \in \mathbf{R} \times S$, then $\beta(T(t, x), y) = \beta(x, e^{t\tau}y)$. This fact and the proof of Proposition 4.10 will serve to prove Proposition 9.2.

9.3. Proposition. *If* $\mu \in M(S, \mathbf{C})$ *and* $\mu^{\#}$ *is pluriharmonic, then*

$$(T^*(\lambda \times \mu))^{\#}(y) = \mu^{\#}(e^{z\tau}y) \tag{9.2}$$

for every y *in* B.

Proof. Since $\mu^{\#}$ is pluriharmonic the right side of the identity (9.1) is equal to the right side of (9.2) which completes the proof of Proposition 9.3.

9.4. If Z is a topological space and if $f: Z \to \mathbf{C}$, then we will denote by $\operatorname{spt}(f)$ the support of f. We will denote by $C_{00}(Z)$ the class of all continuous functions $g: Z \to \mathbf{C}$ such that $\operatorname{spt}(g)$ is compact.

9.5. Proposition. *Let* $\tau \neq 0$, *let* N *be the null space of* τ, *let* $\mu \in M(S, \mathbf{C})$, *and define* $f: S - N \to \mathbf{C}$ *by* $f(x) = \mu^{\#}(e^{z\tau}x)$. *If* $\mu^{\#}$ *is pluriharmonic, then*

$$d(T^*(\lambda \times \mu)) = (1/\sigma(S)) f\, d\sigma.$$

Proof. If $w \in \mathbf{D}$, then by Proposition 9.3 and Proposition 2.10,

$$\int |\mu^{\#}(we^{z\tau}x)|\, d\sigma(x) \leq \sigma(S)\, |T^*(\lambda \times \mu)|\,(S);$$

hence by the Fatou-Lebesgue lemma

$$\int |f|\, d\sigma \leq \sigma(S)\, |T^*(\lambda \times \mu)|\,(S)$$

and hence $f \in L^1(\sigma)$.

If $(g, w) \in C(S) \times \mathbf{D}$, then by Proposition 9.3 and Proposition 2.9,

$$\frac{1}{\sigma(S)} \int g(y)\mu^{\#}(\bar{w}e^{z\tau}y)\, d\sigma(y) = \int g^{\#}(wx)\, d(T^*(\lambda \times \mu))(x).$$

Furthermore if $g \in C_{00}(S - N)$, then

$$\operatorname*{limit}_{w \in \mathbf{D}, w \to 1} \int g(y)\mu^{\#}(\bar{w}e^{z\tau}y)\, d\sigma(y) = \int gf\, d\sigma.$$

Thus if $g \in C_{00}(S - N)$, then by Proposition 2.8,

$$\int g\, d(T^*(\lambda \times \mu)) = \frac{1}{\sigma(S)} \int f\, d\sigma,$$

thus if $E \subset S - N$ and E is σ measurable, then

$$T^*(\lambda \times \mu)(E) = \frac{1}{\sigma(S)} \int_E f\, d\sigma.$$

Furthermore by Proposition 9.3, $(T^*(\lambda \times \mu))^{\#}$ is pluriharmonic, hence by Theorem 1.15, $| T^*(\lambda \times \mu) | (N \cap S) = 0$ which completes the proof of Proposition 9.5.

9.6. The first assertion of Theorem 1.17 follows from Proposition 9.5. The second assertion of Theorem 1.17 follows from Proposition 4.8 and Proposition 9.5. The third assertion of Theorem 1.17 follows from the fact that $\beta > 0$.

10. The proof of Corollary 1.20

10.1. Let $Y = Z^{-1}$ and let $\lambda = (\beta^{Z(0)} \circ Z)Y^*(\mu)$. If $y \in B$, then by Proposition 8.3,

$$\lambda^{\#}(y) = \mu^{\#}(Z(y)). \tag{10.1}$$

We define $g: S - N \to \mathbf{C}$ by $g(x) = \lambda^{\#}(e^{z\tau}x)$. If $x \in S - N$, then

$$Z(x) \in S - Z(S(N));$$

hence $(f \circ Z)(x) = \mu^{\#}((Z \circ e^{z\tau})(x))$. Therefore by (10.1),

$$(f \circ Z)(x) = \lambda^{\#}(e^{z\tau}x) = g(x). \tag{10.2}$$

Hence by Proposition 2.4,

$$\int | f | \beta^{Z(0)} d\sigma = \int | f \circ Z | \, d\sigma = \int | g | \, d\sigma;$$

hence by Theorem 1.17, $f \in L^1(\sigma)$.

If $t \in \mathbf{R}$, then $e^{t\tau}Y(v) = Y(v)$; hence if $x \in \bar{B}$, then

$$\beta(e^{t\tau}x, Y(v)) = \beta(x, e^{-t\tau}Y(v)) = \beta(x, Y(v)).$$

Furthermore $e^{t\tau}Y(E) = Y(E)$; hence by Theorem 1.17

$$\int_{Y(E)} \beta^{Y(v)}\, d\lambda = \frac{1}{\sigma(S)} \int_{Y(E)} \beta^{Y(v)} g\, d\sigma.$$

If h is the characteristic function of E, then $h \circ Z$ is the characteristic function of $Y(E)$; hence by (10.2) and Proposition 2.4,

$$\int_{Y(E)} \beta^{Y(v)} g \, d\sigma = \int h f \beta^{Y(v)} \circ Y \beta^{Z(0)} \, d\sigma. \tag{10.3}$$

If $(x, y) \in \bar{B} \times B$, then by Proposition 8.2, $\beta(x, y) = \beta(Y(x), Y(y)) \, \beta(x, Z(0))$; hence

$$\beta^v = \beta^{Y(v)} \circ Y \beta^{Z(0)}. \tag{10.4}$$

Hence by (10.3),

$$\int_{Y(E)} \beta^{Y(v)} g \, d\sigma = \int_E f \beta^v \, d\sigma.$$

Furthermore

$$\int_{Y(E)} \beta^{Y(v)} \, d\lambda = \int h \circ Z \beta^{Y(v)} \beta^{Z(0)} \circ Z \, dY^*(\mu)$$

$$= \int h \beta^{Y(v)} \circ Y \beta^{Z(0)} \, d\mu;$$

hence by (10.4),

$$\int_{Y(E)} \beta^{Y(v)} \, d\lambda = \int_E \beta^v \, d\mu,$$

and hence (1.1) holds.

The last assertion of Corollary 1.20 follows from the fact that $\beta > 0$.

10.2. Corollary 1.21 follows from Corollary 1.20 and Proposition 4.7.

Reference

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