CARDINAL SPLINE INTERPOLATION IN L_2

BY

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Let $m \geq 1$ be an integer and let S^m denote the class of cardinal spline functions of order m (degree $< m$), i.e., $S \in S^m$ if $S^{(m-2)}$ is a continuous piecewise linear function whose corners are in the set

$$\left\{j + \frac{m}{2} : j = 0, \pm 1, \pm 2, \ldots\right\}.$$

For $n \geq 1$ an integer, let $S_n^m = \{S \in S^m : S(x + n) = S(x) \text{ for all } x\}$.

Let $l_2(n)$ be the space of real n-tuples with the norm

$$\|y\|_2 = \left(\sum_1^n y_i^2\right)^{1/2}.$$

Let $\mathscr{L}_n^m : l_2(n) \to S_n^m$ be defined by

$$(\mathscr{L}_n^m y)(j) = y_j \quad \text{for } j = 1, \ldots, n.$$

A similar definition holds for $\mathscr{L}^m : l_2 \to S^m \cap L_2(-\infty, +\infty)$ where l_2 is the space of doubly-infinite square-summable sequences.

Richards (see reference) has used the functions

$$\psi_m(\theta) = \sin^m\left(\frac{\theta}{2}\right) \Big/ \left(\frac{\theta}{2}\right)^m \tag{1}$$

and

$$\phi_m(\theta) = \sum_{-\infty}^{+\infty} \psi_m(\theta + 2\pi j) \tag{2}$$

to prove:

THEOREM 1 (Richards). *Let $m > 0$ be even. Then*

$$\|\mathscr{L}_n^m\|_2 = \|\mathscr{L}^m\|_2 = 1. \tag{3}$$

More precisely,

$$\|\mathscr{L}_n^m y\|_2 \leq \|y\|_2 \quad \text{for } y \in l_2(n) \tag{4}$$

with equality if and only if $y_1 = y_2 = \cdots = y_n$ and

$$\|\mathscr{L}^m y\|_2 < \|y\|_2 \quad \textit{for } y \in l_2. \tag{5}$$

It is the purpose of this note to extend Richards' results to include the case: $m > 0$ an odd integer.

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In a private communication Richards has observed that

LEMMA. *The validity of* (3), (4), *and* (5) *for* $m > 1$ *odd depends upon the validity of the statement*

$$\phi_{2m}(\theta) \leq \phi_m^2(\theta) \quad \text{for } m > 1 \text{ odd and all } \theta. \tag{6}$$

This lemma is proved by replacing $2m$ in [1] by $2m - 1$ and considering which arguments remain valid. (The only argument which becomes invalid is the sentence including [Richards, Equation (28)].)

We now establish (6).

Set $v = \theta/2\pi$. Then, in view of periodicity and symmetry, (6) is equivalent to

$$\left[\sum_{-\infty}^{+\infty} (-1)^j (v + j)^{-m}\right]^2 \geq \sum_{-\infty}^{+\infty} (v + j)^{-2m} \tag{7}$$

for $0 < v \leq 1/2$ and $m > 1$ odd. Set

$$D_j = (2j - 1 - v)^{-m} - (2j - 1 + v)^{-m} - (2j - v)^{-m} + (2j + v)^{-m},$$

$$R_j = (2j - 1 - v)^{-2m} + (2j - 1 + v)^{-2m} + (2j - v)^{-2m} + (2j + v)^{-2m},$$

$$f(v) = v^{-m} + \sum_1^\infty D_j, \qquad g(v) = v^{-2m} + \sum_1^\infty R_j$$

and

$$L_j = [f(v) + v^{-m} - D_j]D_j + (D_j^2 - R_j).$$

Then the left member of (7) is $f^2(v)$ and the right member of (7) is $g(v)$. By direct expansion,

$$f^2(v) - g(v) = \sum_1^\infty L_j.$$

We shall show that $L_j > 0$.

By elementary calculus and the Binomial theorem,

$$D_j \geq [(2j - 1 - v)^{-m} - (2j - 1 + v)^{-m}](4j - 1)/(4j^2 - v^2)$$

$$\geq 2v(2j - 1 - v)^{1-m}(2j - 1 + v)^{1-m}(4j - 1)/(4j^2 - v^2).$$

Thus, $D_i > 0$ for each i so that

$$f(v) + v^{-m} - D_j > 2v^{-m}.$$

Also,

$$D_j^2 - R_j \geq -4(2j - 1 - v)^{-m}(2j - 1 + v)^{-m}.$$

Thus,

$$L_j \geq [v^{1-m}(2j - 1 - v)(2j - 1 + v)(4j - 1) - (4j^2 - v^2)]/C_j$$

with

$$C_j = (2j - 1 - v)^m(2j - 1 + v)^m(4j^2 - v^2)^{-1}/4 > 0.$$

Since

$$v^{1-m}(2j-1-v)(2j-1+v)(4j-1)-4j^2+v^2$$
$$\geq v^{-2}[(2j-1)^2-v^2](4j-1)-4j^2+v^2$$
$$\geq [(4j-2)^2-1](4j-1)-4j^2>0$$

we have $L_j > 0$.

Thus, (6) is valid for $m > 1$. A direct argument for $m = 1$ completes the proof of the following.

THEOREM 2. *For $m \geq 1$ an integer,*

$$\|\mathscr{L}_n^m\|_2 = \|\mathscr{L}^m\|_2 = 1,$$
$$\|\mathscr{L}^m y\|_2 < \|y\|_2 \quad \textit{for } y \in l_2$$

and

$$\|\mathscr{L}_n^m y\|_2 \leq \|y\|_2 \quad \textit{for } y \in l_2(n)$$

with equality if and only if $y_1 = y_2 = \cdots = y_n$.

REFERENCE

FRANKLIN RICHARDS, *Uniform spline interpolation operators in L_2*, Illinois J. Math,. vol. 18 (1974), pp. 516–521.

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