PERTURBATION OF A SELF ADJOINT DIFFERENTIAL OPERATOR

BY

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Introduction

It is well known that if $f(x) \in L^p[0, \pi]$, $1 < p < \infty$, then, for example, its Fourier sine series converges to $f(x)$ in L^p. The Fourier sine series can, of course, be regarded as the eigenfunction expansion of $f(x)$ with respect to the self adjoint differential operator given by $B(u) = -(d/dx)^2(u)$ with boundary conditions $u(0) = u(\pi) = 0$.

Suppose now that B is a self adjoint differential operator given by $B(u) = (-1)^n(d/dx)^{2m}(u)$, for u in the domain of B. Let K denote a differential operator of the form,

$$p_{2m-2}(x)\left(\frac{d}{dx}\right)^{2m-2} + p_{2m-3}(x)\left(\frac{d}{dx}\right)^{2m-3} + \cdots + p_0(x)$$

where the $p_i(x)$ are bounded on $[0, \pi]$.

It is known (see [1] and [6]) that the eigenfunction expansion for $B + K$ converges in L^p, $1 < p < \infty$. We show, for $2 \leq p < \infty$, that this is implied by the corresponding statement for B. We establish this result by using the gaps in $\sigma(B)$, the spectrum of B, to obtain estimates for

$$\|R(\lambda; B + K) - R(\lambda; B)\|_{L^p \to L^p},$$

the norm of the difference of the resolvent operators, for λ on certain contours in the complex plane. In [7] D. R. Smart also uses the notion of gaps in the spectrum to obtain basically the same perturbation result, although the method of estimating the operator norms is different than ours.

1. Preliminaries

Let L denote the ordinary differential operator, $L = (-1)^m(d/dx)^{2m}$, $x \in I = [0, \pi]$. (We assume L is of even order merely for convenience.) Let U_i, $i = 1, 2, \ldots, 2m$, be independent boundary conditions. Thus we may write

$$U_i(f) = \sum_{j=0}^{2m-1} (a_{ij}f^{(j)}(0) + b_{ij}f^{(j)}(\pi)),$$

a_{ij}, b_{ij} being constants. We assume the boundary conditions are self adjoint and therefore also regular. (See [6].)

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Given a positive integer n we define

$$H^n(I) = \left\{f\colon f \in C^{n-1}(I) \quad \text{and} \quad \left(\frac{d}{dx}\right)^n f \in L^2(I)\right\}.$$

Let B be the operator with domain

$$(B) = \{f\colon f \in H^{2m} \quad \text{and} \quad U_i f = 0, \quad i = 1, 2, \dots, 2m\},$$

with $Bf = Lf$. Since we are assuming the boundary conditions are regular, it follows from [5; p. 64] that the eigenvalues, μ_n, of B satisfy the following asymptotic relationship:

$$\mu_n = n^{2m}\left\{1 + \frac{A}{n} + O(n^{-3/2})\right\}; \ A \text{ is a constant.} \tag{1.1}$$

For $f \in H^n$, n a nonnegative integer, one defines a norm

$$\|f\|_{H^n} = \left(\sum_{j=0}^{n} \left\|\left(\frac{d}{dx_j}\right)^j f\right\|_{L^2}^2\right)^{1/2}.$$

If s is an arbitrary positive number, we define the space H^s as an interpolation space (see [2]) and there is the following known result:

LEMMA 1.1. *If T is a bounded linear mapping from H^{s_1} to H^{t_1} with norm P, and also a bounded mapping from H^{s_2} to H^{t_2} with norm Q, then for $0 < \alpha < 1$, T is a bounded mapping from $H^{\alpha s_1+(1-\alpha)s_2}$ to $H^{\alpha t_1+(1-\alpha)t_2}$ with norm $\leq P^\alpha Q^{1-\alpha}$.*

LEMMA 1.2 (See [4, p. 1686]). *$H^s \subset L^p$ continuously if $1/p > 1/2 - s$.*

2. Estimates of operator norms

Let K be the operator with the same domain as B and

$$Kf = p_{2m-2}(x)\left(\frac{d}{dx}\right)^{2m-2} f + \cdots + p_0(x)f,$$

where the $p_i(x)$ are bounded.

LEMMA 2.1. *Consider the vertical line*

$$l_n = \left\{\lambda\colon \operatorname{Re}\lambda = \frac{\mu_n + \mu_{n+1}}{2}\right\}.$$

Then for sufficiently large n, $l_n \subset \rho(B + K)$, the resolvent set of $B + K$.

Proof. It suffices to show that for λ on l_n, $R(\lambda; B)$ exists and

$$\|K \circ R(\lambda; B)\|_{H^0 \to H^0} < 1,$$

for then we would have that $[I - K \circ R(\lambda; B)]^{-1}$ exists and could invoke the identity:

$$R(\lambda; B + K) = R(\lambda; B)[I - K \circ R(\lambda; B)]^{-1}.$$

We will obtain bounds for $\|K \circ R(\lambda; B)\|_{H^0 \to H^0}$ by means of bounds for $\|R(\lambda; B)\|_{H^0 \to H^{2m-2}}$ (i.e., the norm of the operator $R(\lambda; B)$ from the space $H^0 = L^2$ to the space H^{2m-2}) and bounds for $\|K\|_{H^{2m-2} \to H^0}$.

First let us deal with the operator K. We observe that

$$\left\| \left(\frac{d}{dx}\right)^r f(x) \right\|_{H^s} \leq \|f(x)\|_{H^{r+s}} \quad (r, s \text{ positive integers}).$$

Since we are assuming $p_i(x)$, $i = 0, 1, \ldots, 2m - 2$ is bounded (say $|p_i(x)| < M$) it follows easily that

$$\|K\|_{H^{2m-2} \to H^0} \leq (2m - 1)M. \tag{2.1}$$

We consider $R(\lambda; B)$ for $\lambda = (\mu_n + \mu_{n+1})/2 + iy \in l_n$. Since B is self adjoint the Spectral Theorem (see [4]) tells us that $\|R(\lambda; B)\|_{H^0 \to H^0}$ is the reciprocal of the distance from λ to $\sigma(B)$. We thus have that

$$\|R(\lambda; B)\|_{H^0 \to H^0} = \left[\left(\frac{\mu_{n+1} - \mu_n}{2}\right)^2 + y^2\right]^{-1/2}.$$

For n sufficiently large, we see from (1.1) that

$$\|R(\lambda; B)\|_{H^0 \to H^0} \leq C[(n^{2m-1})^2 + y^2]^{-1/2}. \tag{2.2}$$

It is well known, [4, p. 1744] that $R(\lambda; B)$ is a bounded mapping from H^0 to H^{2m}. We want to estimate the norm of this mapping. By (1.1) we can choose λ_0 real so that $\mu_n > \lambda_0$ for all n. We now express $R(\lambda; B)$ as

$$R(\lambda; B) = R(\lambda_0; B) \circ (\lambda_0 I - B) \circ R(\lambda; B).$$

We have

$$\begin{aligned} \|R(\lambda; B)\|_{H^0 \to H^{2m}} &= \|R(\lambda_0; B) \circ (\lambda_0 I - B) \circ R(\lambda; B)\|_{H^0 \to H^{2m}} \\ &\leq \|R(\lambda_0; B)\|_{H^0 \to H^{2m}} \cdot \|(\lambda_0 I - B) \circ R(\lambda; B)\|_{H^0 \to H^0}. \end{aligned}$$

Spectral theory tells us that

$$\|(\lambda_0 I - B) \circ R(\lambda; B)\|_{H^0 \to H^0} = \sup_{Z \in \sigma(B)} \left|\frac{\lambda_0 - z}{\lambda - z}\right|.$$

We claim that for $\lambda \in l_n$,

$$\sup_{Z \in \sigma(B)} \left|\frac{\lambda_0 - z}{\lambda - z}\right| \leq cn.$$

To see this we note that $\sigma(B)$ is real since B is self adjoint, and therefore for $z \in \sigma(B)$,

$$\left|\frac{\lambda_0 - z}{\lambda - z}\right| \leq \left|\frac{\lambda_0 - z}{\operatorname{Re} \lambda - z}\right|.$$

Thus we can confine our attention to the point $\lambda^* = (\mu_n + \mu_{n+1})/2$. As $z \uparrow \infty$, $|(\lambda_0 - z)/(\lambda^* - z)|$ is monotonically increasing in the interval (λ_0, λ^*) and monotonically decreasing in the interval $(\lambda^*, +\infty)$. It follows easily that

$$\sup_{Z \in \sigma(B)} \left|\frac{\lambda_0 - z}{\lambda^* - z}\right| = \left|\frac{\lambda_0 - \mu_{n+1}}{\lambda^* - \mu_{n+1}}\right| \sim \frac{(n + 1)^{2m}}{(n + 1)^{2m-1}} \leq cn.$$

Thus

$$\|R(\lambda; B)\|_{H^0 \to H^{2m}} \leq c'n, \quad \lambda \in l_n. \tag{2.3}$$

We now invoke Lemma 1.1, taking $s_1 = 0, s_2 = 0, t_1 = 0, t_2 = 2m, \alpha = 1/m$, together with (2.2) and (2.3) to obtain:

$$\|R(\lambda; B)\|_{H^0 \to H^{2m-2}} \leq c[(n^{2m-1})^2 + y^2]^{-1/2m} \cdot n^{(m-1)/m}. \tag{2.4}$$

The right hand side of (2.4) clearly converges to zero as $n \uparrow \infty$. This fact and (2.1) show that $\|KR(\lambda; B)\|_{H^0 \to H^0} < 1$ for $\lambda \in l_n$ if n is sufficiently large. As noted at the beginning of the proof, this establishes the lemma.

LEMMA 2.2. *Let R_n be the square in the complex plane with center at the origin and sides parallel to the axes, and whose right hand vertical side passes through the point $\lambda = (\mu_n + \mu_{n+1})/2$. Then given $\varepsilon > 0$ and $0 \leq t < 1/(2m+1)$, we have that for sufficiently large n, $R_n \subset \rho(B + K)$ and*

$$\int_{R_n} \|R(\lambda; B + K) - R(\lambda; B)\|_{H^0 \to H^{2mt}} |d\lambda| < \varepsilon. \tag{2.5}$$

Proof. We will deal only with the right hand side of R_n, the computations for the other sides being similar, but easier. We note that the right hand side of R_n lies on l_n, which by Lemma 2.1, $\subset \rho(B + K)$. It will thus suffice to establish that for $0 \leq t < 1/(2m + 1)$,

$$\int_{l_n} \|R(\lambda; B + K) - R(\lambda; B)\|_{H^0 \to H^{2mt}} |d\lambda| \to 0 \quad \text{as } n \uparrow \infty. \tag{2.6}$$

We first estimate $\|R(\lambda; B + K) - R(\lambda; B)\|_{H^0 \to H^0}$, $\lambda \in l_n$. We have that

$$R(\lambda; B + K) - R(\lambda; B) = R(\lambda; B)(I - K \circ R(\lambda; B))^{-1} - R(\lambda; B).$$

We have shown that

$$\|K \circ R(\lambda; B)\|_{H^0 \to H^0} \to 0 \quad \text{as } n \uparrow \infty.$$

Now when $\|K \circ R(\lambda; B)\|_{H^0 \to H^0} < 1$ we can expand $(I - K \circ R(\lambda; B))^{-1}$ in a Neumann series, [2, p. 140]. Thus, $(I - K \circ R(\lambda; B))^{-1} = \sum_{j=0}^{\infty} (K \circ R(\lambda; B))^j$, with convergence in the uniform $H^0 \to H^0$ operator topology. So we have

$$\begin{aligned}
&\|R(\lambda; B)(I - K \circ R(\lambda; B))^{-1} - R(\lambda; B)\|_{H^0 \to H^0} \\
&\qquad = \left\| R(\lambda; B) \sum_{j=1}^{\infty} (K \circ R(\lambda; B))^j \right\|_{H^0 \to H^0} \\
&\qquad \leq \|R(\lambda; B)\|_{H^0 \to H^0} \left\| \sum_{j=1}^{\infty} (K \circ R(\lambda; B))^j \right\|_{H^0 \to H^0} \\
&\qquad \leq \|R(\lambda; B)\|_{H^0 \to H^0} \sum_{j=1}^{\infty} \|K \circ R(\lambda; B)\|^j_{H^0 \to H^0} \\
&\qquad \leq c\|R(\lambda; B)\|_{H^0 \to H^0} \|K \circ R(\lambda; B)\|_{H^0 \to H^0}.
\end{aligned}$$

Using (2.1), (2.2), and (2.4), we therefore obtain

$$\|R(\lambda; B + K) - R(\lambda; B)\|_{H^0 \to H^0} \leq c[(n^{2m-1})^2 + y^2]^{-(1+1/m)/2} \cdot n^{(m-1)/m}. \tag{2.7}$$

Now let us estimate $\|R(\lambda; B)(I - K \circ R(\lambda; B)^{-1} - R(\lambda; B)\|_{H^0 \to H^{2m}}$. We have by (2.3),

$$\|R(\lambda; B) \circ (I - K \circ R(\lambda; B))^{-1} - R(\lambda; B)\|_{H^0 \to H^{2m}}$$
$$\leq \|R(\lambda; B)\|_{H^0 \to H^{2m}} \cdot \|(I - K \circ R(\lambda; B))^{-1}\|_{H^0 \to H^0} + \|R(\lambda; B)\|_{H^0 \to H^{2m}} \leq cn.$$

We deduce

$$\|R(\lambda; B + K) - R(\lambda; B)\|_{H^0 \to H^{2m}} \leq cn. \tag{2.8}$$

Interpolating between (2.7) and (2.8) we get for $0 < t < 1$,

$$\|(R(\lambda; B + K) - R(\lambda; B)\|_{H^0 \to H^{2mt}} \leq c[[(n^{2m-1})^2 + y^2]^{-(1+1/m)/2}]^{1-t} \cdot n^t. \tag{2.9}$$

Letting $y = n^{2m-1}y'$ we have that

$$n^{((m-1)/m)(1-t)+t} \int_{-\infty}^{\infty} [(n^{2m-1})^2 + y^2]^{-(1+1/m)(1-t)/2}\, dy$$
$$= n^{((m-1)/m)(1-t)+t-(2m-1)(1+1/m)(1-t)+2m-1}$$
$$\times \int_{-\infty}^{\infty} [1 + (y')^2]^{-(1+1/m)(1-t)/2}\, dy'.$$

Now for $t < 1/(2m + 1)$ we have that the integral on the right converges and that the exponent of n outside the integral is negative. We thus conclude that for $t < 1/(2m + 1)$, (2.6) holds. This concludes the proof of the lemma.

3. Main result

THEOREM 3.1. *Let $2 \leq p < \infty$. If for any $f(x) \in L^p(I)$ the eigenfunction expansion of $f(x)$ with respect to B converges to $f(x)$ in L^p, then the same is true for the eigenfunction expansion with respect to $B + K$.*

Proof. Define $T_{p,2}$ to be the restriction map from $L^p(I)$ to $L^2(I) = H^0(I)$, which is continuous by Holder's inequality. Using Lemma 1.2, we define $S_{1/2,p}$ to be the bounded restriction mapping from $H_{1/2}$ to L^p. Using the identity,

$$[R(\lambda; B)]_{L^p \to L^p} = S_{1/2,p} \circ [R(\lambda; B)]_{L^2 \to H^{1/2}} \circ T_{p,2},$$

and a similar identity involving $R(\lambda; B + K)$, we see that both $R(\lambda; B)$ and $R(\lambda; B + K)$ can be regarded as bounded mappings from L^p to L^p. Now, as is shown in [7], since

$$\frac{1}{2\pi i} \int_{R_n} R(\lambda; B)\, d\lambda \quad \text{and} \quad \frac{1}{2\pi i} \int_{R_n} R(\lambda; B + K)\, d\lambda$$

represent the partial sum operators for the eigenfunction expansion for B and $B + K$ respectively, to demonstrate the theorem, it suffices to prove,

$$\lim_{n \to \infty} \int_{R_n} \|R(\lambda; B + K) - R(\lambda; B)\|_{L^p \to L^p} |d\lambda| = 0. \tag{3.1}$$

We have

$$\|R(\lambda; B + K) - R(\lambda; B)\|_{L^p \to L^p} \le \|S_{1/2, p}\| \cdot \|R(\lambda; B + K) - R(\lambda; B)\|_{L^2 \to H^{1/2}} \cdot \|T_{p, 2}\|.$$

Lemma 2.2 tells us that

$$\lim_{n \to \infty} \int_{R_n} \|R(\lambda; B + K) - R(\lambda; B)\|_{L^2 \to H^{1/2}} |d\lambda| = 0.$$

This clearly implies (3.1). Q.E.D.

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