THE CONTINUUM HYPOTHESIS, INTEGRATION AND DUALS OF MEASURE SPACES

BY

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Introduction

Let S be a set, Σ a σ-algebra of subsets of S, X a Banach space having the Radon-Nikodym property. Let $bv(S, \Sigma, X)$ be the space of all countably additive functions from Σ into X which are of bounded variation. An integral of a function Ψ from Σ into X^* with respect a function μ from Σ into X is defined by taking the limit over the net of Σ-subdivisions of S. It is shown that if cardinality of $ca(S, \Sigma) \leqq 2^{\aleph_0}$ and $2^{\aleph_0} = \aleph_1$, then for each linear functional T on $bv(S, \Sigma, X)$ there is a function Ψ from Σ into X^* such that $T(\mu) = \int_S \Psi \, d\mu$, for all μ in $bv(S, \Sigma, X)$. Also, a space of functions from Σ in X^* is constructed which is linearly isometric to the dual of $bv(S, \Sigma, X)$ via this representation.

2. Results

This paper may be considered as a generalization of [2].

Let S be a set, Σ a σ-algebra of subsets of S and let $ca(S, \Sigma)$ be the space of all real-valued countably additive set functions on Σ. Let H be a maximal subset of $ca(S, \Sigma)$ consisting of mutually singular positive measures.

Let X be a Banach space. Let $bv(S, \Sigma, X)$ be the space of all countably additive functions from Σ into X which are of bounded variation. We consider $bv(S, \Sigma, X)$ as a Banach space under the variation norm.

THEOREM 1. *The space $bv(S, \Sigma, X)$ is isometrically isomorphic to the substitution space $P_{l_1(H)}N_\mu$, where for each μ in H, N_μ is the subspace of $bv(S, \Sigma, X)$ consisting of all measures which are absolutely continuous with respect to μ* [1, p. 31].

Proof. Suppose $\omega \in bv(S, \Sigma, X)$. Then there is a positive measure λ in $ca(S, \Sigma)$ such that $\omega \ll \lambda$. It follows that there is a sequence $\{\mu_n\}_{n=1}^{\infty}$ from H and a sequence $\{\lambda_n\}_{n=1}^{\infty}$ of nonnegative measures such that (1) for each n, $\lambda_n \ll \mu_n$ and (2) $\|\lambda - \sum_{i=1}^{n} \lambda_i\| \to 0$.

Therefore, there is a sequence of disjoint sets $\{B_i\}_{i=1}^{\infty}$ in Σ such that λ_i is concentrated on B_i and $\lambda_j(B_i) = 0$ if $j \neq i$, for all i and j. For each i and each set E in Σ, let $\omega_i(E) = w(E \cap B_i)$. We have

(1) $\omega_i \ll \lambda_i \ll \mu_i$, for each i,
(2) $\|\omega - \sum_{i=1}^{n} \omega_i\| \to 0$, and
(3) $\|\omega\| = \sum_{i=1}^{\infty} \|\omega_i\|$.

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Now, if $f \in P_{l_1(H)}N\mu$, then f is a function from H into $bv(S, \Sigma, X)$ such that for each μ in H, $f(\mu) \in N_\mu$ and $\sum_{\mu \in H} \|f(\mu)\| < \infty$. So, for each such f, let $\phi(f) = \sum_{\mu \in H} f(\mu)$. The mapping ϕ is linear and $\|\phi(f)\| = \|f\|$, and the previous paragraph shows that ϕ is onto.

As a consequence of Theorem 1 and the theory of substitution spaces [1, p. 31], we have:

THEOREM 2. *The dual of $bv(S, \Sigma, X)$ is isometrically isomorphic to the substitution space $P_{l_\infty(H)}N^*_\mu$.*

One of the main objectives of this paper is to represent the dual of $bv(S, \Sigma, X)$ by an integral. This will be done provided X has the Radon-Nikodym property. However, even under this assumption, it seems that the problem of an integral representation is tied up with some problems of general set theory.

For the remainder of this paper two types of integrals are used. First, let us make the following conventions. If $E \in \Sigma$, then we say that D subdivides E provided D is a finite collection of disjoint sets in Σ filling up E and we say that D' refines D provided D' subdivides E and each set in D' is a subset of some set in D.

Let ω map Σ into X, ν map Σ into X^*, and μ map Σ into the nonnegative numbers.

DEFINITION 1. The statement that the form $(d\nu\, d\omega)/d\mu$ is Hellinger integrable over a set E in Σ means there is a number k such that if $\varepsilon > 0$, then there is a subdivision D of E such that if D' refines D, then

$$\left| \sum_{D'} \frac{\nu(B)\omega(B)}{\mu(B)} - k \right| < \varepsilon, \tag{$*$}$$

where the sum is taken over all sets B in D' with $\mu(B) > 0$.

In case the form $(d\nu\, d\omega)/d\mu$ is integrable, the number k of $(*)$ may be denoted by $\int_E (d\nu\, d\omega)/d\mu$.

DEFINITION 2. The statement that the form $\nu\, d\omega$ is integrable over a set E in Σ means there is a number k such that if $\varepsilon > 0$, then there is a subdivision D of E such that if D' refines D, then

$$\left| \sum_{D'} (\nu(B))\omega(B) - k \right| < \varepsilon, \tag{$+$}$$

where the sum is taken over all sets B in D'.

In case the form $\nu\, d\omega$ is integrable, the number k of $(+)$ may be denoted by $\int_E \nu\, d\omega$.

We note that both integrals are linear in ν and in ω. Also, the forms $(d\nu\, d\omega)/d\mu$ and $\nu\, d\omega$ are integrable if and only if they satisfy a corresponding Cauchy type condition.

THEOREM 3. *Suppose $\{\omega_n\}_{n=1}^{\infty}$ is a sequence of functions from Σ into X which are of bounded variation and there is a positive number α such that $\|v(A)\| \leq \alpha\mu(A)$ for every A in Σ with $\mu(A) > 0$. If $\|\omega_n - \omega\| \to 0$ and for each n, $(dv\, d\omega_n)/d\mu$ is integrable, then $(dv\, d\omega)/d\mu$ is integrable. Moreover,*

$$\int_S \frac{dv\, d\omega_n}{d\mu} \to \int_S \frac{dv\, d\omega}{d\mu} \quad \text{and} \quad \left|\int_S \frac{dv\, d\omega}{d\mu}\right| \leq \alpha\|\omega\|.$$

Proof. If D is a subdivision of S, then

$$\begin{aligned}\left|\sum_D \frac{v(B)\omega_n(B)}{\mu(B)} - \sum_D \frac{v(B)\omega(B)}{\mu(B)}\right| &\leqq \sum_D \left|\frac{v(B)(\omega_n(B) - \omega(B))}{\mu(B)}\right| \\ &\leqq \sum_D \frac{\|v(B)\|}{\mu(B)} \|\omega_n(B) - \omega(B)\| \\ &\leq \alpha \sum_D \|\omega_n(B) - \omega(B)\| \\ &\leq \alpha\|\omega_n - \omega\|.\end{aligned}$$

Let $\varepsilon > 0$. Let N be a positive integer so that if $n \geqq N$, then $\|\omega_n - \omega\| < \varepsilon/3\alpha$. Let D' be a subdivision of S such that if D' refines D, then

$$\left|\sum_{D'} \frac{v(B)\omega_N(B)}{\mu(B)} - \sum_D \frac{v(B)\omega_N(B)}{\mu(B)}\right| < \frac{\varepsilon}{3}.$$

It can now be shown that

$$\left|\sum_{D'} \frac{v(B)\omega(B)}{\mu(B)} - \sum_D \frac{v(E)\omega(E)}{\mu(E)}\right| < \varepsilon.$$

Thus, the form $(dv\, d\omega)/d\mu$ is integrable. Also, since each approximating sum to $\int_S (dv\, d\omega)/d\mu$ is bounded in absolute value by $\alpha\|\omega\|$, we have

$$\left|\int_S \frac{dv\, d\omega}{d\mu}\right| \leq A\|\omega\|.$$

Thus, $\int_S (dv\, d\omega_n)/d\mu$ converges to $\int_S (dv\, d\omega)/d\mu$.

In a similar fashion, one can prove:

THEOREM 4. *Suppose that for each n, the form $v\, d\omega_n$ is integrable and v is bounded. If for each n, ω_n is of bounded variation and $\|\omega_n - \omega\| \to 0$, then $v\, d\omega$ is integrable. Moreover,*

$$\left|\int_S v\, d\omega\right| \leq \sup\{v(E) \mid E \text{ in } \Sigma\}\|\omega\| \quad \textit{and} \quad \int_S v\, d\omega_n \to \int_S v\, d\omega.$$

DEFINITION 3. If μ is a positive measure in $ca(S, \Sigma)$, then let $H(\Sigma, \mu, X^*)$, be the space of all additive functions v from Σ into X^* for which there is a number α such that $\|v(E)\| \leq \alpha\mu(E)$, for every E in Σ. Also, for each $v \in H(\Sigma, \mu, X^*)$, let

$$\|v\| = \sup\left\{\frac{\|v(E)\|}{\mu(E)} : \mu(E) > 0\right\}.$$

The space $H(\Sigma, \mu, X^*)$ is a Banach space [3].

We will need the following theorem of J. J. Uhl [3].

THEOREM 5. *Suppose X has the Radon-Nikodym property and μ is a positive measure in $ca(S, \Sigma)$. Then for each $T \in N_\mu^*$, there is only one function v in $H(\Sigma, \mu, X^*)$ such that*

$$T(\omega) = \int_S \frac{dv\, d\omega}{d\mu}, \tag{U}$$

for all ω in $N\mu$. Moreover, if (U) *holds, then $|T| = \|v\|$ and the mapping of N_μ^* into $H(\Sigma, \mu, X^*)$ defined by* (U) *is onto.*

We give an argument here for completeness.

Proof. For each A in Σ and $x \in X$, let $\omega_{(x,A)}(E) = x\mu(E \cap A)$, for all $E \in \Sigma$. Of course, $\omega_{x,A} \in N_\mu$. Let $(v(A))(x) = T(\omega_{(x,A)})$. For each A, $v(A) \in X^*$, since $v(A)$ is linear and

$$|v(A)(x)| = |T(\omega_{(x,A)})| \leq |T|\, \|\omega_{(x,A)}\| \leq |T|\, \|x\|\mu(A).$$

It follows that v is an additive function from Σ into X^*, v is bounded by $|T|\mu(S)$ and for each A, $\|v(A)\| \leq |T|\mu(A)$.

We now show that $T(\omega_{(x,A)}) = \int_S (dv\, d\omega)/d\mu$.

Let D' be a refinement of the subdivision $\{A, A'\}$. Let $\{B_i\}_{i=1}^n$ be all the members of D' which are subsets of A and having positive μ measure. Then

$$\sum_{D'} \frac{(v(B))(\omega_{(x,A)}(B))}{\mu(B)} = \sum_{i=1}^n \frac{(v(B_i))\omega_{(x,A)}(B_i)}{\mu(B_i)}.$$

Since

$$\omega_{(x,A)} = \sum_{j=1}^n \omega_{(x,B_j)},$$

we have

$$\begin{aligned}
\sum_{D'} \frac{v(B)\omega_{(x,A)}(B)}{\mu(B)} &= \sum_{i=1}^n \frac{(v(B_i))((\sum_{j=1}^n \omega_{(x,B_j)})(B_i))}{\mu(B_i)} \\
&= \sum_{i=1}^n \frac{(v(B_i))(\omega_{(x,B_i)}(B_i))}{\mu(B_i)} \\
&= \sum_{i=1}^n \frac{(v(B_i))(x\mu(B_i))}{\mu(B_i)} \\
&= v(A)(x) \\
&= T(\omega_{(x,A)}).
\end{aligned}$$

Thus, $T(\omega) = \int_S (dv\, d\omega)/d\mu$, for all ω in J, the span of the set of all $\omega_{(x,A)}$. But, X has the Radon-Nikodym property means that J is dense in N_μ. Thus, it follows from Theorem 3 that v has the properties listed in the conclusion of Theorem 5.

If v' also has these properties, then for every $x \in X$ and $A \in \Sigma$, $(v(A))(x) = T(\omega_{x,A}) = \int_S (dv'\, d\omega_{x,A})/d\mu = (v'(A))(x)$. Thus, v is unique.

Now, suppose $v \in H(\Sigma, \mu, X^*)$. It follows from the earlier part of the argument for this theorem that for each $x \in X$ and $A \in \Sigma$, the form $(dv\, d\omega_{(x,A)})/d\mu$ is integrable. Again, since X has the Radon-Nikodym property, the space J, the span of all the measures $\omega_{x,A}$ is dense in N_μ. Therefore, by linearity of the form and by Theorem 3, the form $(dv\, d\omega)/d\mu$ is integrable for all $\omega \in N_\mu$ and thus the mapping of N_μ^* into $H(\Sigma, \mu, X^*)$ defined by (U) is onto.

From this point on, it is assumed that X has the Radon-Nikodym property.

The remainder of this paper is based upon the following idea. Given $T \in bv(S, \Sigma, X)$, we will restrict T to the subspace N_μ for each μ in H and obtain the corresponding function v. Then we will "paste together", the functions v/μ, $u \in H$ to obtain Ψ from Σ to X^* so that T may be represented by the form $\Psi\, d\omega$, for all ω in $bv(S, \Sigma, X)$.

In order to describe how these functions are to be pasted together, let μ_1 be some member of H and let Γ be an indexing set for $H - \{\mu_1\}$. Thus, $H = \{\mu_1\} \cup \{\mu_\gamma \mid \gamma \in \Gamma\}$.

DEFINITION 4. A function ϕ from $\{\mu_\gamma \mid \gamma \in \Gamma\}$ into Σ will be said to be a selector for the set $\{\mu_\gamma \mid \gamma \in \Gamma\}$ provided

(1) for each $\gamma \in \Gamma$, μ_γ is concentrated on $B_\gamma = \phi(u_\gamma)$ and μ_1 is concentrated on B'_γ,

(2) if α and γ are in Γ and $B \subseteq B_\gamma \cap B_\alpha$, $\mu_\gamma(B) > 0$, and $\mu_\alpha(B) > 0$, then $\alpha = \gamma$.

THEOREM 6. *If the cardinality of* Γ *is* $\leq \aleph_1$, *then there is a selector for the set* $\{\mu_\gamma \mid \gamma \in \Gamma\}$.

Proof. Let us assume that the set Γ consists of all ordinals α, $2 \leq \alpha < \omega_1$. For each γ and α, $1 \leq \gamma < \alpha < \omega_1$, let $B_{\gamma\alpha}$ be a set in Σ such that $\mu_\gamma(B_{\gamma\alpha}) = 0$ and $\mu_\alpha(B'_{\gamma\alpha}) = 0$. For each α, $1 < \alpha < \omega_1$, let $B_\alpha = \bigcap_{\gamma<\alpha} B_{\gamma\alpha}$. Since each proper initial segment of Γ is countable, we have

$$\mu_\gamma(B_\alpha) = 0 \quad \text{and} \quad \mu_\alpha(B'_\alpha) = 0.$$

So, for each $\gamma \in \Gamma$, μ_γ is concentrated on B_γ and μ_1 is concentrated on B'_γ. Also, if α and γ are in Γ and $B \subseteq B_\gamma \cap B_\alpha$ and $\mu_\alpha(B) > 0$ and $\mu_\gamma(B) > 0$, then $\alpha = \gamma$. Thus, if ϕ is defined by setting $\phi(\mu_\gamma) = B_\gamma$, then ϕ is a selector for $\{\mu_\gamma \mid \gamma \in \Gamma\}$.

Question. If there is a selector for the set $\{\mu_\gamma \mid \gamma \in \Gamma\}$, does it follow that the cardinality of Γ is $\leqq \aleph_1$?

DEFINITION 5. Let μ be a positive measure in $ca(S, \Sigma)$, and ω a function from Σ into a normed linear space Y, and let $B \in \Sigma$. The function ω is said to be μ-additive over B provided that if B_1 and B_2 are disjoint subsets of B, $u_1(B) > 0$ and $\mu(B_2) > 0$, then $\omega(B_1) + \omega(B_2) = \omega(B_1 \cup B_2)$.

THEOREM 7. *Suppose the set* $\{\mu_\gamma \mid \gamma \in \Gamma\}$ *has a selector* ϕ, *with* $\phi(\mu_\alpha) = B_\alpha$, *and* $T \in bv^*(S, \Sigma, X)$. *Then there is only one function* Ψ *from* Σ *into* X^* *such that:*

(1) $\mu_1\Psi$ *is* μ_1*-additive on* S.

(2) $\mu_\alpha\Psi$ *is* μ_α*-additive on* B_α, *for each* $\alpha \in T$.

(3) *If* $\mu_1(B) = 0$ *and there is no* $\gamma \in \Gamma$ *such that both* $\mu_\gamma(B) > 0$ *and* $B \subseteq B_\gamma$, *then* $\Psi(B) = 0$.

(4) (R): $T(\omega) = \int_S \Psi \, d\omega$, *for all* ω *in* $bv(S, \Sigma, X)$.

(5) $|T| = \sup \{|\Psi(B)| : B \in \Sigma\}$.

Proof. Let v_γ be the unique function from Σ into X^* having the properties listed in the conclusion of Theorem 5, for each $\gamma \in \Gamma$ and for 1.

It follows from the properties of a selector, the function Ψ described below is well defined:

$$\Psi(B) = \begin{cases} \dfrac{v_1(B)}{\mu_1(B)} & \text{if } \mu_1(B) > 0 \\ \dfrac{v_\gamma(B)}{\mu_\gamma(B)} & \text{if } B \subseteq B_\gamma \text{ and } \mu_\gamma(B) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that the function Ψ has properties (1), (2), and (3) stated in the conclusion of Theorem 7.

We show that Ψ represents T.

First, assume $\omega \in N_\gamma$, for some $\gamma \in \Gamma$. Then $T(\omega) = \int_S (dv_\gamma \, d\omega)/d\mu_\gamma$. Since μ_γ is concentrated on B_γ, $T(\omega) = \int_{B_\gamma} (dv_\gamma \, d\omega)/d\mu_\gamma$ and ω is zero on all sets lying in B'_γ. Let $\varepsilon > 0$; there is a subdivision D of B_γ such that if D' refines D, then

$$\left| T(\omega) - \sum_{D'} \frac{v_\gamma(B)\omega(B)}{\mu_\gamma(B)} \right| < \varepsilon.$$

If $\omega(B) \neq 0$, then $\mu_\gamma(B) > 0$ and therefore $v_\gamma(B)/\mu_\gamma(B) = \Psi(B)$. So,

$$\left| T(\omega) - \sum_{D'} \Psi(B)\omega(B) \right| < \varepsilon.$$

Therefore, $T(\omega) = \int_S \Psi \, d\omega$, if $\omega \in N_\gamma$, for some γ. A similar argument shows that this representation holds if $\omega \in N_{\mu_1}$. And thus, this representation holds for all ω in J, the span of $N_{\mu_1} \cup (\bigcup_{\gamma \in \Gamma} N_{\mu_\gamma})$. It follows from Theorem 1 that J is dense in $bv(S, \Sigma, X)$ and therefore by Theorem 4, we have

$$\text{(R)} \qquad T(\omega) = \int_S \Psi \, d\omega.$$

for all ω in $bv(S, \Sigma, X)$.

Next, note that for all $B \in \Sigma$, we have $\|\Psi(B)\| \leq |T|$ and that for each $\omega \in bv(S, \Sigma, X)$,

$$|T(\omega)| = \left|\int_S \Psi\, d\omega\right| \leq \sup \{|\Psi(B)|: B \in \Sigma\} \cdot \|\omega\|.$$

Therefore $|T| = \sup \{|\psi(B)|: B \in \Sigma\}$.

Finally, assume Ψ' is a function from Σ into X^* having the properties of Ψ listed in the conclusion of Theorem 7. Let $\gamma \in \Gamma$ and define

$$\nu'_\gamma(E) = 0 \quad \text{if } \mu_\gamma(E) = 0$$

and

$$\nu'_\gamma(E) = \mu_\gamma(E \cap B_\gamma) \cdot \Psi'(E \cap B_\gamma) \quad \text{if } \mu_\gamma(E) > 0.$$

The function $\nu'_{\gamma]} \in H(\Sigma, \mu_\gamma, X^*)$. Also,

$$\int_S \frac{d\nu'_\gamma\, d\omega}{d\mu_\gamma} = \int_S \Psi'\, d\omega = \int_S \Psi\, d\omega = \int_S \frac{d\nu_\gamma\, d\omega}{d\mu_\gamma}$$

for every ω in N_γ. Therefore, $\nu_\gamma = \nu'_\gamma$ and if $B \subseteq B_\gamma$ and $\mu_\gamma(B) > 0$, then

$$\Psi'(B) = \frac{\nu'_\gamma(B)}{\mu_\gamma B} = \frac{\nu_\gamma(B)}{\mu_\gamma(B)} = \Psi(B).$$

There is a similar argument to show that $\Psi'(B) = \Psi(B)$, if $\mu_1(B) > 0$. Therefore, the function Ψ is unique.

In view of Theorem 5, we are lead to the following definition.

DEFINITION 6. Suppose ϕ is a selector for $\{\mu_\gamma \mid \gamma \in \Gamma\}$. The space $M(\Sigma, \phi, X^*)$ consists of all functions Ψ from Σ into X^* which are bounded and have properties (1), (2), and (3) of Theorem 7.

We note that $M(\Sigma, \phi, X^*)$ is a linear space under the usual meaning of addition and scalar multiplication. Also, if we give this space the uniform norm, then $M(\Sigma, \phi, X^*)$ becomes a Banach space.

THEOREM 8. *If $\{\mu_\gamma \mid \gamma \in \Gamma\}$ has a selector ϕ, then each function Ψ in $M(\Sigma, \phi, X^*)$ defines a bounded linear functional on $bv(S, \Sigma, X)$.*

Proof. Let $\gamma \in \Gamma$. Since $\mu_\gamma \cdot \Psi$ is μ_γ-additive and bounded, the function $\mu_\gamma \cdot \Psi$ may be extended from the subsets of B_γ having positive μ_γ-measure to all of Σ, as in the proof of Theorem 7, to be a function ν_γ in $H(\Sigma, \mu_\gamma, X^*)$. By Theorem 5, for each ω in N_{μ_γ}, the form $(d\nu_\gamma\, d\omega)/d\mu_\gamma$ is integrable. Therefore, for each ω in N_{μ_γ}, the form $\Psi\, d\omega$ is integrable. By a similar argument each function ω in N_{μ_1} is Ψ-integrable. By linearity of the integral, the fact that the space J, the span of $N_{\mu_1} \cup \bigcup_{\gamma \in \Gamma} N_{\mu_\gamma}$ is dense in $bv(S, \Sigma, X)$ (Theorem 1), and Theorem 4, we have that for each ω in $bv(S, \Sigma, X)$, the form $\Psi\, d\omega$ is integrable and thus $T(\omega) = \int_S \Psi\, d\omega$ defines a member of $bv^*(S, \Sigma, X)$.

Theorems 7 and 8 are now combined to give:

THEOREM 9. *Suppose X has the Radon-Nikodym property and there is a selector for the set $\{\mu_\gamma \mid \gamma \in \Gamma\}$. Then the spaces $bv^*(S, \Sigma, X)$ and $M(\Sigma, \phi, X^*)$ are isometrically isomorphic by the representation*

$$\text{(R)} \qquad T(\omega) = \int_S \Psi \, d\omega.$$

In connection with Theorem 6, we have

THEOREM 10. *Suppose X has the Radon-Nikodym property, the cardinality of $ca(S, \Sigma)$ is $\leqq 2^{\aleph_0}$ and the continuum hypothesis holds: $2^{\aleph_0} = \aleph_1$. Then the conclusion of Theorem 9 holds.*

For example if S is a separable metric space, Σ is the σ-algebra of all Borel subsets of S, and the continuum hypothesis holds, then we have the representation given by Theorem 9.

Added in proof. It has been pointed out by Joe Diestel that if $bv^*(S, \Sigma, X)$ has a representation via (R), then X has the Radon-Nikodym property. Thus, assuming the continuum hypothesis, a Banach space has the Radon-Nikodym property if and only if the representation given by Theorem 9 holds.

The author would like to close with the following:

Question. Suppose that for each $T \in bv^*(S, \Sigma, X)$, there is some function Ψ from Σ into X^* such that

$$T(\omega) = \int_S \Psi \, d\omega$$

for all $\omega \in bv(S, \Sigma, X)$. Does it follow that there is a maximal set H of mutually singular positive measures from $ca(S, \Sigma)$, a measure $\mu_1 \in H$ and a selector for the set $H - \{\mu_1\}$?

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