QUOTIENTS OF THE AUGMENTATION IDEAL OF A GROUP RING BY POWERS OF ITSELF

BY

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Fundamentals

Let G be a group under multiplication, and let $\mathbf{Z}(G)$ be its group ring over the ring of integers. $\mathbf{Z}(G)$ may be viewed as the set of formal (finite) linear combinations of the elements of G using integer coefficients. The *augmentation mapping* $e\colon \mathbf{Z}(G) \to \mathbf{Z}$ then takes each element of $\mathbf{Z}(G)$ to the sum of its coefficients. Note that e is a ring homomorphism. The kernel of e is called the *augmentation ideal*, and is denoted $\mathscr{G}$. Let $\Theta_n(G) = \mathscr{G}/\mathscr{G}^{n+1}$. $\Theta_n(\mathscr{G})$ is a nilpotent ring, and may be viewed merely as an abelian group under addition. The aim of this paper is to give a method for determining the additive structure of $\Theta_n(G)$, where G is any finitely presented group. Specifically, a presentation for the abelian group $\Theta_n(G)$ is derived from the given presentation of G. This may be used to obtain information about the original group G. An example is given where G is the fundamental group of the complement of a simple link.

Recall that, by definition, $\mathscr{G}$ is the set of all formal linear combinations in $\mathbf{Z}(G)$ whose coefficients add up to 0. Hence, for any $x \in G$, $x - 1$ is an element of $\mathscr{G}$. So it makes sense to define a mapping $d\colon G \to \mathscr{G}$ by $d(x) = x - 1$ for all $x \in G$. Let $q_n\colon \mathscr{G} \to \Theta_n(G)$ denote the ring homomorphism which takes each element of $\mathscr{G}$ to its equivalence class in $\Theta_n(G)$. And call the composite $q_n d = \theta_n$. The mapping $\theta_n\colon G \to \Theta_n(G)$ will play an important role in our study of $\Theta_n(G)$.

There are two special cases in which $\Theta_n(G)$ and the mapping $\theta_n\colon G \to \Theta_n(G)$ are easy to describe; namely, when $n = 0$ and when $n = 1$. It is obvious that:

For any group G, $\Theta_0(G) \cong 0$, and $\theta_0(x) = 0$ for all $x \in G$.

Next, we will describe $\Theta_1(G)$ and θ_1.

LEMMA 1. *For any $x, y \in G$, $d(x)d(y) = d(xy) - d(x) - d(y)$.*

Proof.

$$\begin{aligned} d(x)d(y) &= (x - 1)(y - 1) \\ &= xy - x - y + 1 \\ &= (xy - 1) - (x - 1) - (y - 1) \\ &= d(xy) - d(x) - d(y). \end{aligned}$$

□

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THEOREM 2. *For any group G, $\Theta_1(G)$ is isomorphic (as an abelian group) to G made abelian, and θ_1 corresponds to the canonical epimorphism from G to G made abelian.*

Proof. Note that $\mathscr{G}$ is the free abelian group generated by

$$\{d(x) \mid x \in G - \{1\}\} \quad (d(1) = 0).$$

So $\mathscr{G}^2$ is the subgroup of $\mathscr{G}$ generated by all $d(x)d(y)$, where $x, y \in G$. Thus, by Lemma 1, $\mathscr{G}^2$ is generated by all $d(xy) - d(x) - d(y)$, where $x, y \in G$. So $\Theta_1(G)$ is the free abelian group generated by $\{d(x) \mid x \in G - \{1\}\}$ mod the subgroup generated by all $d(xy) - d(x) - d(y)$. □

The multiplicative structure of $\Theta_1(G)$ is not interesting, since the product of any two elements equals 0.

We will now consider another important mapping associated with $\Theta_n(G)$. Let G^n be the product of n copies of G. Then we define $d^n\colon G^n \to \mathscr{G}$ by $d^n(v) = d(v_1)d(v_2)\cdots d(v_n)$ for all $v \in G^n$.

Hereafter we will use the word "morphism" to mean "group homomorphism".

THEOREM 3. *Let A be an abelian group under addition, and let $h\colon \mathscr{G} \to A$ be a morphism which annihilates $\mathscr{G}^{n+1}$. Then the mapping $hd^n\colon G^n \to A$ is a morphism in each variable when the other variables are held fixed.*

Proof. We will prove the assertion for the first variable.

$$\begin{aligned}
&hd^n(xy, v_2, \ldots, v_n)\\
&\quad = h(d(xy)d(v_2)\cdots d(v_n))\\
&\quad = h((d(x)d(y) + d(x) + d(y))d(v_2)\cdots d(v_n)) \quad \text{(by Lemma 1)}\\
&\quad = h(d(x)d(y)d(v_2)\cdots d(v_n)) + h(d(x)d(v_2)\cdots d(v_n)) + h(d(y)d(v_2)\cdots d(v_n))\\
&\quad = hd^n(x, v_2, \ldots, v_n) + hd^n(y, v_2, \ldots, v_n)
\end{aligned}$$

since $d(x)d(y)d(v_2)\cdots d(v_n) \in \mathscr{G}^{n+1}$. □

Θ_n of a free group

Let F be a free group on a finite set of generators X. And denote by $\mathscr{F}$ the augmentation ideal of F.

THEOREM 4. *$\Theta_n(F)$ is generated under addition by*

$$\bigcup_{1 \le j \le n} \{q_n d^j(v) \mid v \in X^j\}.$$

Proof. Let A be an abelian group under addition, and let $g\colon \Theta_n(F) \to A$ be a morphism which annihilates $\bigcup_{1 \le j \le n} \{q_n d^j(v) \mid v \in X^j\}$. It suffices to prove that g annihilates $\Theta_n(F)$.

Observe that:

5. $gq_n\colon \mathscr{F} \to A$ is a morphism which annihilates $\mathscr{F}^{n+1}$ and

$$\bigcup_{1\le j\le n} \{d^j(v) \mid v \in X^j\}.$$

Now, by Theorem 3, $gq_nd^n\colon F^n \to A$ is a morphism in each variable when the other variables are held fixed. And gq_nd^n annihilates X^n. So gq_nd^n annihilates F^n (since X generates F). That is, gq_n annihilates all elements of the form

$$d(v_1)d(v_2)\cdots d(v_n) \quad \text{where } (v_1, v_2, \ldots, v_n) \in F^n.$$

But these elements generate $\mathscr{F}^n$. So:

6. $gq_n\colon \mathscr{F} \to A$ is a morphism which annihilates $\mathscr{F}^n$ and

$$\bigcup_{1\le j\le n} \{d^j(v) \mid v \in X^j\}.$$

Since we proved 6 from 5 it follows by induction that gq_n annihilates $\mathscr{F}$. And since $q_n\colon \mathscr{F} \to \Theta_n(F)$ is an epimorphism, it follows that g annihilates $\Theta_n(F)$. □

Next we must determine the relations among the generators of $\Theta_n(F)$. We will do this by means of the free differential calculus [2]. Fox's concept of a derivative is defined as follows:

A derivative on $\mathbf{Z}(F)$ is a mapping $\delta\colon \mathbf{Z}(F) \to \mathbf{Z}(F)$ such that

(a) $\delta(p + q) = \delta(p) + \delta(q)$,
(b) $\delta(pq) = \delta(p)e(q) + p\delta(q)$ for all $p, q \in \mathbf{Z}(F)$.

Here e is the augmentation mapping. It is easy to show that:

If δ is a derivative on $\mathbf{Z}(F)$, then $\delta(1) = 0$.

THEOREM 7. *If δ is a derivative on $\mathbf{Z}(F)$, then $\delta(\mathscr{F}^{i+1}) \subset \mathscr{F}^i$ for all positive integers i.*

Proof. Given $p \in \mathscr{F}^i$ and $q \in \mathscr{F}$,

$$\delta(pq) = \delta(p)e(q) + p\delta(q) = p\delta(q) \in \mathscr{F}^i.$$

And $\mathscr{F}^{i+1}$ is generated under addition by $\{pq \mid p \in \mathscr{F}^i, q \in \mathscr{F}\}$. □

THEOREM 8. *If $\delta_1, \delta_2, \ldots, \delta_i$ are derivatives on $\mathbf{Z}(F)$, then the morphism $e\delta_1\delta_2\cdots\delta_i\colon \mathbf{Z}(F) \to \mathbf{Z}$ annihilates $\mathscr{F}^{i+1}$.*

Proof. Applying Theorem 7 repeatedly, we find that

$$\delta_1\delta_2\cdots\delta_i(\mathscr{F}^{i+1}) \subset \mathscr{F}.$$

And e annihilates $\mathscr{F}$. □

Fox has shown in [2] that for each generator $x \in X$ there exists a unique derivative $\partial/\partial x$ on $\mathbf{Z}(F)$ such that $\partial x/\partial x = 1$ and $\partial y/\partial x = 0$ for every other generator $y \in X$. In the theorems that follow, we will denote by $u_1, u_2, \ldots, u_i$ the components of an element $u \in X^i$.

THEOREM 9. *If $u \in X^i$, $v \in X^j$, and $i < j$, then*

$$e \frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} \cdots \frac{\partial}{\partial u_i} d^j(v) = 0.$$

Proof. Note that $d^j(v) \in \mathscr{F}^j \subset \mathscr{F}^{i+1}$. And, by Theorem 8,

$$e \frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} \cdots \frac{\partial}{\partial u_i}$$

annihilates $\mathscr{F}^{i+1}$. □

THEOREM 10. *If $u \in X^i$, $v \in X^j$, and $i \geq j$, then*

$$\frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} \cdots \frac{\partial}{\partial u_i} d^j(v) = \begin{cases} 1 & \textit{if } u = v \\ 0 & \textit{if } u \neq v. \end{cases}$$

Proof. We will proceed by induction on i. Note that for any $x, y \in X$, $(\partial/\partial x)d(y) = \partial y/\partial x$. Hence the assertion is true when $i = 1$.

Now we may assume the assertion is true for a given i. Consider $u \in X^{i+1}$ and $v \in X^j$, where $i \geq j$. We have

$$\begin{aligned} \frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} \cdots \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_{i+1}} d^j(v) &= \frac{\partial}{\partial u_1} \left(\frac{\partial}{\partial u_2} \cdots \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_{i+1}} d^j(v) \right) \\ &= \frac{\partial}{\partial u_1} (1 \quad \text{or} \quad 0) \\ &= 0. \end{aligned}$$

This agrees with the assertion.

Finally we must consider $u \in X^{i+1}$ and $v \in X^{i+1}$. Let $v' = (v_1, v_2, \ldots, v_i)$. Then

$$\begin{aligned} &\frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} \cdots \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_{i+1}} d^{i+1}(v) \\ &\quad = \frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} \cdots \frac{\partial}{\partial u_i} \left[\frac{\partial}{\partial u_{i+1}} (d^i(v')\, d(v_{i+1})) \right] \\ &\quad = \frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} \cdots \frac{\partial}{\partial u_i} \left[\left(\frac{\partial}{\partial u_{i+1}} d^i(v') \right) ed(v_{i+1}) + d^i(v') \frac{\partial}{\partial u_{i+1}} d(v_{i+1}) \right] \\ &\quad = \frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} \cdots \frac{\partial}{\partial u_i} \left[d^i(v') \frac{\partial v_{i+1}}{\partial u_{i+1}} \right]. \end{aligned}$$

If $u = v$, this expression is clearly equal to 1. Otherwise it is 0.

This completes the induction. □

THEOREM 11. *Under addition, $\Theta_n(F)$ is the free abelian group generated by*

$$\bigcup_{1\le j\le n} \{q_n d^j(v) \mid v \in X^j\}.$$

Proof. Consider any $u \in X^i$, where $1 \le i \le n$. By Theorem 8, the morphism

$$e\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}\cdots\frac{\partial}{\partial u_i}: \mathbf{Z}(F) \to \mathbf{Z}$$

annihilates $\mathscr{F}^{i+1}$. Hence the restriction

$$e\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}\cdots\frac{\partial}{\partial u_i}: \mathscr{F} \to \mathbf{Z}$$

can be written as $\pi_u q_n$, where $\pi_u\colon \Theta_n(F) \to \mathbf{Z}$ is a morphism. By 9 and 10 we have

$$e\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}\cdots\frac{\partial}{\partial u_i} d^j(v) = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{if } u \ne v. \end{cases}$$

So

$$\pi_u(q_n d^j(v)) = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{if } u \ne v. \end{cases}$$

Therefore there are no relations among the generators

$$\bigcup_{1\le j\le n} \{q_n d^j(v) \mid v \in X^j\}. \qquad \square$$

At this point it is convenient to adopt a new notation. Given $u \in X^i$, let $c_u = q_n d^i(u)$. (We assume n is known from context.) Let D_u denote the mapping

$$e\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}\cdots\frac{\partial}{\partial u_i} d\colon F \to \mathbf{Z}.$$

And let $\bigcup^n X = \bigcup_{1\le i\le n} X^i$. Then Theorem 11 may be restated as follows:

Under addition, $\Theta_n(F)$ is the free abelian group generated by $\{c_u \mid u \in \bigcup^n X\}$.

Looking at the proof of Theorem 11, we see that

$$D_u = e\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}\cdots\frac{\partial}{\partial u_i} d = \pi_u q_n d = \pi_u \theta_n.$$

And

$$\pi_u(c_v) = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{if } u \ne v. \end{cases}$$

So D_u is the u-coordinate of θ_n. Therefore:

COROLLARY 12. $\theta_n = \sum_{u\in\bigcup^n X} c_u D_u$.

Note that Theorem 11 actually enables us to describe the structure of $\Theta_n(F)$ as a ring. Given $u \in X^i$ and $v \in X^j$, define

$$uv = (u_1, \ldots, u_i, v_1, \ldots, v_j) \in X^{i+j}.$$

Since q_n is a ring homomorphism, the generators of $\Theta_n(F)$ multiply by the rule

$$q_n d^i(u) q_n d^j(v) = q_n d^{i+j}(uv).$$

In other words, $c_u c_v = c_{uv}$.

When $i + j > n$, this product is 0. Hence $\Theta_n(F)$ is the truncated polynomial ring (with height $n + 1$, with **Z**-coefficients, and with no constant terms) in the noncommuting variables $\{c_x \mid x \in X\}$.

Computation of $\Theta_n(G)$ from a presentation of G

Let F be a free group on a finite set of generators X. Let R be a finite subset of F, and let E be the smallest normal subgroup of F containing R. Let $G = F/E$. Then G is a finitely presented group, with generators X and relations R. Our aim is to compute the additive structure of $\Theta_n(G)$. We will approach the problem by way of $\Theta_n(F)$ and $\theta_n: F \to \Theta_n(F)$, which are already known.

Throughout this paper, the word "ideal" will always mean a two-sided ideal. Call $M(E)$ the ideal of $\mathscr{F}$ generated by $\{d(r) \mid r \in R\}$.

THEOREM 13. *$d(y) \in M(E)$ for all $y \in E$.*

Proof. Note that $M(E)$ is also an ideal of $\mathbf{Z}(F)$. So $\mathbf{Z}(F)/M(E)$ is a ring, and we have a ring homomorphism

$$f: \mathbf{Z}(F) \to \mathbf{Z}(F)/M(E) \quad \text{with } f(1) = 1.$$

Now F sits in $\mathbf{Z}(F)$, and f acts as a group homomorphism on F. So

$$\{x \in F \mid f(x) = 1\}$$

is a normal subgroup of F. And $f(r) = 1$ for all $r \in R$. Thus $f(y) = 1$ for all $y \in E$. That is, $d(y) \in M(E)$ for all $y \in E$. □

Call g the ring homomorphism from $\mathscr{F}$ onto $\mathscr{G}$ induced by the canonical morphism from F onto G.

THEOREM 14. *$M(E)$ is the kernel of g.*

Proof. $M(E)$ is clearly contained in the kernel of g. Moreover, given $x \in F$ and $y \in E$, we have

$$d(xy) - d(x) = d(x)d(y) + d(y)$$

by Lemma 1. Hence, by Theorem 13, $d(xy) - d(x)$ belongs to $M(E)$. But

$$\{d(xy) - d(x) \mid x \in F, \quad y \in E\}$$

generates the kernel of g under addition. So $M(E)$ is the entire kernel. □

The morphism from F onto G also induces a ring homomorphism from $\mathscr{F}^{n+1}$ onto $\mathscr{G}^{n+1}$, and a ring homomorphism h from $\Theta_n(F)$ onto $\Theta_n(G)$. Call $N_n(E)$ the kernel of h.

THEOREM 15. *$N_n(E)$ is the ideal of $\Theta_n(F)$ generated by $\{\theta_n(r) \mid r \in R\}$.*

Proof. Consider the following diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathscr{F}^{n+1} & \longrightarrow & \mathscr{G}^{n+1} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M(E) & \longrightarrow & \mathscr{F} & \xrightarrow{g} & \mathscr{G} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle q_n} & & \downarrow & & \\
0 & \longrightarrow & N_n(E) & \longrightarrow & \Theta_n(F) & \xrightarrow{h} & \Theta_n(G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The three rows and the second and third columns are exact. Hence, by a variation of the Nine Lemma, the first column is exact. Thus $q_n\colon \mathscr{F} \to \Theta_n(F)$ takes $M(E)$ onto $N_n(E)$. And, since $M(E)$ is generated by $\{d(r) \mid r \in R\}$, $N_n(E)$ is generated by $\{q_n d(r) \mid r \in R\}$. □

COROLLARY 16. *$N_n(E)$ is generated under addition by*

(i) *all $\theta_n(r)$, where $r \in R$,*
(ii) *all $c_u\theta_n(r)$, where $r \in R$ and $u \in \bigcup^{n-1} X$,*
(iii) *all $\theta_n(r)c_v$, where $r \in R$ and $v \in \bigcup^{n-1} X$,*
(iv) *all $c_u\theta_n(r)c_v$, where $r \in R$, $u \in X^i$, $v \in X^j$, and $i + j < n$.*

Since $\Theta_n(G) \cong \Theta_n(F)/N_n(E)$, Corollary 16 gives a presentation for $\Theta_n(G)$. The generators of $\Theta_n(G)$ are the generators of $\Theta_n(F)$; the relations of $\Theta_n(G)$ are the generators of $N_n(E)$. Since the presentation is finite, standard methods may be used to determine the structure of $\Theta_n(G)$ as an abelian group.

In principle, the ring structure of $\Theta_n(G)$ can also be found from this presentation. But less is known about the structure of nilpotent rings than is known about abelian groups. This appears to be a more difficult problem.

Example 17. Let $X = \{x\}$ and let $R = \{x^9\}$. Then F is isomorphic to the group of integers $\mathbf{Z}$, and G is isomorphic to $\mathbf{Z}_9$. We will compute $\Theta_4(\mathbf{Z}_9)$ as an abelian group. It turns out that

$$D_{(x,x,\ldots,x)}(x^n)(s \text{ entries}) = C(n, s).$$

Using Corollary 12, we obtain

$$\theta_4(x^9) = 9c_x + 36c_{(x,x)} + 84c_{(x,x,x)} + 126c_{(x,x,x,x)}.$$

Since $\Theta_4(F)$ is a commutative ring in this case, the generators listed in 16 boil down to $\theta_4(x^9)$, $c_x\theta_4(x^9)$, $c_{(x,x)}\theta_4(x^9)$, and $c_{(x,x,x)}\theta_4(x^9)$. Thus $N_4(E)$ is generated by

$$9c_x + 36c_{(x,x)} + 84c_{(x,x,x)} + 126c_{(x,x,x,x)},$$

$$9c_{(x,x)} + 36c_{(x,x,x)} + 84c_{(x,x,x,x)},$$

$$9c_{(x,x,x)} + 36c_{(x,x,x,x)} \quad \text{and} \quad 9c_{(x,x,x,x)}.$$

To find the canonical form of $\Theta_4(\mathbf{Z}_9)$, form the matrix

$$\begin{bmatrix} 9 & 36 & 84 & 126 \\ 0 & 9 & 36 & 84 \\ 0 & 0 & 9 & 36 \\ 0 & 0 & 0 & 9 \end{bmatrix}.$$

By performing elementary row and column operations (using integer coefficients only), we obtain

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 27 & 0 \\ 0 & 0 & 0 & 27 \end{bmatrix}.$$

Thus $\Theta_4(\mathbf{Z}_9) \cong \mathbf{Z}_3 + \mathbf{Z}_3 + \mathbf{Z}_{27} + \mathbf{Z}_{27}$.

Example 18. Let $X = \{x, y, z\}$ and let $R = \{r_1, r_2\}$, where

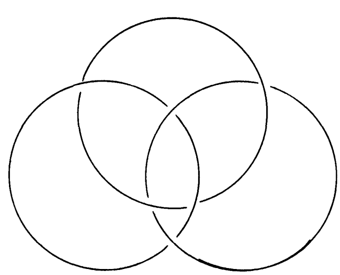
$$r_1 = yzy^{-1}xyz^{-1}y^{-1}zx^{-1}z^{-1} \quad \text{and} \quad r_2 = zxz^{-1}yzx^{-1}z^{-1}xy^{-1}x^{-1}.$$

Then F is a free group on three generators, and G is isomorphic to the fundamental group of the complement of the link in the figure. We will compute $\Theta_3(G)$. Using Corollary 12, we obtain

$$\theta_3(r_1) = -c_{(x,y,z)} + c_{(y,z,x)} - c_{(z,y,x)} + c_{(x,z,y)},$$

$$\theta_3(r_2) = c_{(z,x,y)} - c_{(y,z,x)} - c_{(x,z,y)} + c_{(y,x,z)}.$$

Note that the generators listed in parts (ii), (iii), and (iv) of 16 are 0. Hence $N_3(E)$ is generated by $\theta_3(r_1)$ and $\theta_3(r_2)$. Only six of the thirty-nine generators of



$\Theta_3(F)$ occur in $\theta_3(r_1)$ or $\theta_3(r_2)$. So $\Theta_3(G)$ is isomorphic to $33\mathbf{Z}$ plus $6\mathbf{Z}$ mod the subgroup generated by $\theta_3(r_1)$ and $\theta_3(r_2)$. In matrix form, this is

$$\begin{bmatrix} -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{bmatrix},$$

which becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus $\Theta_3(G) \cong 33\mathbf{Z} + 4\mathbf{Z} \cong 37\mathbf{Z}$.

Remark on links. Suppose G is the fundamental group of the complement of a link in Euclidean 3-space. Call G_n the nth lower central subgroup of G. ($G_1 = G$, $G_n = [G, G_{n-1}]$ for all $n > 1$.) It is shown in [7] that G/G_n is an isotopy invariant of the given link. Let $G \cong F/E$, where F, E, X, and R are as before. We may then obtain a presentation for G/G_n by combining the relations of G with all of the elements of F_n. It is shown in [2] that $D_u(r) = 0$ for all $r \in F_n$, $u \in \bigcup^{n-1} X$. Hence, by 12, $\theta_{n-1}(r) = 0$ for all $r \in F_n$. So by 16, $\Theta_{n-1}(G/G_n) \cong \Theta_{n-1}(G)$. Therefore $\Theta_{n-1}(G)$ is an isotopy invariant of the link, for all positive integers n.

In Example 18, we found that $\Theta_3(G) \cong 37\mathbf{Z}$. But the fundamental group of the complement of a trivial link with 3 components is a free group F on 3 generators. And $\Theta_3(F) \cong 39\mathbf{Z}$. So the link in this example is not isotopically trivial. Moreover, deeper information about this link may be obtained by computing $\Theta_n(G)$ for larger n. It is our hope that this approach will be fruitful in studying the isotopy properties of more difficult links.

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