ON THE DIMENSION OF VARIETIES OF SPECIAL DIVISORS, II[1]

BY

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1. Introduction

In [1], we defined the analytic spaces $\mathscr{G}_n^r$, the universal analytic spaces of special divisors. We derived the equations which define the tangent space at a point of $\mathscr{G}_n^r - \mathscr{G}_n^{r+1}$. Let X be a compact Teichmuller surface of genus g and suppose s_0 is the module point of X on T_g, the Teichmuller space. Let D be a divisor on X of degree n and dimension r and let i denote the index of specialty of D. With the notation of [1], the tangent space to $\mathscr{G}_n^r$ at (s_0, D) is defined by ir equations $E_{j,k}$, where $j = 1, \ldots, r$ and $k = 1, \ldots, i$, in the $3g - 3 + n$ unknowns $s_1, \ldots, s_n, b_1, \ldots, b_{3g-3}$. The coefficient of b_m in $E_{j,k}$ is given by evaluating $\alpha_{j,k}$, a certain quadratic differential depending on D, at a point Q_m on X, which is chosen to satisfy certain requirements.

PROPOSITION 1. *Suppose* (s_0, D) *is in* $\mathscr{G}_n^r - \mathscr{G}_n^{r+1}$ *and suppose that* $ir \leq 3g - 3$. *Put* $\tau = (r + 1)(n - r) - rg$. *Then if all the* $\alpha_{j,k}$ *are linearly independent, the dimension of the tangent space to* $\mathscr{G}_n^r$ *at* (s_0, D) *is* $3g - 3 + \tau + r$ *and* (s_0, D) *is a smooth point of* $\mathscr{G}_n^r$.

Proof. [1].

In [1], we showed that $\mathscr{G}_n^1 - \mathscr{G}_n^2$, if nonempty, is smooth of pure dimension $3g - 3 + \tau + 1$. In this paper, by explicit computations, we show that $\mathscr{G}_n^2$ (resp. $\mathscr{G}_n^3$) has a component of dimension $3g - 3 + \tau + 2$ (resp. $3g - 3 + \tau + 3$) if τ is nonnegative. Our computations are based on examples given by Meis [2].

2. Meis's work

In [2], Meis demonstrates the existence of special divisors for the case $r = 1$. Since this monograph is rather difficult to obtain, we will review his method in some detail.

His proof proceeds by considering the universal analytic space of special divisors $\mathscr{G}_n^1$ over the Teichmuller space T_g and explicitly exhibiting a special fiber of dimension $\tau + 1$ in the case in which n is the minimum integer such that τ is nonnegative. He may then conclude that a component of $\mathscr{G}_n^1$ has dimension $3g - 3 + \tau + 1$ and that this component maps surjectively down to T_g.

Hence he shows that for an arbitrary Riemann surface X and n any integer such that τ is nonnegative, the analytic subspace G_n^1 of the nth symmetric product of X is nonempty of dimension at least $\tau + 1$. His methods also show that for a generic surface, G_n^1 has a component of dimension $\tau + 1$ if n is the minimum integer such that τ is nonnegative.

We present Meis's examples below, and will use them in the following sections. Suppose g is given and $r = 1$. Then the minimum n such that τ is nonnegative is

$$n = \begin{cases} \dfrac{g + 2}{2} & \text{if } g \text{ is even} \\[2ex] \dfrac{g + 3}{2} & \text{if } g \text{ is odd.} \end{cases}$$

So, the $r = 1$ case breaks up naturally into even and odd genus subcases. Meis gives one class of even genus surfaces and one class of odd genus surfaces.

Even genus case. Suppose $g = 2m$. Consider the Riemann surface of the algebraic function

$$y^{m+1} = (x - 1)(x - 2)(x - 3)(x - 4)^m(x - 5)^m(x - 6)^m.$$

This surface has $m + 1$ sheets and ramification points of order m over the points $x = 1, 2, \ldots, 6$. By the Riemann-Hurwitz formula, the surface thus has genus $2m$. Meis shows that a basis for the holomorphic differentials on this surface is given by

$$d\zeta_k = \frac{(x - 4)^{k-1}(x - 5)^{k-1}(x - 6)^{k-1}\, dx}{y^k}, \quad k = 1, \ldots, m,$$

$$d\zeta_{k+m} = x\, d\zeta_k, \quad k = 1, \ldots, m.$$

One can easily compute the order of vanishing of the differentials at the ramification points and at the points over $x = 0$ and $x = \infty$ (and these are the only points where the differentials might vanish). To do this, notice that a local parameter at the point $x = j$, for $j = 1, 2, \ldots, 6$ is $(x - j)^{1/(m+1)}$; a local parameter at the points over $x = 0$ is x; and a local parameter at the points over $x = \infty$ is $1/x$. Then express $d\zeta_i$ as $f_i(t)\, dt$, where t is a local parameter at the point of interest, and see what the order of vanishing of $f_i(t)$ is at $t = 0$. Meis obtains the following table for the order of vanishing of the differentials at the point(s) over the given value of x:

$x =$	1	2	3	4	5	6	0	∞
$d\zeta_k$	$m - k$	$m - k$	$m - k$	$k - 1$	$k - 1$	$k - 1$	0	1
$d\zeta_{k+m}$	$m - k$	$m - k$	$m - k$	$k - 1$	$k - 1$	$k - 1$	1	0

for $k = 1, 2, \ldots, m$.

Odd genus case. Suppose $g = 2m + 1$. Consider the Riemann surface of the algebraic function

$$y^3 = \frac{\prod\limits_{i=1}^{m+2} (x - i)}{\prod\limits_{j=m+3}^{2m+2} (x - j)}.$$

This surface has 3 sheets and ramification points of order 2 over $x = 1, 2, \ldots,$ $2m + 2$ and over $x = \infty$. Thus this surface has genus $2m + 1$. Meis shows that a basis for the holomorphic differentials is given by:

$$d\zeta_k = \frac{x^{k-1}\, dx}{y^2 \prod\limits_{j=1}^{2m+2} (x - j)}, \quad k = 1, \ldots, m + 1,$$

$$d\zeta_{k+m+1} = y\, d\zeta_k, \quad k = 1, \ldots, m.$$

Meis obtains the following table for the order of vanishing of the differentials:

	$x =$	1	$m + 3$	0	∞
$1 \le k \le m + 1$	$d\zeta_k$	0	1	$k - 1$	$3(m + 1 - k)$
$1 \le k \le m$	$d\zeta_{k+m+1}$	1	0	$k - 1$	$3(m + 1 - k) - 2$

For examples of special divisors with $r = 1$ and n the minimum integer such that τ is nonnegative, Meis takes for even g the $(g + 2)/2$ points over $x = 0$ and for odd g the $(g + 3)/2$ ramification points over $x = 1, 2, \ldots, m + 2$ (note that $m + 2 = (g + 3)/2$).

3. The $r = 2$ case

In this section, given g and n such that $3(n - 2) - 2g$ is nonnegative, we will construct a divisor D on Meis's surface of genus g such that D is of degree n and dimension 2 and such that the quadratic differentials $\alpha_{j,k}$ associated to D are linearly independent.
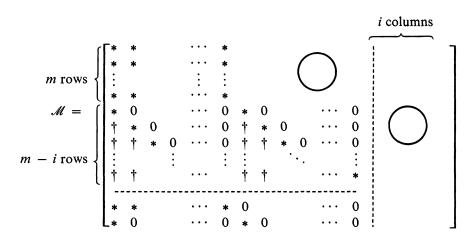
Even genus case. Suppose that $g = 2m$ and that n is given such that $3(n - 2) - 2g$ is nonnegative. Consider Meis's Riemann surface of genus g, as described in the previous section. Our divisor D will consist of the following points:

   (1) the $(g + 2)/2\ (=m + 1)$ points over $x = 0$, denoted $P_1, P_2, \ldots, P_{m+1}$,

   (2) the (ramification) point over $x = 6$, denoted P_{m+2}, with multiplicity $m - i$, where $i = 2 + g - n$, and

   (3) the point over $x = 5$, denoted P_{m+3}.

Note that the assumption that $3(n - 2) - 2g$ is nonnegative implies that i must be less than m. Let $d\zeta_k, k = 1, 2, \ldots, g$, be Meis's basis of differentials.

It is easy to see from the table of vanishing of the $d\zeta_k$ in Section 2 that our divisor D is of index specialty i and hence of dimension 2.

One may also see from this table that the matrix $\mathcal{M}$ of [1, Section 4] which is associated to D has the form

$$i \text{ columns}$$

where $*$ is nonzero and $\dagger$ may be nonzero. The last row arises from P_{m+3} and the next to the last row arises from P_{m+1}, after a possible renumbering of $P_1, \ldots, P_{m+1}$ so as to insure that the leading minor of order m of $\mathcal{M}$, which arises from $P_1, \ldots, P_m$, is nonzero. This is possible since, as Meis shows, the divisor $P_1 + \cdots + P_{m+1}$ is of (projective) dimension 1.

Note that the m differentials which vanish at P_{m+1} vanish only simply there. Thus the quadratic differentials $d\tau_{P_{m+1,0}} d\zeta_{n+k-2}$ (see [1]), for $k = 1, \ldots, i$, will each have a simple pole at P_{m+1} (since $d\tau_{P_{m+1,0}}$ has a pole of order 2 there).

Now suppose there existed a linear dependence relation among the $\alpha_{j,k}$, say

$$a_1 \alpha_{1,1} + a_2 \alpha_{1,2} + \cdots + a_i \alpha_{1,i} + a_{i+1} \alpha_{2,1} + \cdots + a_{2,i} = 0. \qquad (*)$$

By definition of the $\alpha_{j,k}$ we have

$$\alpha_{1,k} = d\zeta_{n+k-2} \left(\sum_{j=1}^{m} \mu_j \, d\tau_{P_{J,0}} + \sum_{v=0}^{m-i-1} \bar{\mu}_v \, d\tau_{P_{m+2,v}} + (-1)^{n-1} \mu \, d\tau_{P_{m+1,0}} \right)$$

where μ_j, $\bar{\mu}_v$ are $\pm$ minors of order $n - 2$ of $\mathcal{M}$ and μ is the nonzero leading minor of order $n - 2$ of $\mathcal{M}$. The $\alpha_{1,k}$, for $k = 1, \ldots, i$, will each have a simple pole at P_{m+1}, since they contain $d\zeta_{n+k-2} \, d\tau_{P_{m+1,0}}$ with nonzero coefficient and all other terms are regular at P_{m+1}. But the $\alpha_{2,k}$ will all be finite at P_{m+1}, since they don't contain $d\tau_{P_{m+1,0}}$ at all.

Therefore, the relation $(*)$ will imply the existence of a linear dependence relation among the $\alpha_{1,k}$, $k = 1, \ldots, i$, and a linear dependence relation among the $\alpha_{2,k}$, $k = 1, \ldots, i$. If $(*)$ is nontrivial, then at least one of these relations

will be nontrivial. But the $\alpha_{j,k}$ for fixed j are linearly independent by the remark preceding Theorem 3 of [1]. Hence all the $\alpha_{j,k}$ are linearly independent.

Odd genus case. This case is quite similar. Suppose that $g = 2m + 1$ and that n is given such that $3(n - 2) - 2g$ is nonnegative. Consider Meis's surface of genus g. Let P_0 and P_0' denote two of the three points over $x = 0$. Our divisor D will consist of the following points:

(1)   the $m + 2$ ramification points over $x = 1, 2, \ldots, m + 2$, which we will denote by $P_1, \ldots, P_{m+2}$,

(2)   the point P_0 with multiplicity $m - i$, and

(3) P_0'.

The divisor $P_1 + \cdots + P_{m+2}$ was Meis's example of a member of a g_{m+2}^1. Similarly to the even genus case, we may assume that the next to the last row in the matrix $\mathcal{M}$ arises from P_{m+2}. The last row in $\mathcal{M}$ arises from P_0'.

Now, the last m differentials in Meis's basis vanish simply at P_{m+2}, hence the quadratic differentials $d\tau_{P_{m+2},0} \, d\zeta_{n+k-2}$, for $k = 1, \ldots, i$, will each have a pole at P_{m+2}. We may apply the same reasoning as in the even genus case to conclude that a linear dependence relation among all the $\alpha_{j,k}$ would imply the existence of a linear dependency among those arising from a fixed row (i.e., fixed j), but these are linearly independent.

THEOREM 1. $\mathcal{G}_n^2$ *has a component of dimension* $3g - 3 + \tau + 2$ *for any* n *and* g *such that* τ *is nonnegative.*

Proof.   Let g and n be given such that τ is nonnegative. Let X denote Meis's surface of genus g and let $s_0 \in T_g$ denote the module point of X. Consider the divisor D on X exhibited above. We have shown that the quadratic differentials $\alpha_{j,k}$, for $j = 1, 2$, and $k = 1, 2, \ldots, 2 + g - n$, associated to D are linearly independent. It follows from Proposition 1 that the tangent space to $\mathcal{G}_n^2$ at (s_0, D) has dimension $3g - 3 + \tau + 2$. Since every component of $\mathcal{G}_n^2$ has dimension at least this number [1], we may conclude that $\mathcal{G}_n^2$ has a component of dimension $3g - 3 + \tau + 2$.

4. The case $r = 3$

Given g and n such that $4(n - 3) - 3g$ is nonnegative, we will construct a divisor D on Meis's surface of genus g such that D is of degree n and dimension 3 and such that the quadratic differentials $\alpha_{j,k}$ associated to D are linearly independent.

Even genus case. Suppose $g = 2m$ and suppose n is given such that $4(n - 3) - 3g$ is nonnegative. Consider Meis's Riemann surface of genus g. Our divisor D will consist of the following points:

(1) $P_1, \ldots, P_{m+1}$,

(2) P_{m+2} with multiplicity $m - i$, where now $i = 3 + g - n$,

(3) P_{m+3} and the point over $x = 4$, denoted P_{m+4}.

It is easy to see that D is of degree n and index of specialty i.

To insure that the leading minor μ of order $n - 3$ of the matrix $\mathcal{M}$ is non-zero, we take for the last three rows of $\mathcal{M}$ those arising from P_{m+1}, P_{m+3} and P_{m+4}. ($\mathcal{M}$ has a form analogous to that described in the preceding section.)

Now suppose there exists a linear relation of the form

$$a_1\alpha_{1,1} + \cdots + a_i\alpha_{1,i} + a_{i+1}\alpha_{2,1} + \cdots + a_{3i}\alpha_{3,i} = 0. \qquad (*)$$

We will show that all the coefficients in this relation are zero by considering the order of the $\alpha_{j,k}$ at P_{m+1}, P_{m+3} and P_{m+4}.

Only those $\alpha_{j,k}$ with $j = 1$ will contain $d\tau_{P_{m+1},0} \, d\zeta_{n+k-3}$ and the coefficient of this term in each $\alpha_{1,k}$ will be nonzero (namely $(-1)^{n-1}\mu$). Hence the $\alpha_{1,k}$ will each have a pole at P_{m+1}, while the $\alpha_{2,k}$ and the $\alpha_{3,k}$ will all be regular there. By (*), this implies that we must have

$$a_1\alpha_{1,1} + \cdots + a_i\alpha_{1,i} = 0.$$

But the $\alpha_{j,k}$ are linearly independent for a fixed j, so $a_1 = a_2 = \cdots = a_i = 0$.

It is quite a bit more complicated to show that the other coefficients in (*) are zero. Recall that the order of $d\zeta_{n+k-3}$ at P_{m+3} and P_{m+4} is $m - (i - k + 1)$, for $k = 1, \ldots, i$. Now, the $\alpha_{2,k}$ will each contain $d\tau_{P_{m+3},0} \, d\zeta_{n+k-3}$ with coefficient $(-1)^{n-1}\mu$, hence will each have order $m - (i - k + 1) - 2$ at P_{m+3} (all differentials of the second kind except for $d\tau_{P_{m+3},0}$ are regular at P_{m+3}). The $\alpha_{3,k}$ will each have order at least $m - (i - k + 1)$ at P_{m+3}, since they do not contain $d\tau_{P_{m+3},0}$. The converse situation will hold at P_{m+4}.

Consider the following table of order of vanishing of the $\alpha_{j,k}$ at the points P_{m+3} and P_{m+4}:

at	P_{m+3}	P_{m+4}		P_{m+3}	P_{m+4}
$\alpha_{2,1}$	$m - i - 2$	$\geq m - i$	$\alpha_{3,1}$	$\geq m - i$	$m - i - 2$
$\alpha_{2,2}$	$m - i - 1$	$\geq m - i + 1$	$\alpha_{3,2}$	$\geq m - i + 1$	$m - i - 1$
$\alpha_{2,3}$	$m - i$	$\geq m - i + 2$	$\alpha_{3,3}$	$\geq m - i + 2$	$m - i$
$\alpha_{2,4}$	$m - i + 1$	$\geq m - i + 3$	$\alpha_{3,4}$	$\geq m - i + 3$	$m - i + 1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\alpha_{2,i}$	$m - 3$	$\geq m - 1$	$\alpha_{3,i}$	$\geq m - 1$	$m - 3$

Recall that (*) has become

$$a_{i+1}\alpha_{2,1} + \cdots + a_{2i}\alpha_{2,i} + a_{2i+1}\alpha_{3,1} + \cdots + a_{3i}\alpha_{3,i} = 0. \qquad (*)$$

Observing the orders at P_{m+3}, we see that we must have $a_{i+1} = a_{i+2} = 0$, since $\alpha_{2,1}$ and $\alpha_{2,2}$ have lower order at P_{m+3} than any of the other $\alpha_{j,k}$. We needn't have that $a_{i+3} = 0$ though, since we may have that the order of $\alpha_{3,1}$ is $m - i$ at P_{m+3} and $\alpha_{3,1}$ and $\alpha_{2,3}$ may "cancel" each other.

However, now considering the orders at P_{m+4}, we see that we must have $a_{2i+1} = a_{2i+2} = 0$. But, going back to the situation at P_{m+3}, this implies that $a_{i+3} = a_{i+4} = 0$. And this in turn implies that $a_{2i+3} = a_{2i+4} = 0$ (going back to P_{m+4}). By continuing to go back and forth in this manner, we can show that all the coefficients in (*) must be 0, establishing the linear independence of the $\alpha_{j,k}$.

Odd genus case. Suppose $g = 2m + 1$ and suppose n is given such that $4(n - 3) - 3g$ is nonnegative. Consider Meis's surface of genus g. Our divisor D will consist of the following points:

(1) $P_1, \ldots, P_{m+2}$,

(2) P_0 with multiplicity $m - i$, where $i = 3 + g - n$, and

(3) the other two points over $x = 0$.

By a completely analogous argument to that in the even genus case, one can show that the $\alpha_{j,k}$ associated to D are linearly independent.

Applying Proposition 1 as before, we have

THEOREM 2. $\mathscr{G}_n^3$ *has a component of dimension* $3g - 3 + \tau + 3$ *for any n and g such that* τ *is nonnegative.*

We were unable to use Meis's examples to construct special divisors of higher dimension such that we could show that the associated quadratic differentials were linearly independent. We were also unable to provide other examples of surfaces to deal with higher dimension divisors.

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