ON ISOCLINIC CLOSURE: CORRECTION TO "ELLIPTIC SPACES IN GRASSMANN MANIFOLDS"[1]

BY

DANIEL B. SHAPIRO AND JOSEPH A. WOLF

Let $\mathbf{F}$ be a field $\mathbf{R}$ (real), $\mathbf{C}$ (complex) or $\mathbf{K}$ (quaternion), $\mathbf{F}^k$ the positive definite left unitary vector space of dimension k over $\mathbf{F}$, and $\mathbf{G}_{n,k}(\mathbf{F})$ the Grassmann manifold of n-dimensional $\mathbf{F}$-subspaces of $\mathbf{F}^k$ with its usual structure as a Riemannian symmetric space. If W is a subspace of $\mathbf{F}^k$ then π_W denotes the orthogonal projection $\mathbf{F}^k \to W$. Subspaces B, B' of the same dimension in $\mathbf{F}^k$ are *isoclinic* if $\pi_B: B' \to B$ is an $\mathbf{F}$-unitary similarity. For example, the connected totally geodesic submanifolds $\mathbf{B}$ in $\mathbf{G}_{n,k}(\mathbf{F})$ such that any two distinct elements of $\mathbf{B}$ have zero intersection as subspaces of $\mathbf{F}^k$, have the property that the elements of $\mathbf{B}$ are pairwise isoclinic [2, Theorems 2 and 4], [3, Theorem 2]. In [2] and [3] one finds a complete classification of all such submanifolds $\mathbf{B}$.

After writing out that classification, Wolf considered an arbitrary subset $\mathbf{A}$ of $\mathbf{G}_{n,k}(\mathbf{F})$ whose elements are pairwise isoclinic, and in [3, Section 7] he claimed to define an operation of "isoclinic closure" enlarging $\mathbf{A}$ to a totally geodesic submanifold $\mathbf{A}_* = \mathbf{B} \subset \mathbf{G}_{n,k}(\mathbf{F})$ of the type described above. That isoclinic closure operation depended in an essential manner on the following property:

(*) If $X, B, B' \in \mathbf{G}_{n,k}(\mathbf{F})$ are pairwise isoclinic with $B \neq B'$ then $Z = \pi_{B \oplus B'}(X)$ either is 0 or is n-dimensional and isoclinic to X.

If $k = 2n$ then (*) follows from the Hurwitz equations; see [2, Theorem 1]. Professor Y.-C. Wong [4; Chapter III, Section 11] and Daniel Shapiro (unpublished) gave examples showing that (*) fails for $(n, k) = (2, 6)$, and thus fails whenever n is even and $k \geq 3n$.

Here is a counterexample to (*) for $k = 2n + 1$, thus for all (n, k) with $k > 2n$. Let $\{e_1, \ldots, e_n; f_1, \ldots, f_n; u\}$ be an orthornormal basis of $\mathbf{F}^{2n+1}$ and define the linear spans

$$B = \{e_1, \ldots, e_n\}_{\mathbf{F}} \quad \text{and} \quad B' = \{e_1 + f_1, \ldots, e_n + f_n\}_{\mathbf{F}},$$

and

$$X = \{x_1, \ldots, x_n\}_{\mathbf{F}} \quad \text{where } x_1 = e_1 + f_1 + 2\sqrt{2}\, u$$

and

$$x_j = e_j - 3f_j \quad \text{for } 2 \leq j \leq n.$$

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[1] Illinois J. Math., vol. 7 (1963), pp. 447–462.

Then B and B' are clearly isoclinic, and the **F**-hermitian inner products of the x_i are

$$x_1 \cdot x_1 = 1 + 1 + 8 = 10 = 1 + 9 = x_j \cdot x_j \quad \text{for } 2 \leq j \leq n$$

and

$$x_i \cdot x_j = 0 \quad \text{for } i \neq j.$$

Since $e_j - 3f_j = -(e_j + f_j) + 2(e_j - f_j)$, the orthogonal projection $\pi_B: X \to B$ sends x_1 to e_1 and x_j to e_j for $2 \leq j \leq n$, and the orthogonal projection $\pi_{B'}: X \to B'$ sends x_1 to $e_1 + f_1$ and x_j to $-(e_j + f_j)$ for $2 \leq j \leq n$. These are similarities, so X, B and B' are pairwise isoclinic, while $\pi_{B \oplus B'}: X \to B \oplus B'$ sends x_1 to $e_1 + f_1$ and x_j to $e_j - 3f_j$ for $2 \leq j \leq n$, which is not a similarity.

While we no longer have the isoclinic closure operation, we can still define *isoclinically closed set* as a subset $\mathbf{A} \subset \mathbf{G}_{n,k}(\mathbf{F})$ whose elements are pairwise isoclinic and satisfy the following:

If B, $B' \in \mathbf{A}$ with $B \neq B'$, then $\mathbf{A}$ contains the isoclinic sphere on $B \oplus B'$ constructed in [2, Chapter I] *from* $\{X \in \mathbf{A}: X \subset B \oplus B'\}$.

This agrees with the definition of isoclinically closed set in [3, Section 7] when the latter happens to make sense. Except for that definition we must throw out everything in [3, Section 7]. The notions of reducibility and support in [3, Section 8] remain valid, and [3, Lemma 10] is true for isoclinically closed sets:

Let B be an isoclinically closed set of pairwise isoclinic n-dimensional subspaces of $\mathbf{F}^k$. *Then* $\mathbf{B} = \mathbf{B}^1 \cup \cdots \cup \mathbf{B}^m$ *where the* $\mathbf{B}^i$ *are isoclinically closed and irreducible, and* $\operatorname{supp} \mathbf{B}^i \perp \operatorname{supp} \mathbf{B}^j$ *for $i \neq j$. In the topology induced from* $\mathbf{G}_{n,k}(\mathbf{F})$, *the* $\mathbf{B}^i$ *are the topological components of* $\mathbf{B}$.

Thus the main result of [3, Chapter II], which is Theorem 4 in Section 9 there, is correct as stated and proved.

References

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2. J. A. Wolf, *Geodesic spheres in Grassmann manifolds*, Illinois J. Math., vol. 7 (1963), pp. 425–446.
3. ———, *Elliptic spaces in Grassmann manifolds*, Illinois J. Math., vol. 7 (1963), pp. 447–462.
4. Y.-C. Wong, *Geometry of n-planes in Euclidean and pseudo-Euclidean spaces and differential geometry of Grassmann manifolds and Cartan domains*,

University of California
Berkeley, California