GEOMETRY OF INTEGRAL SUBMANIFOLDS OF A CONTACT DISTRIBUTION

BY

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1. A differentiable $(2n + 1)$-dimensional manifold M is said to be a *contact manifold* if it carries a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. This condition, roughly speaking, means that the $2n$-dimensional "(tangent) subbundle" D defined by $\eta = 0$ is as far from being integrable as possible. In particular, the maximum dimension of an integral submanifold of D is n [3]. However, not much seems to be known about the immersion of such submanifolds into the ambient space, especially from Riemannian point of view. Thus we consider in this paper a normal contact metric (Sasakian) manifold, especially one with constant ϕ-sectional curvature, and study the immersion of its n-dimensional integral submanifolds.

The main result of this paper (Theorem 4.2) is that a compact minimal integral submanifold of a Sasakian space form M is totally geodesic if the square of the length of the second fundamental form is bounded by

$$\frac{n\{n(\tilde{c} + 3) + \tilde{c} - 1\}}{4(2n - 1)}$$

where $\tilde{c}$ is the ϕ-sectional curvature of M. In addition to giving other properties of integral submanifolds, we give examples in Section 5 of totally geodesic and minimal nontotally geodesic integral submanifolds.

2. Let M be a contact manifold with contact form η. It is well known that a contact manifold carries an *associated almost contact metric structure* (ϕ, ξ, η, G) where ϕ is a tensor field of type (1, 1), ξ a vector field, and G a Riemannian metric satisfying

$$\phi^2 = -I + \xi \otimes \eta, \quad \eta(\xi) = 1, \quad G(\phi X, \phi Y) = G(X, Y) - \eta(X)\eta(Y) \quad (2.1)$$

and

$$\Phi(X, Y) = G(X, \phi Y) = d\eta(X, Y). \quad (2.2)$$

The existence of tensors ϕ, ξ, η, G on a differentiable manifold M satisfying equations (2.1) is equivalent to a reduction of the structural group of the tangent bundle to $U(n) \times 1$ [2].

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Let $\tilde{\nabla}$ denote the Riemannian connection of G. Then M is a *normal contact metric* (*Sasakian*) *manifold* if

$$(\tilde{\nabla}_X \phi)Y = G(X, Y)\xi - \eta(Y)X \tag{2.3}$$

in which case we have

$$\tilde{\nabla}_X \xi = -\phi X. \tag{2.4}$$

A plane section of the tangent space $T_m M$ at $m \in M$ is called a *ϕ-section* if it is spanned by vectors X and ϕX orthogonal to ξ.

The sectional curvature $\tilde{K}(X, \phi X)$ of a ϕ-section is called a *ϕ-sectional curvature*. A Sasakian manifold is called a *Sasakian space form* and denoted $M(\tilde{c})$ if it has constant ϕ-sectional curvature equal to $\tilde{c}$; in this case the curvature transformation $\tilde{R}_{XY} = [\tilde{\nabla}_X, \tilde{\nabla}_Y] - \tilde{\nabla}_{[X,Y]}$ is given by

$$\begin{aligned} \tilde{R}_{XY} = \tfrac{1}{4}(\tilde{c} + 3)\{&G(Y, Z)X - G(X, Z)Y\} \\ + \tfrac{1}{4}(\tilde{c} - 1)\{&\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ G(X, Z)\eta(Y)\xi - G(Y, Z)\eta(X)\xi \\ &+ \Phi(Z, Y)\phi X - \Phi(Z, X)\phi Y + 2\Phi(X, Y)\phi Z\}. \end{aligned} \tag{2.5}$$

Let $\iota \colon N \to M$ be an immersed submanifold of codimension p. If G denotes the metric on M, the induced metric g is given by $g(X, Y) \circ \iota = G(\iota_* X, \iota_* Y)$. For simplicity we shall henceforth not distinguish notationally between X and $\iota_* X$. Let ∇ and $\tilde{\nabla}$ denote the Riemannian connections of g and G, respectively, $\nabla^\perp$ the connection in the normal bundle, and $\xi_1, \ldots, \xi_p$ a local field of orthonormal normal vectors. Then the Gauss-Weingarten equations are

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad \tilde{\nabla}_X \xi_\alpha = -A_\alpha X + \nabla^\perp_X \xi_\alpha,$$

where σ is the second fundamental form and the A_α's the Weingarten maps. Decomposing σ we have $\sigma(X, Y) = \sum_\alpha h^\alpha(X, Y)\xi_\alpha$ where the tensors h^α satisfy $h^\alpha(X, Y) = g(A_\alpha X, Y)$ and are symmetric. Letting R denote the curvature of ∇, the Gauss equation is

$$\begin{aligned} g(R_{XY}Z, W) = G(\tilde{R}_{XY}Z, W) &+ G(\sigma(X, W), \sigma(Y, Z)) \\ &- G(\sigma(X, Z), \sigma(Y, W)). \end{aligned} \tag{2.6}$$

Finally for the second fundamental form σ, we define the covariant derivative $'\nabla$ with respect to the connection in the (tangent bundle) $\oplus$ (normal bundle) by

$$('\nabla_X \sigma)(Y, Z) = \nabla^\perp_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).$$

3. Let M be a contact manifold, then the "(tangent) subbundle" D defined by $\eta = 0$ admits integral submanifolds up to and including dimension n but of no higher dimension [3]. It is also shown in [3] that in order for r linearly independent vectors $X_1, \ldots, X_r \in T_m M$ to be tangent to an r-dimensional

integral submanifold of D, it is necessary and sufficient that $\eta(X_i) = 0$ and $d\eta(X_i, X_j) = 0$, $i, j = 1, \ldots, r$. Moreover such integral submanifolds are quite abundant in the sense that given $X \in T_m M$ belonging to D, there exists an r-dimensional integral submanifold ($1 \leq r \leq n$) of D through m such that X is tangent to it.

We first give a simple characterization of an integral submanifold of D in terms of an associated almost contact metric structure.

PROPOSITION 3.1. *Let $\iota: N \to M$ be an immersed submanifold. N is an integral submanifold of D if and only if every tangent vector X belongs to D and ϕX is normal.*

Proof. If N is an integral submanifold of D and X and Y arbitrary vectors on N, then $0 = d\eta(X, Y) = G(X, \phi Y)$ and so ϕY is normal. Conversely for X belonging to D, $\eta(X) = 0$. Also since ϕX and ϕY are normal for X and Y tangent, $d\eta(X, Y) = G(X, \phi Y) = 0$ and N is an integral submanifold.

In this paper we concentrate on integral submanifolds of D of dimension n. Let $\iota: N \to M$ be an integral submanifold and $X_1, \ldots, X_n$ a local orthonormal basis of vector fields on N. Then we define a local field of orthonormal vectors ξ_α, $\alpha = 0, 1, \ldots, n$ by $\xi_0 = \xi$ and $\xi_i = \phi X_i$, $i = 1, \ldots, n$.

A contact manifold whose associated structure satisfies equation (2.4) is called *K-contact*, a somewhat weaker notion than that of a Sasakian structure (equation (2.3)).

PROPOSITION 3.2. *For an integral submanifold of a K-contact manifold, the second fundamental form in the direction ξ vanishes.*

Proof. $h^0(X, Y) = G(\tilde{\nabla}_X Y, \xi) = -G(Y, \tilde{\nabla}_X \xi) = G(Y, \phi X) = 0$.

Let $\omega^1, \ldots, \omega^n, \omega^{1*}, \ldots, \omega^{n*}, \omega^0 = \eta$ be the dual basis of X_i, ϕX_i, ξ, $i = 1, \ldots, n$. Then the first structural equation of Cartan for M is

$$d\omega^A = -\sum_{B=0}^{2n} \omega^A_B \wedge \omega^B, \quad n + 1 = 1^*, \text{ etc.},$$

where (ω^A_B) is a real representation of a skew-Hermitian matrix and hence we have $\omega^{i*}_j = \omega^{j*}_i$. Now as $\omega^\alpha = 0$ along N we have $\sum_B \omega^\alpha_B \wedge \omega^B = 0$ in which the ω^α_i give the second fundamental form, i.e.

$$\omega^{i*}_j = \sum_k h^i{}_{jk}\omega^k, \quad \omega^0_i = \sum_j h^0{}_{ij}\omega^j \tag{3.1}$$

where $h^\alpha{}_{ij} = h^\alpha(X_i, X_j)$. We now obtain the following algebraic proposition.

PROPOSITION 3.3. *Let N be an immersed submanifold of an almost contact manifold M (structural group $U(n) \times 1$) such that the condition of Proposition 3.1 holds. Then the Weingarten maps A_i, $i = 1, \ldots, n$ satisfy*

(1) $A_i X_j = A_j X_i$,

(2) $\operatorname{tr}(\sum_i A_i^2)^2 = \sum_{i,j} (\operatorname{tr} A_i A_j)^2$.

Proof. From (3.1) and the fact that $\omega_j^{i*} = \omega_i^{j*}$ we have $h^i{}_{jk} = h^j{}_{ik}$, but $h^\alpha{}_{jk} = h^\alpha(X_j, X_k) = g(A_\alpha X_j, X_k)$ giving (1). For (2) we have

$$\operatorname{tr}\left(\sum_i A_i^2\right)^2 = \sum \operatorname{tr} A_i^2 A_j^2 = \sum h^i{}_{kl} h^i{}_{lm} h^j{}_{mh} h^j{}_{hk}$$
$$= \sum h^k{}_{il} h^m{}_{li} h^m{}_{jh} h^k{}_{hj} = \sum (\operatorname{tr} A_k A_m)^2$$

where the sums are over all repeated indices.

4. In this section we study n-dimensional integral submanifolds which are minimally immersed in a Sasakian space form $M(\tilde{c})$. Let N denote the submanifold and ι the immersion. Since $\eta(X) = 0$ for X tangent to N, we have from equation (2.5) and the Gauss equation (2.6)

$$\begin{aligned} g(R_{XY}Z, W) &= \tfrac{1}{4}(\tilde{c} + 3)(g(X, W)g(Y, Z) - g(X, Z)g(Y, W)) \\ &\quad + \sum_\alpha (g(A_\alpha X, W)g(A_\alpha Y, Z) - g(A_\alpha X, Z)g(A_\alpha Y, W)) \end{aligned} \tag{4.1}$$

and hence the sectional curvature $K(X, Y)$ of N determined by an orthonormal pair X, Y is

$$K(X, Y) = \tfrac{1}{4}(\tilde{c} + 3) + \sum_\alpha (g(A_\alpha X, X)g(A_\alpha Y, Y) - g(A_\alpha X, Y)^2). \tag{4.2}$$

Moreover the Ricci tensor S and the scalar curvature ρ of N are given by

$$\begin{aligned} S(X, Y) &= \tfrac{1}{4}(n - 1)(\tilde{c} + 3)g(X, Y) \\ &\quad + \sum_\alpha (\operatorname{tr} A_\alpha)g(A_\alpha X, Y) - \sum_\alpha g(A_\alpha X, A_\alpha Y) \end{aligned}$$

and

$$\rho = [\tfrac{1}{4}n(n - 1)](\tilde{c} + 3) + \sum_\alpha (\operatorname{tr} A_\alpha)^2 - \|\sigma\|^2$$

where $\|\sigma\|^2 = \sum_\alpha \operatorname{tr}(A_\alpha^2) = \sum_{\alpha, i, j} h^\alpha{}_{ij} h^\alpha{}_{ij}$ is the square of the length of the second fundamental form. In particular, if the immersion is minimal,

$$S(X, Y) = \tfrac{1}{4}(n - 1)(\tilde{c} + 3)g(X, Y) - \sum_\alpha g(A_\alpha X, A_\alpha Y), \tag{4.3}$$

$$\rho = [\tfrac{1}{4}n(n - 1)](\tilde{c} + 3) - \|\sigma\|^2. \tag{4.4}$$

THEOREM 4.1. *Let N be an integral submanifold of a Sasakian space form $M(\tilde{c})$ which is minimally immersed. Then the following are equivalent:*

(a) *N is totally geodesic,*

(b) $K = \tfrac{1}{4}(\tilde{c} + 3)$,

(c) $S = \tfrac{1}{4}(n - 1)(\tilde{c} + 3)g$,

(d) $\rho = \tfrac{1}{4}n(n - 1)(\tilde{c} + 3)$.

Proof. That (a) implies (b), (c), and (d) is immediate from (4.2), (4.3), and (4.4), respectively. That (c) and (d) each imply (a) is also immediate. For (b) implies (a), let X_1 be an arbitrary unit vector and choose $X_2, \ldots, X_n$ such that $X_1, X_2, \ldots, X_n$ is an orthonormal basis. Then

$$S(X_1, X_1) = \sum_{i=2}^{n} K(X_1, X_i) = \tfrac{1}{4}(\tilde{c} + 3)(n - 1)$$

which is (c).

LEMMA 4.1. *Let N be a minimal integral submanifold of a Sasakian space form $M(\tilde{c})$. Then*

$$\begin{aligned} \tfrac{1}{2}\Delta\|\sigma\|^2 &= \|'\nabla\sigma\|^2 + \sum_{i,j} \operatorname{tr}(A_iA_j - A_jA_i)^2 - \sum_{i,j} (\operatorname{tr} A_iA_j)^2 \\ &\quad + \tfrac{1}{4}(n(\tilde{c} + 3) + \tilde{c} - 1)\|\sigma\|^2 \\ &= \|'\nabla\sigma\|^2 + 2\sum_{i,j} \operatorname{tr}(A_iA_j)^2 - 3\sum_{i,j} (\operatorname{tr} A_iA_j)^2 \\ &\quad + \tfrac{1}{4}(n(\tilde{c} + 3) + \tilde{c} - 1)\|\sigma\|^2. \end{aligned}$$

Proof. In the same way as in [1], we have the following formula:

$$\begin{aligned} \tfrac{1}{2}\Delta\|\sigma\|^2 &= \|'\nabla\sigma\|^2 + \sum_{\alpha,\beta} \operatorname{tr}(A_\alpha A_\beta - A_\beta A_\alpha)^2 - \sum_{\alpha,\beta} (\operatorname{tr} A_\alpha A_\beta)^2 \\ &\quad + \sum (4\tilde{R}^\alpha{}_{\beta ij}h^\alpha{}_{jk}h^\beta{}_{ik} - \tilde{R}^\alpha{}_{k\beta k}h^\alpha{}_{ij}h^\beta{}_{ij} + 2\tilde{R}^i{}_{jkj}h^\alpha{}_{il}h^\alpha{}_{kl} + 2\tilde{R}^i{}_{jkl}h^\alpha{}_{il}h^\alpha{}_{jk}) \end{aligned}$$

where $\tilde{R}^A{}_{BCD}$ are the components of the curvature tensor of $\tilde{\nabla}$. Using equation (2.5) the last term on the right hand side becomes

$$\tfrac{1}{4}(n(\tilde{c} + 3) + \tilde{c} - 1)\|\sigma\|^2$$

giving the first equality. The second follows from the first by Proposition 3.3.

LEMMA 4.2 [1]. $\operatorname{tr}(A_\alpha A_\beta - A_\beta A_\alpha)^2 \geq -2(\operatorname{tr} A_\alpha^2)(\operatorname{tr} A_\beta^2)$.

THEOREM 4.2. *Let N be a compact minimal integral submanifold of a Sasakian space form $M(\tilde{c})$, $\tilde{c} > -3$. If*

$$\|\sigma\|^2 < \frac{n\{n(\tilde{c} + 3) + \tilde{c} - 1\}}{4(2n - 1)}$$

then N is totally geodesic.

Proof. Let $\Lambda = (\operatorname{tr} A_iA_j)$. Then Λ is a symmetric $n \times n$ matrix defined with respect to an orthonormal basis $e_1, \ldots, e_n$ at some point $p \in M^n$. The corresponding matrix defined with respect to another orthonormal basis is congruent to Λ. Thus, without loss of generality, we may assume that

$$\operatorname{tr} A_iA_j = 0 \quad \text{for} \quad i \neq j.$$

From Lemma 4.1 we have

$$\begin{aligned} \tfrac{1}{2}\Delta\|\sigma\|^2 &= \|'\nabla\sigma\|^2 + \sum_{i,j} \operatorname{tr}(A_iA_j - A_jA_i)^2 - \sum_{i} (\operatorname{tr} A_i^2)^2 \\ &\quad + \tfrac{1}{4}(n(\tilde{c} + 3) + \tilde{c} - 1)\|\sigma\|^2; \end{aligned}$$

but using Lemma 4.2

$$\begin{aligned}\tfrac{1}{2}\,\Delta\|\sigma\|^2 &\geq -2\sum_{i\neq j}(\operatorname{tr} A_i^2)(\operatorname{tr} A_j^2) - \sum_i (\operatorname{tr} A_i^2)^2 \\ &\quad + \tfrac{1}{4}(n(\tilde c + 3) + \tilde c - 1)\|\sigma\|^2 \\ &= \frac{1}{n}\sum_{i<j}(\operatorname{tr} A_i^2 - \operatorname{tr} A_j^2)^2 - \left(2 - \frac{1}{n}\right)\left(\sum_i \operatorname{tr} A_i^2\right)^2 \\ &\quad + \tfrac{1}{4}(n(\tilde c + 3) + \tilde c - 1)\|\sigma\|^2 \\ &\geq -\left(2 - \frac{1}{n}\right)\|\sigma\|^4 + \tfrac{1}{4}(n(\tilde c + 3) + \tilde c - 1)\|\sigma\|^2 \\ &= \frac{2n-1}{n}\|\sigma\|^2\left(\frac{n^2(\tilde c + 3) + n(\tilde c - 1)}{4(2n-1)} - \|\sigma\|^2\right).\end{aligned}$$

Thus we have $\Delta\|\sigma\|^2 \geq 0$, but $\int_N \Delta\|\sigma\|^2 * 1 = 0$ so that $\Delta\|\sigma\|^2 = 0$ and hence $\|\sigma\| = 0$ giving the result.

COROLLARY. *Let N be a complete minimal integral surface in a 5-dimensional Sasakian space form $M(\tilde c)$. If the sectional curvature of N is greater than 1/3, N is totally geodesic.*

Proof. Since N is complete and its sectional curvature greater than 1/3, N is compact. The result now follows from equation (4.4) and the theorem.

THEOREM 4.3. *Let N be a minimal integral submanifold of a Sasakian space form $M(\tilde c)$. If N is a space form of constant curvature c, then either $c = (\tilde c + 3)/4$, in which case N is totally geodesic, or $c \leq 1/(n+1)$ with equality if and only if $\tilde\nabla\sigma = 0$.*

Proof. Since N has constant curvature c, $\rho = n(n-1)c$ and equation (4.4) gives

$$\|\sigma\|^2 = n(n-1)(\tfrac{1}{4}(\tilde c + 3) - c) \quad \text{and} \quad c \leq \tfrac{1}{4}(\tilde c + 3).$$

Also equation (4.1) becomes

$$\sum_\alpha (h^\alpha{}_{ik}h^\alpha{}_{jl} - h^\alpha{}_{il}h^\alpha{}_{jk}) = (c - \tfrac{1}{4}(\tilde c + 3))(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

Multiplying both sides by $\sum_m h^m{}_{il}h^m{}_{jk}$ and summing on i, j, k and l, we have

$$\sum_{h,m} \operatorname{tr}(A_hA_m)^2 - \sum_{h,m}(\operatorname{tr} A_hA_m)^2 = (c - \tfrac{1}{4}(\tilde c + 3))\|\sigma\|^2. \tag{4.5}$$

Moreover N is Einstein, so $S = (\rho/n)g$ and equations (4.3) and (4.4) give

$$\sum_{i,k} h^i{}_{jk}h^i{}_{kl} = \left(\tfrac{1}{4}(n-1)(\tilde c + 3) - \frac{\rho}{n}\right)\delta_{jl} = \frac{\|\sigma\|^2}{n}\delta_{jl}$$

which is equivalent to $\sum_{i,k} h^j{}_{ik} h^l{}_{ki} = (\|\sigma\|^2/n)\delta_{jl}$ by Proposition 3.3 and so

$$\operatorname{tr} A_j A_l = \frac{\|\sigma\|^2}{n}\delta_{jl}. \tag{4.6}$$

Substituting (4.5) and (4.6) into the second equation of Lemma 4.1 we have

$$0 = \|'\nabla\sigma\|^2 + 2(c - \tfrac{1}{4}(\tilde{c} + 3))\|\sigma\|^2 - \frac{\|\sigma\|^4}{n} + \tfrac{1}{4}(n(\tilde{c} + 3) + \tilde{c} - 1)\|\sigma\|^2$$

or

$$\|'\nabla\sigma\|^2 = n(n^2 - 1)(c - \tfrac{1}{4}(\tilde{c} + 3))\left(c - \frac{1}{n + 1}\right)$$

from which the result follows.

5. In this section we give examples of some integral submanifolds of Sasakian space forms.

Consider the space C^{n+1} of $n + 1$ complex variables and let J denote its usual almost complex structure. Let

$$S^{2n+1} = \{z \in C^{n+1}: |z| = 1\}.$$

We give S^{2n+1} its usual contact structure as follows. For every $z \in S^{2n+1}$ and $X \in T_z S^{2n+1}$, set $\xi = -Jz$ and $\phi X = JX$. Let η be the dual 1-form of ξ and G the standard metric on S^{2n+1}. Then (ϕ, ξ, η, G) is a Sasakian structure on S^{2n+1}. Let L be an $(n + 1)$-dimensional linear subspace of C^{n+1} passing through the origin and such that JL is orthogonal to L. Then $S^{2n+1} \cap L$ satisfies the condition of Proposition 3.1 and so is an integral submanifold of D for the manifold S^{2n+1}. Clearly $S^{2n+1} \cap L$ is an n-sphere imbedded as a totally geodesic submanifold of S^{2n+1}.

For a second example of a totally geodesic submanifold, consider R^5 with its usual contact structure $\eta = \frac{1}{2}(dx^5 - x^3\,dx^1 - x^4\,dx^2)$. Then D is spanned by $X_1 = (\partial/\partial x^1) + x^3(\partial/\partial x^5)$, $X_2 = (\partial/\partial x^3)$, $X_3 = (\partial/\partial x^2) + x^4(\partial/\partial x^5)$, $X_4 = (\partial/\partial x^4)$. The distinguished vector field ξ is $2(\partial/\partial x^5)$, G is given by

$$\eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{4} dx^i \otimes dx^i$$

and ϕ can be found from $d\eta$ and G. With respect to the structure (ϕ, ξ, η, G), it is well known that R^5 is a Sasakian space form of constant ϕ-sectional curvature equal to -3. Let X, Y be independent linear combinations of the X_i, having constant coefficients, such that Y is orthogonal to ϕX. Computing $[X, Y]$ we find $[X, Y] = 0$ so that X and Y determine an integral surface N on which we may choose coordinates u and v such that $X = \iota_*(\partial/\partial u)$ and $Y = \iota_*(\partial/\partial v)$. Thus N has coordinates u and v such that $\partial/\partial u$ and $\partial/\partial v$ form an orthonormal basis with respect to the induced metric and hence N is flat. Therefore, since $\tilde{c} = -3$, Theorem 4.1 shows that N is totally geodesic.

Finally we give an example of an integral submanifold of a Sasakian space form which is minimal but not totally geodesic. Let

$$S^5 = \{z \in C^3 : |z| = 1\}$$

be the 5-dimensional sphere with the Sasakian structure described above. If we write $z = (z^1, z^2, z^3)$, the equations $|z^1| = |z^2| = |z^3| = 1/\sqrt{3}$ give an imbedding of a 3-dimensional torus T^3 in S^5 which is minimal [1]. Moreover ξ is tangent to T^3, and for X orthogonal to ξ and tangent to T^3, ϕX is normal to T^3 in S^5. Viewing T^3 as a cube with opposite faces identified, ξ is just a "diagonally pointing" vector field. Now consider a 2-dimensional torus T^2 imbedded in T^3 by $\sum_\alpha \log(\sqrt{3})z^\alpha = 2k\pi\sqrt{(-1)}$ where the logarithm is the multi-valued one and k is an integer. Then T^2 is orthogonal to ξ in T^3 and hence an integral submanifold of S^5. Since $\nabla_X \xi = -\phi X$, T^2 is totally geodesic in T^3 and hence minimal and not totally geodesic in S^5.

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