MODULES WITH WAISTS[1]

BY

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When one studies the finitely generated modules over an Artin ring, one basically studies the indecomposable ones. In such a study one tries to understand

(1) the structure of the indecomposable modules,
(2) how to construct new indecomposable modules from old and
(3) whether or not there are a finite number of nonisomorphic indecomposable modules.

In this context, we introduce a new class of indecomposable modules which we call modules having waists. Section 1 develops the basic theory of modules having waists. In Section 2 we restrict our study to waists in radical^2 = 0 Artin rings. Sections 3 and 4 deal with the construction of new modules from old with emphasis on the construction of modules having waists.

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Section 1

Let Λ be a ring. All modules are left unitary modules unless otherwise stated. We say a Λ-module M *has a waist* if there is a nontrivial proper submodule M' of M such that every submodule of M contains M' or is contained in M'. In this case, we say that *M' is a waist in M*. We immediately have the following.

(a) If M has a waist then M is indecomposable.
(b) If M' is a waist in M and $X \subset_{\neq} M' \subset_{\neq} Y \subseteq M$ then M'/X is a waist in Y/X. In particular, Y/X is indecomposable.
(c) If M has a waist and Λ is left Noetherian, then M has a unique maximal waist if M is finitely generated.
(d) If M'' is a waist in M' and if M' is a waist in M then M'' is a waist in M.
(e) If M'' is a waist in M' and $M' \subseteq M$ with M'/M'' a waist in M/M'' then M' and M'' are both waists in M.

We henceforth assume Λ is a left Artin ring with radical $\mathbf{r}$. Recall the definition of the lower Loewy series for a Λ-module M. It is the sequence of submodules

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$0 \subseteq S_0(M) \subseteq S_1(M) \subseteq \cdots \subseteq S_n(M) = M$ where $S_0(M) = \text{soc}(M)$ and $S_i(M) = \pi_i^{-1}(\text{soc}(M/S_{i-1}(M)))$ with $\pi_i: M \to M/S_{i-1}(M)$ the canonical surjection. We keep this notation for the rest of the paper. The following proposition shows that a module has at most a finite number of different waists and that these waists must be of a special form.

PROPOSITION 1.1. *Suppose M' is a waist in M. Then*

(1) $M' = \mathbf{r}^i M$ *for some* $i \geq 1$
(2) $M' = S_j(M)$ *for some* $j \geq 0$.

If $M' = \mathbf{r}^i M = S_j(M)$ then $i + j + 1 =$ Loewy length of M, which we denote by $ll(M)$. (Recall that $ll(M) =$ smallest n such that $\mathbf{r}^n M = 0$.)

Proof. Suppose $\mathbf{r}^i M \subseteq M'$ but $\mathbf{r}^{i-1} M \not\subseteq M'$. Then M' is contained in $\mathbf{r}^{i-1}M$. Since $\mathbf{r}^{i-1}M/\mathbf{r}^i M$ is a semisimple module with $M'/\mathbf{r}^i M$ a proper submodule, it follows that $M'/\mathbf{r}^i M$ is not a waist in $\mathbf{r}^{i-1}M/\mathbf{r}^i M$. By (b) above, $M'/\mathbf{r}^i M = 0$ and (1) is proven. One may similarly prove (2). The final statement is true for arbitrary modules and is also easily proven.

As a consequence of Proposition 1.1, we get:

COROLLARY 1.2. *If M' is a waist in M,* $\text{soc}(M') = \text{soc}(M)$.

It is easy to see that the class of modules having waists includes nonsimple modules having either a unique maximal submodule or a unique minimal submodule. Hence nonsimple indecomposable projective, injective, and uniserial modules all have waists.

In studying modules having waists the following result is useful.

THEOREM 1.2. *Let Λ be a left Artin ring. Suppose M' is a nontrivial submodule of M. Then the following statements are equivalent.*

(1) *M' is a waist in M.*
(2) *If $X \subsetneq M' \subsetneq Y \subseteq M$ then Y/X is indecomposable.*
(3) *$M'/\mathbf{r}M'$ is a waist in $M/\mathbf{r}M'$.*
(4) *If $\pi: M \to M/M'$ is the canonical surjection, then M' is a waist in $\pi^{-1}(\text{soc}(M/M'))$.*

Proof. (1) $\Rightarrow$ (2) by (b) above.

(2) $\Rightarrow$ (3). Let $\overline{N}$ be a submodule of $M/\mathbf{r}M'$. Let $N \subseteq M$ such that $\mathbf{r}M' \subseteq N$ and $N/\mathbf{r}M' = \overline{N}$. Assume $\overline{N}$ is not contained in $M'/\mathbf{r}M'$. We want to show $\overline{N}$ contains $M'/\mathbf{r}M'$. It suffices to show $M' \subseteq N$. Now

$$N \cap M' \subseteq M' \subsetneq N + M'.$$

Assume $N \cap M' \neq M'$. Then $N + M'/N \cap M' \simeq N/N \cap M' \oplus M'/N \cap M'$. By (2), since $N + M'/N \cap M'$ is indecomposable, $N/N \cap M' = (0)$. Thus $N \subseteq M'$, a contradiction.

(3) $\Rightarrow$ (4). Let $X \subseteq \pi^{-1}(\text{soc}(M/M'))$. Suppose X is not contained in M'. By (3), $X + \mathbf{r}M' \supseteq M'$. Thus $X \cap M' + \mathbf{r}M' = M'$. By Nakayama's Lemma, $X \cap M' = M'$ and hence $X \supseteq M'$.

(4) $\Rightarrow$ (1). Let $X \subseteq M$. Suppose X is not contained in M'. If we show that $X \cap \pi^{-1}(\text{soc}(M/M'))$ is not contained in M', then $X \cap \pi^{-1}(\text{soc}(M/M'))$ contains M' and thus X contains M'. But $\pi(X + M') \neq 0 \Rightarrow \text{soc}(M/M') \cap \pi(X + M) \neq 0$. It follows that there exists $x \in X$, $x \notin M'$ such that $x \in \pi^{-1}(\text{soc}(M/M'))$.

We now proceed to study waists M' in M with the property that $M'/\mathbf{r}M'$ is a simple Λ-module. These special kinds of waists play an important role in the study of waists in Artin algebras Λ such that $\Lambda/\mathbf{r}^2$ is of finite representation type. Recall that an Artin ring is of *finite representation type* if it has only a finite number of nonisomorphic finitely generated indecomposable modules. In Section 2 we show that if $\Lambda/\mathbf{r}^2$ is of finite representation type and if M has a waist, then either $M/\mathbf{r}M$ is simple or M has a waist M' with $M'/\mathbf{r}M'$ simple. For convenience we denote $X/\mathbf{r}X$ by top (X), where X is a Λ-module.

PROPOSITION 1.3. *Suppose M' is a proper submodule of M such that* top (M') *is simple. Then:*

(1) *M' is a waist in M if and only if* $\text{soc}(M/\mathbf{r}M') \cong \text{top}(M')$.

(2) *Suppose $\mathbf{r}M' \neq 0$. Let $\pi: M \to M/\mathbf{r}M'$. Then $\mathbf{r}M'$ is a waist in M if and only if $\mathbf{r}M'$ is a waist in* $\pi^{-1}(\text{soc}(M/\mathbf{r}M'))$.

Proof. (1) M' is a waist in M if and only if $M'/\mathbf{r}M'$ is a waist in $M/\mathbf{r}M'$ if and only if $\text{top}(M') = \text{soc}(M'/\mathbf{r}M') = \text{soc}(M/\mathbf{r}M')$. The last equivalence follows because top (M') is simple.

(2) This is a special case of Theorem 1.2 (4).

Note that it can easily happen that M' is contained in M with top (M') simple and $\mathbf{r}M'$ is a waist in M and yet M' is not a waist in M.

If Λ is an Artin algebra, i.e., Λ is a finitely generated module over its center C which is an Artin ring, then there is a duality, D, between left Λ-modules and right Λ-modules. Namely, if E is the C-injective envelope of $C/\text{rad}(C)$ then $D(X) = \text{Hom}_C(X, E)$ for X either a left or right Λ-module. One may easily check that if Λ is an Artin algebra then M' is a waist in M if and only if $D(M/M')$ is a waist in $D(M)$.

We conclude this section by showing that under suitable conditions, the lengths of modules having waists are bounded. Since nonsimple indecomposable injective modules always have waists, namely their socles, we assume that indecomposable injectives are finitely generated and hence of finite length. If X is a finitely generated Λ-module, $l(X)$ = length of X.

PROPOSITION 1.4. *Let Λ be a left Artin ring. Assume each indecomposable injective Λ-module is finitely generated. Let M be a Λ-module having a waist. Then*

$$l(M) + 2 \leq \max\{l(P) + l(E)\}$$

where P (resp. E) is an indecomposable projective (resp. injective) Λ-module.

Proof. Choose an indecomposable projective P such that there is a nonzero map $f: P \to M/M'$. This lifts to $g: P \to M$. By choice, $g(P)$ is not contained in M'. Thus $M' \subsetneq g(P)$ and hence $l(M') \leq l(P) - 1$.

Now let M'' be a maximal proper submodule of M'. Then M'/M'' is a waist in M/M''. Since M'/M'' is simple, $M'/M'' = \operatorname{soc}(M/M'')$. Let E be the Λ-injective envelope of M'/M''. Then there is an injection $M/M'' \to E$ and hence $l(M/M'') \leq l(E)$. Thus $l(M/M') \leq l(E) - 1$ and we conclude $l(M) \leq l(P) + l(E) - 2$.

COROLLARY 1.5. *If Λ is an Artin algebra and M has a waist, then the length of M is bounded. In particular $l(M) + 2 \leq \max \{l(P) + l(Q)\}$ where P (resp. Q) is an indecomposable left (resp. right) projective Λ-module.*

Proof. The proof follows from the fact that the duality D preserves length and if E is an indecomposable left injective, then $D(E)$ is an indecomposable right projective.

COROLLARY 1.6. *If Λ is an Artin algebra such that every indecomposable Λ-module is either simple or has a waist, then Λ is of finite representation type.*

Proof. It follows that the lengths of the indecomposable Λ-modules are bounded. The result follows from [2].

Section 2

We now study modules having waists in Artin rings Λ with $\mathbf{r}^2 = 0$. This is especially of interest because, by Theorem 1.2, in an arbitrary Artin ring Λ if M' is a waist in M with $M' = \mathbf{r}^i M = S_j(M)$ then

(1) $M'/\mathbf{r}M'$ is a waist in $\mathbf{r}^{i-1}M/\mathbf{r}M'$ and
(2) $M'/S_{j-1}(M)$ is a waist in $S_{j+1}(M)/S_{j-1}(M)$.

Thus, given a module which has a waist in an Artin ring Λ, it induces in general two different modules having waists over $\Lambda/\mathbf{r}^2$. Finally, if we consider the method of constructing new modules having waists discussed in Section 3, we can, at times, knowing the modules which have a waist for $\Lambda/\mathbf{r}^2$, create new ones of larger Loewy lengths.

If Λ is a left Artin ring with $\mathbf{r}^2 = 0$ and M has a waist, then it follows that $\mathbf{r}M$ is the unique waist in M. We begin the study by refining Theorem 1.2 to get a better classification of waists in Artin rings whose radical has square zero.

THEOREM 2.1. *Let Λ be a left Artin ring with $\mathbf{r}^2 = 0$. Let M be a Λ-module and $\pi: M \to M/\mathbf{r}M$ be the canonical surjection. Then the following statements are equivalent:*

(1) *M has a waist.*
(2) *If S is a simple summand of $M/\mathbf{r}M$ then $\pi^{-1}(S)$ is indecomposable.*
(3) *If X is a maximal submodule of $\mathbf{r}M$ then M/X is indecomposable.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) follow from Theorem 2.1.

(2) $\Rightarrow$ (1). Suppose $X \subseteq M$ and X is not contained in $\mathbf{r}M$. Let S be a simple summand of $\pi(X) \subseteq \pi(M)$. Then $\pi^{-1}(S) = X'$ is indecomposable. Now $\mathbf{r}M \subseteq X'$. It suffices to show $X' \subseteq X$. Consider the following exact commutative diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbf{r}M \cap X & \to & \mathbf{r}M & \to & Z & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X' \cap X & \to & X' & \to & X'/X' \cap X & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S & \to & S & \to & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $Z = \operatorname{coker}(\mathbf{r}M \cap X \to \mathbf{r}M)$. If $Z \neq 0$, since $\mathbf{r}M$ is semisimple, we get a splitting $X' = (X' \cap X) \oplus Z$. Thus $Z = 0$ implies $\mathbf{r}M \subseteq X$.

(3) $\Rightarrow$ (1). Let $X \subseteq M$ and suppose X is not contained in $\mathbf{r}M$. If $\mathbf{r}M$ is not contained in X, choose a maximal proper submodule Z of $\mathbf{r}M$ containing $X \cap \mathbf{r}M$. Then M/Z is indecomposable. Now $Z \simeq (X \cap \mathbf{r}M) \oplus Z'$ for some semisimple module Z'. We get the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X \cap \mathbf{r}M & \to & Z & \to & Z' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X & \to & M & \to & M/X & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X/(X \cap \mathbf{r}M) & \to & M/Z & \to & (M/X)/Z' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Noting that $X/(X \cap \mathbf{r}M)$ is contained in $M/\mathbf{r}M$ we see that $M/Z \to M/\mathbf{r}M$ induces a splitting of the bottom row. Thus, the indecomposability of M/Z implies $X/X \cap \mathbf{r}M = 0$ and we are done.

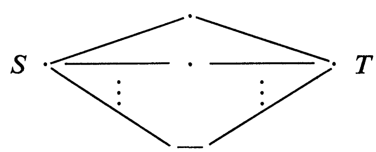
We assume for the rest of this section that Λ is an Artin algebra with $\mathbf{r}^2 = 0$.

THEOREM 2.2. *Suppose Λ is of finite representation type. Then every module which has a waist has a simple top or simple socle.*

Proof. Assume that M is a Λ-module having a waist such that top (M) and soc (M) are both not simple. Note that soc $(M) = \mathbf{r}M$ since M is indecomposable.

Case 1. $M/\mathbf{r}M$ has two nonisomorphic simple summands, say S and T. Let P be the projective cover of S and P' be the projective cover of T. Then, using

Theorem 1.2, it is not hard to see that $\mathbf{r}M$ is a summand of both $\mathbf{r}P$ and $\mathbf{r}P'$. Now Λ is of finite representation type if and only if $\Lambda/\mathbf{r} + \mathbf{r}$ is also [1] where $\Lambda/\mathbf{r} + \mathbf{r}$ denotes the trivial extension of $\Lambda/\mathbf{r}$ by $\mathbf{r}$. W. Müller [8] associates to $\Lambda/\mathbf{r} + \mathbf{r}$ a weakly symmetric self-injective ring Γ of radical$^3 = 0$ and a diagram $\underline{D}(\Gamma)$ which classifies whether or not $\Lambda/\mathbf{r} + \mathbf{r}$ is of finite type. Since $\mathbf{r}M$ is not simple, it follows that $\underline{D}(\Gamma)$ has a subdiagram which is a cycle, namely



Simple modules in $\mathbf{r}M$

This contradicts Λ's being of finite representation type [8, Lemma 4.7].

Case 2. $\mathbf{r}M$ has two nonisomorphic summands. Apply Case 1 to $D(M)$.

Case 3. $M/\mathbf{r}M$ is a direct sum of at least two copies of one simple module, say S, and $\mathbf{r}M$ is a direct sum of at least two copies of one simple module, say T. As above $T \oplus T \subseteq \mathbf{r}P$ where P is the projective cover of S. One may similarly show that $S \oplus S \subseteq E/\mathrm{soc}\,(E)$, where E is the injective envelope of T. Take Γ as in Case 1 and then one has a subdiagram in $\underline{D}(\Gamma)$ of the following form:

$$(\underset{S}{\cdot} = \underset{\underline{T}}{\cdot},\ \underset{T}{\cdot} = \underset{\underline{S}}{\cdot}).$$

This again implies Λ is not of finite representation type.

It is worthwhile noting that the converse is false. There are $\mathbf{r}^2 = 0$ Artin algebras of infinite representation type such that every module which has a waist has a simple top or simple socle. We will also see in Section 4 that in general there are modules which have waists in $\mathbf{r}^2 = 0$ Artin algebras with nonsimple tops and socles.

As an application we give the following description of modules which have waists which we mentioned in Section 1. Suppose Γ is an Artin algebra such that $\Gamma/\mathbf{r}^2$ is of finite representation type. It follows from Theorem 2.2 and remarks made at the beginning of this section that if M' is a waist in M, then at least one of the following modules must be simple: top (M), soc (M), top (M'), soc (M/M'). Thus if M has a waist M' and top (M) is not simple, then M has a waist M'' with top (M'') simple. For, either M' has that property or we can take M'' to be soc (M) if it is simple or if not, take

$$M'' = \pi^{-1}(\mathrm{soc}\,(M/M')) \quad \text{where } \pi\colon M \to M/M'.$$

In the proof of the next proposition we make use of the existence and nature of "almost split exact sequence" for Artin algebras. The definitions and relevant properties can be found in [3]. One may also give a proof of the proposition using techniques found in W. Müller's work, but it is more lengthy.

PROPOSITION 2.3. *Let Λ be an Artin algebra with $\mathbf{r}^2 = 0$. If there exists an indecomposable projective module of length ≥ 4 then there exists an indecomposable Λ-module having a nonsimple top and socle.*

Proof. Let P be an indecomposable projective such that $l(P) \geq 4$. Let $S = \operatorname{top}(P)$. Since Λ is an Artin algebra there is an almost split short exact sequence $0 \to M \to X \to S \to 0$. It follows that M is indecomposable. From [3] one can show that X is the Λ-injective envelope of $\mathbf{r}P$. Now $l(\mathbf{r}P) \geq 3$. From this it follows easily that $l(\operatorname{top}(M)) \geq 2$ and $l(\operatorname{soc}(M)) = l(\mathbf{r}M) \geq 3$.

We now put the last few results together to prove the following.

THEOREM 2.4. *Let Λ be an Artin algebra with $\mathbf{r}^2 = 0$. Then the following statements are equivalent:*

(1) *Every indecomposable left Λ-module has a simple top or simple socle.*

(2) *Every indecomposable left Λ-module has a waist or is simple.*

(3) *Every indecomposable left Λ-module is either projective, injective, or uniserial.*

(1′) *Every indecomposable right Λ-module has a simple top or simple socle.*

(2′) *Every indecomposable right Λ-module has a waist or is simple.*

(3′) *Every indecomposable right Λ-module is either projective, injective, or uniserial.*

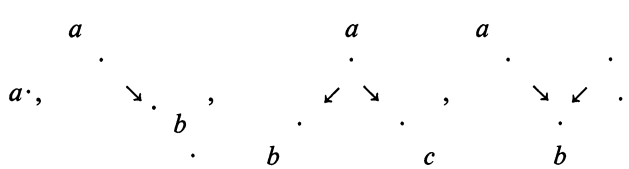
Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are clear.

(2) $\Rightarrow$ (1). By 1.6, Λ has finite representation type. Then (1) follows from 2.2.

(2) $\Rightarrow$ (3). By 2.3 and 2.2, it follows that every indecomposable projective has length ≤ 3. By duality (2′) holds and hence every indecomposable injective has length ≤ 3. By (1), every indecomposable is a factor of an indecomposable projective or a submodule of an indecomposable injective. Thus if M is indecomposable and not projective or injective, $l(M) \leq 2$ and (3) holds.

The equivalences involving (1′), (2′), and (3′) follow by duality.

We end this section with some applications. If one considers the separated diagrams for an Artin algebra Λ with $\mathbf{r}^2 = 0$ (see [4]), one sees that Λ satisfies Theorem 2.4 if and only if the separated diagram for Λ is composed of disjoint copies of the following types of diagrams:



More generally, if Λ is a factor ring of a tensor algebra associated to a k-species $\mathscr{S} = (K_i, {}_iM_j)_{i,j \in \mathscr{I}}$, with k a field and each $K_i = k$, then each nonsimple

Λ-module has a waist if the diagram associated to $\mathscr{S}$ (see [6], [7] for definitions) is composed of disjoint diagrams of the following types:

$$\underset{a_1}{\cdot} \to \underset{a_2}{\cdot} \to \cdots \to \underset{a_m}{\cdot}, \quad m \geq 1,$$

$$\underset{a_n}{\cdot} \to \underset{a_{n-1}}{\cdot} \to \cdots \to \underset{a_0}{\cdot} \leftarrow \underset{b_1}{\cdot} \leftarrow \cdots \leftarrow \underset{b_m}{\cdot}, \quad n \geq 1,\ m \geq 1,$$

$$\underset{a_n}{\cdot} \leftarrow \underset{a_{n-1}}{\cdot} \leftarrow \cdots \leftarrow \underset{a_0}{\cdot} \to \underset{b_1}{\cdot} \to \cdots \to \underset{b_m}{\cdot}, \quad n \geq 1,\ m \geq 1.$$

This can be seen using Gabriel's description [5] of the indecomposable representations of the diagrams and their relations to modules [7].

Finally, if one considers the ring

$$\Lambda = \begin{pmatrix} k & & & 0 \\ k & k & & \\ k & k & k & \\ k & k & 0 & k \end{pmatrix},$$

a subring of the full 4×4 lower triangular matrix ring over a field k, we find that Λ is isomorphic to a tensor algebra whose associated diagram is

$$\begin{array}{ccc} & \downarrow & \\ & \cdot & \\ \swarrow & & \searrow \\ \cdot & & \cdot \end{array}$$

Again using [5], [7], one sees that for this ring every indecomposable has a simple top or simple socle and hence the converse to the above description is not true.

Section 3

We begin by describing a general technique of creating new modules from old.

THEOREM 3.1. *Let R be an arbitrary ring. Let $A \subseteq B$ and $C \subseteq D$ be R-modules and suppose there is an isomorphism $\alpha: B/A \to C$. Then the following statements are equivalent.*

(1) *There is a module X with $B \subseteq X$ and an isomorphism $\beta: X/A \to D$ such that the following diagram commutes:*

$$\begin{array}{ccc} X/A & \xrightarrow[\beta]{\approx} & D \\ \uparrow\cup & & \uparrow\cup \\ B/A & \xrightarrow[\alpha]{\approx} & C \end{array}.$$

(2) *If ϕ: $\operatorname{Ext}_R^1(D, A) \to \operatorname{Ext}_R^1(C, A)$ is induced from $C \subseteq D$ then the exact sequence*

$$0 \longrightarrow A \longrightarrow B \xrightarrow{\alpha \cdot \pi} C \longrightarrow 0, \quad \textit{where } \pi: B \longrightarrow B/A,$$

is in the image of ϕ.

(3) *If* ψ: $\operatorname{Ext}_R^1(D/C, B) \to \operatorname{Ext}_R^1(D/C, C)$ *is induced from*

$$B \xrightarrow{\alpha\cdot\pi} C$$

then the exact sequence $0 \to C \to D \to D/C \to 0$ *is in the image of* ψ.

Proof. Follows from standard homological techniques and consideration of the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\alpha\cdot\pi} & C & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & X & \longrightarrow & D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & D/C & \xrightarrow{=} & D/C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Given $A \subseteq B$, $C \subseteq D$, and $\alpha\colon B/A \to C$ as in Theorem 3.1, if conditions (1)–(3) hold, we say that *we can paste B and D by* α. We call *X the pasted module.*

COROLLARY 3.2. *Given* $A \subseteq B$, $C \subseteq D$, *and* $\alpha\colon B/A \to C$. *Then if either the R-projective dimension of* $D/C \leq 1$ *or the R-injective dimension of* $A \leq 1$ *then we can paste B and D by* α. *In particular, if R is hereditary we can always paste modules.*

We hasten to remark that even if both B and D have waists, the pasted module X need not even be indecomposable. Nevertheless the following proposition gives a way to create new waists from old.

PROPOSITION 3.3. *Let* Λ *be a left Artin ring with radical* $\mathbf{r}$. *Let M and N be* Λ*-modules with* $ll(M) = m$ *and* $ll(N) = n$.

(1) *Suppose that there is an isomorphism* α; $M/\mathbf{r}M \to \mathbf{r}^{n-1}N$ *and we can paste M and N by* α. *If N has a waist then so does the pasted module X.*

(2) *Suppose that there is an isomorphism* $\beta\colon M/S_{m-2}(M) \to \operatorname{soc}(M)$ *and we paste M and N by* β. *If M has a waist then so does the pasted module X.*

Proof. (1) Let N' be a waist in N. We have $\beta\colon X/\mathbf{r}M \approx N$. Let $X' = \beta^{-1}(N')$. Then X' is a waist in X.

(2) The proof is analogous.

As an important application we have:

COROLLARY 3.4. *Let* Λ *be a left Artin ring. Let M and N be* Λ*-modules such that*

(i) $M/\mathbf{r}M$ *is simple and*

(ii) *there is an isomorphism* $\alpha\colon M/\mathbf{r}M \to \operatorname{soc}(N)$.

If we can paste M and N by α *then the pasted module has a waist or is simple.*

Proof. Assume M or N is not simple. Say M is not simple. Then M has a waist and the conditions of 3.3 (2) are satisfied. If N is not simple, the conditions of 3.3 (1) are satisfied.

We give an example of how these results may be applied. Let Λ be a hereditary Artin algebra such that $\Lambda/\mathbf{r}^2$ is of finite representation type. Let S be a simple Λ-module. Let P be the projective cover of S and E be the injective envelope of S. By Corollary 3.2 and Corollary 3.4 there is a module X_S having a waist such that $P \subseteq X_S$ and $X_S/\mathbf{r}P \simeq E$. From Section 2 we see that if Y is a Λ-module with nonsimple top and nonsimple socle having a waist, then Y is a submodule of a factor module of X_S for some simple module S.

Section 4

We now give a technique for constructing waists in $\mathbf{r}^2 = 0$ left Artin rings Λ which have nonsimple tops and non-simple socles.

THEOREM 4.1. *Let S_i, $i = 1, \ldots, n$ and T_j, $j = 1, \ldots, m$ be simple Λ-modules. Let $X = \coprod_{i=1}^n S_i$ and $Y = \coprod_{j=1}^m T_j$.* Assume *either that the S_i's are nonisomorphic simple or the T_j's are nonisomorphic simple. Let P_i be the projective cover of S_i, $i = 1, \ldots, n$ and let E_j be the injective envelope of T_j, $j = 1, \ldots, m$. Finally assume X is a summand of each E_j/T_j for all j and Y is a summand of each $\mathbf{r}P_i$ for all i. Then there exists a module M having a waist so that $Y \cong \mathbf{r}M$ and $X \cong M/\mathbf{r}M$.*

Proof. We only do the case where the S_i's are all nonisomorphic. We choose $U_i \subseteq \mathbf{r}P_i$ so that $\mathbf{r}(P_i/U_i) \cong Y$ for each i. Let $\bar{P}_i = P_i/U_i$. Let $\bar{P} = \coprod_{i=1}^n \bar{P}_i$. Define a morphism $f\colon \coprod_{i=1}^{n-1} Y \to \bar{P}$ by

$$f(y_1, \ldots, y_{n-1}) = (y_1, y_1 + y_2, \ldots, y_{n-2} + y_{n-1}, y_{n-1}).$$

Then f is a monomorphism. Let $M = \operatorname{coker} f$. Thus we get the following commutative diagram:

$$(*) \qquad \begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z & \longrightarrow & \mathbf{r}\bar{P} & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z & \xrightarrow{f} & \bar{P} & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \bar{\pi}} & & \downarrow{\scriptstyle \pi} & & \\
 & & 0 & \longrightarrow & X & \xrightarrow{=} & X & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $Z = \coprod_{i=1}^{n-1} Y$ (since $\mathbf{r}\bar{P} \cong \coprod_{i=1}^n Y$ and $\bar{P}/\mathbf{r}\bar{P} \cong X$).

It remains to show Y is a waist in M since, if so, $Y = \mathbf{r}M$ and $X \cong M/\mathbf{r}M$. By Theorem 2.1 it suffices to show that if S is a simple summand of $M/\mathbf{r}M$ then $\pi^{-1}(S)$ is indecomposable. By distinctness, we may suppose $S = S_i$ for some $1 \leq i \leq n$. Now (*) induces

$$(**)\qquad \begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z & \longrightarrow & \mathbf{r}\bar{P} & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z & \xrightarrow{f} & \bar{\pi}^{-1}(S_i) & \longrightarrow & \pi^{-1}(S_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & S_i & \xrightarrow{=} & S_i & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Now $\bar{\pi}^{-1}(S_i) = \mathbf{r}\bar{P}_1 \oplus \cdots \oplus \mathbf{r}\bar{P}_{i-1} \oplus \bar{P}_i \oplus \mathbf{r}\bar{P}_{i+1} \oplus \cdots \oplus \mathbf{r}\bar{P}_n$. But

$$g\colon \bar{\pi}^{-1}(S_i) \to Z$$

given by

$$g(z_1, z_2, \ldots, z_n) = (z_1, z_2 - z_1, \ldots, z_{i-1} - z_{i-2}, \\ z_{i+1} - z_{i+2}, z_{i+2} - z_{i+3}, \ldots, z_{n-1} - z_n, z_n)$$

splits f in the middle row of (**). Thus by the Krull-Schmitt Theorem $\pi^{-1}(S_i) \cong \bar{P}_i$ and we are done.

One may check that the ring

$$\Lambda = \begin{pmatrix} k & & & 0 \\ 0 & k & & \\ k & k & k & \\ k & k & 0 & k \end{pmatrix}, \quad k \text{ a field,}$$

has $\mathbf{r}^2 = 0$ and has a module having a waist with a nonsimple top and nonsimple socle. Hence Λ is of infinite representation type. To see this, let C_1, C_2, C_3, C_4 be the simples corresponding to the idempotents

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then take $S_1 = C_1$, $S_2 = C_2$, $T_1 = C_3$, and $T_2 = C_4$. It is easy to check that the hypotheses of the theorem are satisfied.

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