A LATTICE PROPERTY OF POST'S SIMPLE SET

BY

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In his classic paper [9] Post not only posed his famous problem but also introduced concepts and techniques which he hoped would lead to its solution. Although the problem was not solved along the lines envisioned by Post, his methods have been extensively analyzed for their own interest, as have the particular sets constructed by Post. In [1] it is shown that Post's hypersimple set can be either (Turing) complete or incomplete depending on which enumeration of the r.e. sets is used for its construction. In [3] the analogous result is obtained for Post's simple (nonhypersimple) set with respect to truth-table completeness. In [7], [5] it is shown that Post's simple set is Turing (in fact weak truth-table) complete for any enumeration. We prove another result in this line by showing that Post's simple set is not contained in any maximal set.

The first example of a coinfinite r.e. set not contained in any maximal set was given by D. A. Martin [4]. A more "natural" example of such a set was given by R. W. Robinson [10, Corollary 7], who showed that the deficiency set of a creative set is not contained in any maximal set. Such a deficiency set is a dense simple set [10, Corollary 6], whereas we prove that Post's simple set is not contained in any dense simple set.

In order to state our result in a strong and recursively invariant form we recall a notion from [7]. A simple set A is called *strongly effectively simple* (*s.e.s.*) if there is a recursive function g such that whenever $W_e \subseteq \bar{A}$, every element of W_e is less than $g(e)$. (Here W_e is the eth r.e. set in some effective enumeration.) Post's simple set is s.e.s. with $g(e) = 2e + 1$. A set A is called *dense* [6] if $\bar{A}$ is finite or if for every recursive function h it is the case that $a_n \geq h(n)$ for all but finitely many n, where a_n is the $(n + 1)$st element of $\bar{A}$ in ascending order. It is easy to see that all maximal sets are dense [6, p. 298]. In fact all hyperhypersimple sets are dense [4], although this latter result is much more difficult to prove. Since every r.e. coinfinite superset of a s.e.s. set is s.e.s., the following result implies that Post's simple set is not contained in any maximal (or even hyperhypersimple) set. This result was also used in [8, Corollary 2] to show that no s.e.s. set has a regressive complement.

THEOREM 1. *No s.e.s. set is dense.*

Proof. Suppose for a contradiction that A is both s.e.s. and dense. Let g witness that A is s.e.s., and let a_n be the $(n + 1)$st element of $\bar{A}$ in ascending order. Before giving the actual proof we sketch it in a style intended to clarify its motivation.

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For $i \leq j$ we define an r.e. set $W_{f(i,j)}$ (f recursive) in order to be able to use g to show that $a_i \geq h(j+1)$, provided $a_i \geq h(j)$ and $a_{i+1} \geq h(j+1)$. Here h is a recursive function such that $\lim_j h(j) = \infty$ and $h(j) \geq gf(i,j)$ for $i \leq j$. This goal can be achieved by finding s such that $a_i^s \geq h(j)$ and $a_{i+1}^s \geq h(j+1)$ and then defining $W_{f(i,j)} = \{a_i^s\}$. Then $gf(i,j) \leq h(j) \leq a_i^s$ so $gf(i,j)$ fails to exceed all elements of $W_{f(i,j)}$. It follows that $a_i^s \in A$, so $a_i \geq a_{i+1}^s \geq h(j+1)$ as required. If we now assume for simplicity that $a_j \geq h(j)$ for *all* j, it is easy to show that $a_i \geq h(j)$ whenever $i \geq j$, in contradiction to the assumption that A is coinfinite. (The proof is an induction in the order $S_0, S_1, S_2, \ldots$, where S_j is the sequence of statements $a_j \geq h(j), a_{j-1} \geq h(j), \ldots, a_0 \geq h(j)$.)

To convert the above discussion into a proof it is necessary to show the existence of the appropriate recursive functions f and h and carry out the induction referred to. By the recursion theorem it is permissible to use an index of f in the definition of a partial recursive function f. From an index of f we may calculate an index of the partial recursive function h where

$$h(j) = \sup_{i \leq j} gf(i,j) + j$$

(and $h(j)$ is understood to be undefined if $f(i,j)$ is undefined for any $i \leq j$). If $i \leq j$, $h(j)$ is defined, $h(j+1)$ is defined, and there is an s such that $a_i^s \geq h(j)$ and $a_{i+1}^s \geq h(j+1)$, define $f(i,j)$ so that $W_{f(i,j)} = \{a_i^t\}$, where t is the least s such that $a_i^s \geq h(j)$ and $a_{i+1}^s \geq h(j+1)$. If the hypothesis of the preceding sentence fails, choose $f(i,j)$ so that $W_{f(i,j)} = \emptyset$.

Observe that instructions for enumerating $W_{f(i,j)}$ may be uniformly obtained from i, j, and an index of f, and make sense without the assumption that f (and h) are total. Hence, by the recursion theorem, there exist total recursive f, h as above. Since A is dense, there is a number j_0 such that $a_j \geq h(j)$ for $j \geq j_0$. For $j \geq j_0$, let C_j be the conjunction of the statements $a_i \geq h(j)$ for $j_0 \leq i \leq j$. We prove C_j by induction on j starting with $j = j_0$. C_{j_0} just says $a_{j_0} \geq h(j_0)$, which is immediate by choice of j_0. We now assume C_j ($j \geq j_0$) and prove C_{j+1}. It is shown that $a_i \geq h(j+1)$ by a descending induction on i, starting with $i = j+1$ and ending with $i = j_0$. Again $a_{j+1} \geq h(j+1)$ comes from the choice of j_0. Now assume $a_{i+1} \geq h(j+1)$ in order to prove $a_i \geq h(j+1)$, where $j_0 \leq i < j+1$. Then $i \leq j$ so from the induction assumption C_j it follows that $a_i \geq h(j)$. It now follows that $W_{f(i,j)} = \{a_i^s\}$ for some s such that $a_i^s \geq h(j)$ and $a_{i+1}^s \geq h(j+1)$ and hence, as in the initial sketch of the argument, that $a_i \geq h(j+1)$ as required. The fact that all the C_j's hold contradicts the assumption that A is coinfinite, so the proof of the theorem is complete.

We remark that, in contrast to Theorem 1, there exists a dense simple (in fact maximal) set which is effectively simple. (A simple set A is called *effectively simple* [12] if there is a recursive function g such that $|W_e| < g(e)$ whenever $W_e \subseteq \bar{A}$.) The construction of a maximal, effectively simple set is an easy variation of the maximal set construction [11, Chapter 12].

The next result is a sort of converse to Theorem 1.

THEOREM 2. *Every r.e., coinfinite, and nondense set is contained in a s.e.s. set.*

Proof. Let A be r.e., coinfinite, and nondense. Let a_n be the $(n + 1)$st element of $\bar{A}$ in increasing order. Since A is not dense, there is a recursive function h such that $h(n) \geq a_n$ for infinitely many n. Without loss of generality we assume that h is strictly increasing.

The desired s.e.s. superset of A will be $A \cup B$, where B is obtained by Post's simple set construction [8] with $2e$ replaced by $h(2e)$. That is, for each e enumerate W_e until the first time, if ever, a number $x_e \geq h(2e)$ occurs. Let $B = \{x_e : x_e \text{ exists}\}$. Let $C = A \cup B$. Obviously C is an r.e. superset of A.

To see that $\bar{C}$ is infinite, consider n such that $h(n) \geq a_n$. If n is even, say $n = 2e$, then every element of B less than $h(n)$ has the form x_i for $i < e$ and so there are at least $2e - e = e$ nonmembers of C less than $h(2e)$. If n is odd, say $n = 2e - 1$, then again B has at most e elements less than $h(2e)$ and thus at most e elements less than $h(2e - 1)$, since $h(2e - 1) < h(2e)$. Hence there are at least $(2e - 1) - e = e - 1$ nonmembers of C less than $h(2e - 1)$. Since there are infinitely many such n, it follows that $\bar{C}$ is infinite.

Finally C is s.e.s. Suppose $W_e \subseteq \bar{C}$, so x_e does not exist. Then every element of W_e is less than $h(2e)$, so g witnesses that A is s.e.s., where $g(e) = h(2e)$.

The first corollary contrasts with Theorem 1.

COROLLARY 1. *There exists an r-maximal s.e.s. set.*

Proof. By the proof of [2, Theorem 8] there is an r-maximal set A with no dense simple superset. Thus, in particular A is not dense, so it has a s.e.s. superset C, which is r-maximal because it contains A.

The next two corollaries are immediate consequences of Theorems 1 and 2.

COROLLARY 2. *A coinfinite r.e. set is dense if and only if it has no s.e.s. superset.*

COROLLARY 3. *Every coinfinite r.e. set which is not dense has a coinfinite r.e. superset which has no dense simple superset.*

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