SOME CONDITIONS FOR UNIFORM *H*-CONVEXITY

BY

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A compact set K in $\mathbf{C}^n$ is called *uniformly H-convex* if there exist a sequence $\{D_k\}_{k=1}^{\infty}$ of domains of holomorphy and a constant r, $0 < r < 1$, such that

(a) D_k contains all points at distance $< r/k$ from K, and
(b) all points of D_k have distance $< 1/k$ from K.

This terminology is due to Čirka [1] who proved several propositions concerning uniform approximation by holomorphic functions on uniformly H-convex sets, although the condition itself appears earlier in the paper by Hörmander and Wermer [2].

If K is the closure of a bounded, strongly pseudoconvex domain then K is known to be uniformly H-convex. However, it is not known whether this remains true for the closure of an arbitrary bounded domain of holomorphy with smooth, but not strongly pseudoconvex, boundary.

Let $D \subset \mathbf{C}^n$ be a domain with C^3 boundary. We denote by $n(z)$ the unit exterior normal to ∂D at z. For each $t > 0$ we consider the set D_t, defined by

$$D_t = D \cup \{z + sn(z): z \in \partial D, 0 \leq s < t\}.$$

It is well known that if t is sufficiently small then D_t is a domain with C^2 boundary. We call D a special domain of holomorphy if D_t is a domain of holomorphy for all sufficiently small t. Convex domains and strongly pseudoconvex domains with smooth boundary are special, and it is clear that the closure of a special domain of holomorphy is uniformly H-convex. The purpose of this note is to characterize the special domains of holomorphy by means of a boundary condition.

It will be convenient for us to work entirely in the underlying real vector space $\mathbf{R}^{2n}$. We suppose that $D = \{\rho < 0\}$ where ρ is a real-valued C^3 function on a neighborhood of $\bar{D}$ satisfying the condition grad $\rho \neq 0$ on ∂D. Such a function will be referred to as a defining function for D. If $z \in \partial D$, the tangent space to ∂D at z, denoted $T_z(\partial D)$, is the set of vectors normal to grad $\rho(z)$. The holomorphic tangent space to ∂D at z, denoted $A_z(\partial D)$, is the subspace of $T_z(\partial D)$ consisting of vectors v such that $Jv \in T_z(\partial D)$, where J is the orthogonal transformation on $\mathbf{R}^{2n}$ corresponding to multiplication by $\sqrt{-1}$ in $\mathbf{C}^n$.

Let $H_\rho(z)$ denote the $2n \times 2n$ matrix $(\partial^2\rho/\partial x_i\, \partial x_j(z))$ and let $L_\rho(z)$ be the matrix $\frac{1}{4}\{H_\rho(z) + {}^tJH_\rho(z)J\}$. The Levi form for ∂D at z is the bilinear form defined on $A_z(\partial D)$ by the matrix $L_\rho(z)$. (A simple computation shows that this definition is consistent with the usual definition of the Levi form as a hermitian

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form on a complex vector space.) The domain D is a domain of holomorphy if and only if the Levi form is positive semidefinite on $A_z(\partial D)$ for each $z \in \partial D$.

Since ∂D is compact we can find $t_0 > 0$ such that the matrix $(I + tH_\rho(z))$ is invertible for all t, $0 \leq t \leq t_0$. Henceforth we will assume that $t \in [0, t_0]$.

PROPOSITION 1. *Let D be a bounded domain of holomorphy in $\mathbf{C}^n$ and let ρ be a C^3 defining function for D satisfying* $|\text{grad } \rho(z)| = 1$ *for all $z \in \partial D$. Then D is a special domain of holomorphy if and only if there exists $t_1 > 0$ such that, if $0 \leq t \leq t_1$ the matrix*

$$L^t(z) = H_\rho(z)(I + tH_\rho(z))^{-1} + {}^tJH_\rho(z)(I + tH_\rho(z))^{-1}J$$

defines a positive semidefinite form on $A_z(\partial D)$ for each $z \in \partial D$.

Proposition 1 is stated in terms of a particular defining function for D. However the following corollary gives a necessary condition independent of the choice of defining function.

COROLLARY 1. *If D is a special domain of holomorphy and ρ is any C^2 defining function for D then $\langle L_\rho(z)v, v\rangle = 0$ for $v \in A_z(\partial D)$ implies $\langle H_\rho(z)v, w\rangle = 0$ for all $w \in A_z(\partial D)$.* (Here $\langle\ ,\ \rangle$ denotes the usual scalar product on $\mathbf{R}^{2n}$).

Remark. It is easy to find examples of pseudoconvex hypersurfaces which do not satisfy the condition of Corollary 1. For instance, if S is the surface $x_2 = x_1 y_1$ in $\mathbf{C}^2$ (where $z_j = x_j + iy_j$, $j = 1, 2$) then the Levi form is identically zero on S but the real Hessian form is not identically zero on the holomorphic tangent space.

The next corollary gives a sufficient condition for D to be a special domain of holomorphy.

COROLLARY 2. *Let D be a bounded domain of holomorphy in $\mathbf{C}^n$ with a C^3 defining function ρ satisfying* $|\text{grad } \rho(z)| = 1$ *for all $z \in \partial D$. Suppose that $\partial D = E_1 \cup E_2$ where E_1 and E_2 are closed sets satisfying the following conditions:*

(i) *if $z \in E_1$ and $v \in T_z(\partial D)$ then $\langle H_\rho(z)v, v\rangle \geq 0$;*

(ii) *if $z \in E_2$, $v \in A_z(\partial D)$ and $\langle L_\rho(z)v, v\rangle = 0$ then $H_\rho(z)v = 0$;*

(iii) *there is a constant $C > 0$ such that if $z \in E_2$ and $\lambda(z)$ is any nonzero eigenvalue of the form defined by $L_\rho(z)$ on $A_z(\partial D)$ then $\lambda(z) \geq C$.*

Then D is a special domain of holomorphy.

It follows from the proof of Corollary 1 given below that condition (ii) is necessary for D to be special, given that $|\text{grad } \rho(z)| = 1$ on ∂D. Also note that as special cases of Corollary 2 one can deduce that strongly pseudoconvex domains and convex domains with C^3 boundaries are special.

For the proof of Proposition 1 we introduce the following notation. Let

$z_0 \in \partial D$. Choose a parametrization for ∂D near z_0, i.e., a C^3 mapping $\phi = (\phi_1, \ldots, \phi_{2n})$ of a neighborhood U of 0 in $\mathbf{R}^{2n-1}$ into a neighborhood V of z_0 such that

(a) $\phi(0) = z_0$,
(b) $d\phi$ has rank $2n - 1$ at each point of V, and
(c) $\partial D \cap V = \phi(U)$.

Let $u_1, \ldots, u_{2n-1}$ denote the coordinates in $\mathbf{R}^{2n-1}$. Then the vectors $v^1, \ldots, v^{2n-1}$ defined by

$$v^\alpha = \left[\left(\frac{\partial \phi_1}{\partial u_\alpha}\right)(0), \ldots, \left(\frac{\partial \phi_{2n}}{\partial u_\alpha}\right)(0)\right]$$

form a basis for the tangent space $T_{z_0}(\partial D)$. For t sufficiently small the mapping ϕ' defined by

$$\phi'(u) = \phi(u) + tn(\phi(u)), \quad u \in U,$$

is a parametrization of ∂D_t near $z'_0 = z_0 + tn(z_0)$. Consequently, if we let

$$w^\alpha = v^\alpha + t\left(\frac{\partial(n \circ \phi)}{\partial u_\alpha}\right)(0)$$

then $\{w^1, \ldots, w^{2n-1}\}$ is a basis for $T_{z_0'}(\partial D_t)$. Here $\partial(n \circ \phi)/\partial u_\alpha$ is the vector whose jth component is $\partial(n_j \circ \phi)/\partial u_\alpha$.

Now it is straightforward to verify, using where necessary the fact that $|\text{grad } \rho| = 1$, that

(1) $H_\rho(z_0)(v^\alpha) = (\partial(n \circ \phi)/\partial u_\alpha)(0)$ and
(2) if σ is any C^2 defining function for D_t then $\text{grad } \sigma(z'_0) = |\text{grad } \sigma(z'_0)|$ $\text{grad } \rho(z_0)$.

In particular,

(3) $T_{z_0}(\partial D) = T_{z_0'}(\partial D_t)$

from which it follows that

(4) $A_{z_0}(\partial D) = A_{z_0'}(\partial D_t)$.

Finally one has the following identity

(5) if $w \in T_{z_0'}(\partial D_t)$ then $\langle H_0(z'_0)w^\alpha, w\rangle = |\text{grad } \sigma(z'_0)|\langle H_\rho(z_0)v^\alpha, w\rangle$.

Indeed, let $w = (w_1, \ldots, w_{2n})$ and write $a(z')$ for $|\text{grad } \sigma(z')|$. Then, using the property

$$\sum w_j\left(\frac{\partial \sigma}{\partial x_j} \circ \phi'\right)(0) = 0$$

one obtains

$$\begin{aligned}\langle H_\sigma(z'_0)w^\alpha, w\rangle &= \sum_{i,j}\left(\frac{\partial^2\sigma}{\partial x_i\,\partial x_j}\right)\left(\frac{\partial\phi'_i}{\partial u_\alpha}\right)w_j\\ &= \sum_j w_j\left(\frac{\partial}{\partial u_\alpha}\right)\left[\left(\frac{\partial\sigma}{\partial x_j}\right)\circ\phi'\right](0)\\ &= a(z'_0)\sum_j w_j\left(\frac{\partial}{\partial u_\alpha}\right)\left[(a\circ\phi')^{-1}\left(\frac{\partial\sigma}{\partial x_j}\circ\phi'\right)\right]\\ &= a(z'_0)\sum_j w_j\left(\frac{\partial}{\partial u_\alpha}\right)\left(\frac{\partial\rho}{\partial x_j}\circ\phi\right)(0)\\ &= |\mathrm{grad}\ \sigma(z'_0)|\ \langle H_\rho(z_0)v^\alpha, w\rangle.\end{aligned}$$

Proof of Proposition 1. Let $v = \sum b_\alpha w^\alpha$, $w = \sum c_\alpha w^\alpha$. Then

$$\langle H_\sigma(z'_0)v, w\rangle = \sum_{\alpha,\beta}\langle H_\sigma(z'_0)w^\alpha, w^\beta\rangle b_\alpha c_\beta.$$

From (5),

$$\langle H_\sigma(z'_0)w^\alpha, w^\beta\rangle = |\mathrm{grad}\ \sigma(z'_0)|\langle H_\rho(z_0)v^\alpha, w^\beta\rangle.$$

But (1) implies $v^\alpha = (1 + tH_\rho(z_0))^{-1}w^\alpha$. Thus

(6) $\langle H_\sigma(z'_0)v, w\rangle = |\mathrm{grad}\ \sigma(z'_0)|\langle H_\rho(z_0)(1 + tH_\rho(z_0))^{-1}v, w\rangle.$

Since $L_\sigma(z'_0) = H_\sigma(z'_0) + {}^tJH_\sigma(z'_0)J$, Proposition 1 is established.

Proof of Corollary 1. Observe that for any symmetric matrix A,

(7) $A(I + tA)^{-1} = A - tA^2(I + tA)^{-1},$

and also $A^2(I + tA)^{-1}$ is positive for small t. Suppose now that $|\mathrm{grad}\ \rho(z)| = 1$ for $z \in \partial D$ and that $\langle H_\rho(z)v, v\rangle + \langle H_\rho(z)Jv, Jv\rangle = 0$. If D is special then Proposition 1 implies that

$$0 = \langle H_\rho(z)^2(1 + tH_\rho(z))^{-1}v, v\rangle + \langle H_\rho(z)^2(1 + tH_\rho(z))^{-1}Jv, Jv\rangle$$

from which it follows that $H_\rho(z)v = 0$.

If we do not assume $|\mathrm{grad}\ \rho(z)| = 1$ then $\rho = g\rho'$ where $|\mathrm{grad}\ \rho'(z)| = 1$, and ρ' is a defining function for D. A straightforward calculation shows that $\langle H_\rho(z)v, w\rangle = g(z)\langle H_{\rho'}(z)v, w\rangle$, since v and w are orthogonal to grad $\rho'(z)$. But $H_{\rho'}(z)v = 0$ by the preceding argument, which completes the proof.

Proof of Corollary 2. If $z \in E_1$ then $H_\rho(z)$ maps $T_z(\partial D)$ into $T_z(\partial D)$. (This follows from the assumption that $|\mathrm{grad}\ \rho(z)|$ is constant on ∂D.) Since by (i), $H_\rho(z)$ is positive semidefinite on $T_z(\partial D)$ it follows that $H_\rho(z)(I + tH_\rho(z))^{-1}$ is positive semidefinite for sufficiently small t, independent of $z \in E_1$ by compactness.

If $z \in E_2$ we choose an orthonormal basis $w^1, \dots, w^{2n-2}$ for $A_z(\partial D)$ such that, if $w = \sum b_\alpha w^\alpha$ then $\langle L_\rho(z)w, w\rangle = \sum \lambda_\alpha b_\alpha^2$ with $\lambda_\alpha = 0$ or $\lambda_\alpha \geq C$. Write $w = w_1 + w_2$ where $\langle L_\rho(z)w_1, w_1\rangle = 0$ and $w_2 = \sum b_{\alpha,2} w^\alpha$ where $b_{\alpha,2} = 0$ if $\lambda_\alpha = 0$. Then $\langle L_\rho(z)w_2, w_2\rangle \geq C|w_2|^2$. Also, by (ii), $H_\rho(z)w_1 = 0$. Finally, observe that L_ρ commutes with J. Arguing as in the proof of Corollary 1 one obtains

$$\begin{aligned} 4\langle L^t(z)w, w\rangle &= 4\langle L_\rho(z)w, w\rangle - t\langle H_\rho(z)^2(I + tH_\rho)(I + tH_\rho(z))^{-1}w, w\rangle \\ &\quad - t\langle H_\rho(z)^2(I + tH_\rho(z))^{-1}Jw, Jw\rangle \\ &= 4\langle L_\rho(z)w_2, w_2\rangle - t\langle H_\rho(z)^2(I + tH_\rho(z))^{-1}w_2, w_2\rangle \\ &\quad - t\langle H_\rho(z)^2(I + tH_\rho(z))^{-1}Jw_2, Jw_2\rangle \\ &\geq (4C - \gamma(t))\|w_2\|^2 \end{aligned}$$

which can be made nonnegative by choosing t small independent of $z \in E_2$.

References

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