CERTAIN QUOTIENT SPACES ARE COUNTABLY SEPARATED

BY

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In honor of S. M. Ulam's sixty-fifth birthday

1. Introduction

In this paper we make the standard conventions that all topological groups G are locally compact and have a countable basis for their topologies, all multiplier representations of G are unitary and Borel, and all Hilbert spaces are separable.

The purpose of this paper is to prove the following theorem.

THEOREM 1.1. *Let G be a locally compact group, K an open normal subgroup of G, and α a multiplier for G. Suppose that K has only Type I α-representations. Let $\hat{K}^{\alpha}$ be the α-dual space of K. Then $\hat{K}^{\alpha}/G$ is countably separated if G has only Type I α-representations.*

Note that the assumption that K have only Type I α-representations is superfluous, for if G has only Type I α-representations, then any open subgroup of G has only Type I α-representations (Kallman [6, Proposition 2.1]).

This theorem generalizes a result of C. Moore (see Auslander-Moore [1, Corollary to Theorem 9, p. 110]), who proved this result with the additional assumption that G/K is Abelian. It will prove to be of great importance in a forthcoming paper of the author (Kallman [6]). Theorem 1.1 will be proved gradually in a sequence of intermediate propositions. The ideas which go into the following proof seem to have little overlap with the ideas in Auslander-Moore [1]. The present proof seems to be relatively short and straightforward, and avoids completely the extremely difficult measure-theoretic arguments of Auslander-Moore [1].

In Section 2 we prove a very simple corollary to a beautiful theorem of K. Kunugui [9] in set theory.[2] As far as the author knows, this is the first time that Kunugui's result has been used in group representations. This corollary is of vital importance for the proof of Theorem 1.1. Section 3 is devoted to the study of certain Borel sets and mappings. We prove Theorem 1.1 in Section 4 in a sequence of elementary lemmas. In Section 5 we present a slight generalization of a result of Kaniuth [7], which is a useful supplement to Theorem 1.1. We follow M. Smith [11] quite closely here.

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[2] See also Arsenin [12] and Larman [13].

See the two books by Dixmier ([2] and [3]), Auslander-Moore [1], and Mackey [10] for the basic results and notation in operator theory and group representations which we will use.

It is a pleasure to thank D. Mauldin for pointing out Kunugui's result.

2. A corollary to a theorem of Kunugui

Let Y be a locally compact Hausdorff space with a countable basis for its topology. Let $\mathbf{C}(Y)$ be the family of all closed subsets of Y. $\mathbf{C}(Y)$ is a compact Hausdorff space in a natural topology (Fell [4]). Moore has noted that $\mathbf{C}(Y)$ is a complete separable metric space (Auslander-Moore [1, pp. 67–68]). His argument goes as follows. Y is a complete separable metric space. Let $d(\cdot, \cdot)$ be a complete metric on Y defining the topology of Y. For each y in Y, let $b(y)$ be the supremum of those numbers r such that the closed unit ball $B(y, r)$ of radius r at y is compact. Let

$$d_y(B) = \inf(1, b(y)/2, d(y, B))$$

where $d(y, B)$ is the distance from y to B. For fixed y, $B \to d_y(B)$ is continuous. If $[y_n \mid n \geq 1]$ is a countable dense set in Y, let $d_n(B) = d_{y_n}(B)$. Then the $d_n(\cdot)$ separate points in $\mathbf{C}(Y)$ and define an embedding of $\mathbf{C}(Y)$ into a countable product of unit intervals. A mapping f from a Borel space to $\mathbf{C}(Y)$ is a Borel mapping if and only if $d_n(f(\cdot))$ is a Borel mapping for each n.

PROPOSITION 2.1. *Let X be a standard Borel space and Y a locally compact Hausdorff space with a countable basis for its topology. Let*

$$\pi_1 : X \times Y \to X \quad \textit{and} \quad \pi_2 : X \times Y \to Y$$

be the natural coordinate projections. Let C be a Borel subset of $X \times Y$ such that $\pi_2(\pi_1^{-1}(x) \cap C)$ is closed for all x in X. Then the mapping

$$x \to \pi_2(\pi_1^{-1}(x) \cap C)$$

of $X \to \mathbf{C}(Y)$ is a Borel mapping.

Proof. For each x in X, let $C(x) = \pi_2(\pi_1^{-1}(x) \cap C)$. $C(x)$ is an element of $\mathbf{C}(Y)$. Fix y in Y. It suffices to show that $x \to d_y(C(x))$ is a Borel mapping. Fix $\varepsilon > 0$. We only need show that $[x \mid d_y(C(x)) < \varepsilon]$ is a Borel subset of X. But

$$d_y(C(x)) = \min(1, b(y)/2, d(y, B))$$

If $\varepsilon > \min(1, b(y)/2)$,

$$[x \mid d_y(C(x)) < \varepsilon] = X.$$

Suppose that $\varepsilon \leq \min(1, b(y)/2)$. Then $B(y, \varepsilon)$, the closed unit ball about y

of radius ε, is compact, and $d_y(C(x)) < \varepsilon$ if and only if $d(y, C(x)) < \varepsilon$. Choose a sequence $0 < \varepsilon_1 < \varepsilon_2 < \cdots \uparrow \varepsilon$. Then

$$[x \mid d_y(C(x)) < \varepsilon] = [x \mid d(y, C(x)) < \varepsilon] = \bigcup_{n \geq 1} \pi_1(\pi_2^{-1}(B(y, \varepsilon_n)) \cap C).$$

But for each $n \geq 1$, $\pi_2^{-1}(B(y, \varepsilon_n)) \cap C$ is a Borel subset of $X \times Y$, and

$$\pi_2^{-1}(B(y, \varepsilon_n)) \cap \pi_1^{-1}(x) \cap C$$

is compact for each x in X. Hence, by the main result of Kunugui [9],

$$\pi_1(\pi_2^{-1}(B(y, \varepsilon_n)) \cap C)$$

is a Borel subset of X. Hence, $[x \mid d_y(C(x)) < \varepsilon]$ is always a Borel subset of X. Q.E.D.

3. Some preliminaries

The purpose of this section is to recall a few facts about multiplier representations of groups and C^*-algebras and to combine some facts in the literature to show that a certain set is a Borel set.

Recall the following well known facts. Let G be a locally compact group. A multiplier for G is a Borel mapping ω of $G \times G$ into the complex numbers of modulus one such that: (1) $\omega(e, a) = \omega(a, e) = 1$ for all a in G; (2) $\omega(ab, c) \cdot \omega(a, b) = \omega(a, bc)\omega(b, c)$ for all a, b, c in G. A Borel mapping π of G into the unitary operators on a separable Hilbert space is called an ω-representation of G in case $\pi(ab) = \omega(a, b)\pi(a)\pi(b)$ for all a, b in G. Let $L^1(G, \omega)$ be the Banach space $L^1(G)$ with the multiplication

$$(fg)(t) = \int f(s)g(s^{-1}t)\omega(s, s^{-1}t)^{-1}\, ds$$

and with the *-operation $f^*(t) = \bar{f}(s^{-1})\omega(s^{-1}, s)\Delta(s^{-1})$. One easily checks that $L^1(G, \omega)$ is a Banach *-algebra, and that if π is an ω-representation of G, then $\pi(f) = \int f(s)\pi(s)\, ds$ is a *-representation of $L^1(G, \omega)$. Conversely, one may show that each *-representation of $L^1(G, \omega)$ arises in this manner from a unique ω-representation of G. Furthermore, the ω-representation of G is irreducible (nondegenerate) if and only if the corresponding representation of $L^1(G, \omega)$ is irreducible (nondegenerate). Denote by $C^*(G, \omega)$ the C^*-completion of $L^1(G, \omega)$. Thus, there is, in the usual manner, a one-to-one correspondence between the ω-representations of G and the *-representations of $C^*(G, \omega)$, with irreducible (nondegenerate) representations corresponding to irreducible (nondegenerate) representations. Denote by $\hat{G}^\omega$ the equivalence classes of irreducible unitary ω-representations of G. Identify $\hat{G}^\omega$ as a topological and Borel space with $C^*(G,\hat{\ }\omega)$.

For each $n = 1, 2, \ldots, \infty$, let $\mathbf{H}_n$ be a fixed n-dimensional separable Hilbert space. If $\mathbf{A}$ is a separable C^*-algebra, denote by $\text{Rep}_n(\mathbf{A})$ the set of representations of **A** into the bounded linear operators on $\mathbf{H}_n$. Rep_n (**A**) is a complete separable metric space in a natural topology. Irr_n (**A**), the irreducible elements of Rep_n (**A**), and the nondegenerate elements of Rep_n (**A**) are both G_δ's in Rep_n (**A**). $\text{Rep}_n (G, \omega)$ ($\text{Irr}_n (G, \omega)$) is similarly defined, and is identified as a topological and Borel space with the nondegenerate elements of $\text{Rep}_n (C^*(G, \omega))$ (with $\text{Irr}_n (C^*(G, \omega))$).

LEMMA 3.1. *The mapping* $\pi \to \pi \mid K$ *of* $\text{Rep}_n (G, \omega)$ *into* $\text{Rep}_n (K, \omega)$ *is continuous if* K *is a closed subgroup of* G.

Proof. This follows easily from a trivial generalization of Dixmier [3, Proposition 18.1.9, p. 316]. The details are left to the reader. Q.E.D.

LEMMA 3.2. *Let* **A** *be a separable* C^**-algebra,* $1 \leq p, q \leq \infty$, r *an integer,* M *a topological space,* $m \to \pi_m$ *a mapping of* M *into* Rep_p (**A**), *and* $m \to \pi'_m$ *a mapping of* M *into* Rep_q (**A**). *Then the set of* m *in* M *such that the interlacing number of* π_m *and* π'_m *is less than or equal to* r *is a* G_δ *in* M.

Proof. This is a simple generalization of Dixmier [3, Lemme 3.7.3, p. 76], and the details are left to the reader. Q.E.D.

Let Rep (G, ω) be the union of the $\text{Rep}_n (G, \omega)$ with the natural topology and Borel structure, and let Irr (G, ω) be the union of the $\text{Irr}_n (G, \omega)$ with the natural topology and Borel structure.

LEMMA 3.3. *The subset of* $\text{Irr} (K, \omega) \times \text{Irr} (G, \omega)$ *of all* (a, b) *such that* a *is a direct summand of* $b \mid K$ *is a Borel set.*

Proof. Let $1 \leq p, q \leq \infty$. It suffices to show that the subset of $\text{Irr}_p (K, \omega) \times \text{Irr}_q (G, \omega)$ of all (a, b) such that a is direct summand of $b \mid K$ is a Borel set. Let

$$M = \text{Irr}_p (K, \omega) \times \text{Irr}_q (G, \omega), \quad \pi_{(a,b)} = a, \quad \pi'_{(a,b)} = b \mid K.$$

Both $m \to \pi_m$ and $m \to \pi'_m$ are continuous. Take $r = 0$ in Lemma 3.2. Lemma 3.3 now follows since a is a direct summand of $b \mid K$ if and only if the interlacing number between a and $b \mid K$ is greater than or equal to 1. Q.E.D.

Note that if b is in $\hat{G}^\omega$ and a is in $\hat{K}^\omega$, we may unambiguously discuss whether or not a is a direct summand of $b \mid K$. In the following proposition we assume that K and G have only Type I ω-representations.

PROPOSITION 3.4. *The subset of* $\hat{K}^\omega \times \hat{G}^\omega$ *of all* (a, b) *such that* a *is a direct summand of* $b \mid K$ *is a Borel set.*

Proof. Since both K and G have only Type I ω-representations, there exist Borel sets B and C, in Irr (K, ω) and Irr (G, ω) respectively, such that $\hat{K}^\omega$ and $\hat{G}^\omega$ are Borel isomorphic to B and C, respectively. Let D be the Borel set described in Lemma 3.3. Then the set of (a, b) such that a is a direct summand of $b \mid K$ may be identified with $(B \times C) \cap D$, and hence is a Borel set. Q.E.D.

4. Proof of Theorem 1.1

In this section we gradually prove Theorem 1.1 in a sequence of lemmas.

If H is a closed subgroup of G and a is an equivalence class of representations of H, we denote by $U(a, H, G)$ the equivalence class of representations of G induced by a from H to G. Let x be an element of $\hat{K}^\alpha$. Since G/K is discrete, note that

$$[b \mid b \text{ in } \hat{G}^\alpha, \quad b \mid K \text{ has } x \text{ as a direct summand}] \\ = [U(a, G_x, G) \mid a \text{ in } \hat{G}_x^\alpha, \quad a \mid K = \text{a multiple of } x],$$

as follows easily from an application of Mackey theory.

Let $\pi: G_x \to G_x/K$ be the natural quotient mapping. There exists a multiplier ω_1 for G_x, an element b of $\hat{G}_x^{\omega_1}$, and a multiplier ω_2 of G_x/K such that $\theta: a \to a(\pi(\cdot)) \otimes b$ is a one-to-one mapping of $(G_x/K)^{\wedge \omega_2}$ onto $[a \mid a \text{ in } \hat{G}_x^\alpha, a \mid K =$ a multiple of $x]$.

LEMMA 4.1. *$a \to \theta(a)$ is continuous in the Fell topology.*

Proof. Let $a_\gamma \to a$ in $(G_x/K)^{\wedge \omega_2}$. Let ξ be in Rep $(G_x/K, \omega_2)$ be such that $\hat{\xi} = a$, and let η be in Rep (G_x, ω_1) such that $\hat{\eta} = b$. Let ξ act on the Hilbert space $\mathbf{H}$, and let η act on the Hilbert space $\mathbf{K}$. $\xi(\pi(\cdot)) \otimes \eta(\cdot)$ acts on $\mathbf{H} \otimes \mathbf{K}$. Choose nonzero vectors v in $\mathbf{H}$ and w in $\mathbf{K}$. It suffices to show that we may approximate

$$\langle (\xi(\pi(\cdot)) \otimes \eta(\cdot))(v \otimes w), v \otimes w \rangle = \langle \xi(\pi(\cdot))v, v \rangle \langle \eta(\cdot)w, w \rangle = \psi(\cdot)$$

uniformly on compact sets by positive definite functions associated with the $\theta(a_\gamma)$ (make a simple modification in Dixmier [3, Proposition 18.1.5, p. 315]). Let C be a compact set in G. $\pi(C)$ is a finite set in G_x/K. For large γ, choose a positive definite function ϕ_γ on G_x/K associated with a_γ which is close to $\langle \xi(\pi(\cdot))v, v \rangle$ on $\pi(C)$. For such a γ, $\psi_\gamma(\cdot) = \phi_\gamma(\pi(\cdot))\langle \eta(\cdot)w, w \rangle$ is a positive definite function associated with $\theta(a_\gamma)$, and $\psi_\gamma(\cdot)$ approximates $\psi(\cdot)$ uniformly on C. Q.E.D.

LEMMA 4.2. *$(G_x/K)^{\wedge \omega_2}$ is compact in the Fell topology.*

Proof. $C^*(G_x/K, \omega_2)$ has an identity, namely the Dirac delta function at the identity. Hence, $(G_x/K)^{\wedge \omega_2}$ is compact in the Fell topology by Dixmier [3, Proposition 3.3.7, p. 64]. Q.E.D.

Since G has only Type I α-representations and has a countable basis for its topology, there exists a countable ordinal γ and an ascending family $[U_\beta \mid \beta \leq \gamma]$ of open sets in $\hat{G}^\alpha$ such that $U_0 = \emptyset$, $U_\gamma = G$, if $\beta \leq \gamma$ is a limit ordinal, then $U_\beta = \bigcup_{\beta' < \beta} U_{\beta'}$, and if β is nonzero and not a limit ordinal, then $U_\beta - U_{\beta-1}$ is a dense locally compact Hausdorff subset of $\hat{G}^\alpha - U_{\beta-1}$ with a countable basis for its topology (see Dixmier [3, 4.5.7, pp. 94–95]).

LEMMA 4.3. *For each x in* $\hat{K}^\alpha$, *there exists an ordinal* β *such that*

$$[b \mid b \text{ in } \hat{K}^\alpha, \quad b \mid K \text{ has } x \text{ as a direct summand}] \cap (U_{\beta+1} - U_\beta)$$

is closed in $(U_{\beta+1} - U_\beta)$, *and if* $\beta' > \beta$, *then*

$$[b \mid b \text{ in } \hat{G}^\alpha, \quad b \mid K \text{ has } x \text{ as a direct summand}] \cap (\hat{G}^\alpha - U_{\beta'}) = \emptyset.$$

Proof. For brevity of notation, let

$$C(x) = [b \mid b \text{ in } \hat{G}^\alpha, \quad b \mid K \text{ has } x \text{ as a direct summand}].$$

Using previous notation and results,

$$\begin{aligned} C(x) &= [U(a, G_x, G) \mid a \text{ in } \hat{G}_x^\alpha, \ a \mid K = \text{a multiple of } x] \\ &= [U(\theta(a), G_x, G) \mid a \text{ in } (G_x/K)^{\wedge \omega_2}]. \end{aligned}$$

But $(G_x/K)^{\wedge \omega_2}$ is compact, θ is continuous, and induction is continuous (Fell [5, Theorem 4.1]). Hence, $C(x)$ is compact in the Fell topology. Consider $[\beta' \mid \beta' \leq \gamma, C(x) \cap (\hat{G}^\alpha - U_{\beta'}) = \phi]$. Let β be the smallest such β'. β is not a limit ordinal, for if it were $U_\beta = \bigcup_{\beta' < \beta} U_{\beta'}$. Since $C(x)$ is compact, there would exist $\beta' < \beta$ such that $C(x)$ is contained in $U_{\beta'}$, a contradiction. Hence, β has a predecessor $\beta - 1$.

$$C(x) \cap (U_\beta - U_{\beta-1}) \neq \emptyset.$$

$C(x) \cap (U_\beta - U_{\beta-1}) = C(x) \cap (\hat{G}^\alpha - U_{\beta-1})$, a compact subset of $\hat{G}^\alpha$, for $\hat{G}^\alpha - U_{\beta-1}$ is closed. Since $U_\beta - U_{\beta-1}$ is open in $\hat{G}^\alpha - U_{\beta-1}$, $C(x) \cap (U_\beta - U_{\beta-1})$ is compact in $U_\beta - U_{\beta-1}$. Since $U_\beta - U_{\beta-1}$ is Hausdorff in its relative topology, $C(x) \cap (U_\beta - U_{\beta-1})$ is closed in $(U_\beta - U_{\beta-1})$. Q.E.D.

Use the same notation as in Lemma 4.3.

LEMMA 4.4. *For each x in* $\hat{K}^\alpha$ *and ordinal* $\beta < \gamma$,

$$C(x) \cap (U_{\beta+1} - U_\beta)$$

is an F_σ *in* $U_{\beta+1} - U_\beta$.

Proof. Use previous notation. For a in $(G_x/K)^{\wedge \omega_2}$, let $f(a) = U(\theta(a), G_x, G)$. $f(\cdot)$ is continuous. Hence $f^{-1}(U_{\beta+1} - U_\beta)$ is the intersection of a closed set and an open set in $(G_x/K)^{\wedge \omega_2}$. Hence,

$$f^{-1}(U_{\beta+1} - U_\beta) = \bigcup_{n \geq 1} C_n,$$

where each C_n is compact in $(G_x/K)^{\wedge \omega_2}$. But as in Lemma 4.3, each $f(C_n)$ is compact and therefore closed in $U_{\beta+1} - U_\beta$. Q.E.D.

Proof of Theorem 1.1. Let B be that subset of $\hat{K}^\alpha \times \hat{G}^\alpha$ consisting of all pairs (a, b) such that $b \mid K$ has a as a direct summand. Let $\pi_1 \colon \hat{K}^\alpha \times \hat{G}^\alpha \to \hat{K}^\alpha$

be the natural projection onto the first coordinate, and $\pi_2\colon \hat{K}^\alpha \times \hat{G}^\alpha \to \hat{G}^\alpha$ the natural projection onto the second coordinate. Let $\beta < \gamma$ and consider

$$B \cap (\hat{K}^\alpha \times (U_{\beta+1} - U_\beta)).$$

This is a Borel subset of $\hat{K}^\alpha \times \hat{G}^\alpha$, each of whose vertical sections is an F_σ by Lemma 4.4. Hence,

$$\pi_1(B \cap (\hat{K}^\alpha \times (U_{\beta+1} - U_\beta))) = C_\beta$$

is a Borel subset of $\hat{K}^\alpha$ by the main theorem of Kunugui [9]. Let

$$D_\beta = C_\beta - \bigcup_{\beta' > \beta} (C_\beta \cap C_{\beta'}).$$

Since γ is countable, each D_β is a G-invariant Borel subset of $\hat{K}^\alpha$. Again, since γ is countable and $\hat{K}^\alpha = \bigcup_{\beta<\gamma} D_\beta$, it suffices to show that each D_β/G is countably separated. Consider

$$\pi_1^{-1}(D_\beta) \cap (\hat{K}^\alpha \times (U_{\beta+1} - U_\beta)) \cap B,$$

a Borel subset of $D_\beta \times (U_{\beta+1} - U_\beta)$. For each x in D_β,

$$E(x) = \pi_2(\pi_1^{-1}(x) \cap (\hat{K}^\alpha \times (U_{\beta+1} - U_\beta)) \cap B)$$

is closed in $U_{\beta+1} - U_\beta$ by Lemma 4.3. Hence, $x \to E(x)$ is a Borel mapping of D_β into $\mathrm{C}(U_{\beta+1} - U_\beta)$ which is constant on G-orbits. An elementary application of Mackey theory shows that $x \to E(x)$ also separates those orbits. Hence, D_β/G is countably separated. Q.E.D.

5. A slight variant of a result of Kaniuth

This section may be regarded as an addendum to the previous sections. Let G be a locally compact group, α a multiplier for G, and K an open normal subgroup of G. When does G have only Type I α-representations? K must have only Type I α-representations (Kallman [6, Proposition 2.1]) and $\hat{K}^\alpha/G$ must be countably separated. Furthermore, using previous notation, each G_x/K must have only Type I ω_2-representations. The purpose of this section is to show that for this to be the case, G_x/K must be almost Abelian.

Let G be a countable discrete group, and let ω be a multiplier for G. We denote by U^I the left regular ω-representation of G, the ω-representation of G induced by I (see Mackey [10, p. 274] for the definition of the right regular ω-representation). Recall that U^I acts on the Hilbert space of all square-summable functions on G by $(U^I(a)f)(c) = f(a^{-1}c) \cdot \omega(c^{-1}, a)^{-1}$. Let δ denote the Dirac delta function at the identity of G. One computes that $T \to \rho(T) = \langle T\delta, \delta \rangle$ is a faithful finite normal trace on the von Neumann algebra generated by U^I, and that each T in this von Neumann algebra has an expansion $T = \sum_{a \in G} \lambda_a U^I(a)$ which converges in the trace norm. The λ_a are uniquely determined, and formal sum and product correspond to operator sum and product.

PROPOSITION 5.1. *Suppose U^I is Type* I. *Then G has a normal Abelian subgroup of finite index.*

It is well known that the converse of this proposition is false in general. This proposition was proved by Kaniuth [7] for ω trivial. We follow Martha Smith [11]. The proof, modulo some very simple modifications, is the same. We give some of the details for the convenience of the reader.

Let Δ denote the normal subgroup of G consisting of all elements with only a finite number of conjugates. G is called an *FC* group in case $G = \Delta$.

LEMMA 5.2. *Let T be an element of $\mathbf{R}(U^I)$, the von Neumann algebra generated by U^I. Then T is central if and only if*

$$\lambda_{b^{-1}ab} = \lambda_a \omega(b, b^{-1})^{-1}\omega(b, b^{-1}ab)\omega(ab, b^{-1})$$

for all a and b in G, where $T = \sum_{a \in G} \lambda_a U^I(a)$.

This lemma is a computation, using the easily verified fact that $U^I(b)^{-1} = U^I(b^{-1})\omega(b, b^{-1})$. Using this lemma, it follows easily that if T is central in $\mathbf{R}(U^I)$, then λ_a is nonzero only if a is in Δ.

The following lemma is modeled on Theorem 5 of Kaplansky [8].

LEMMA 5.3. *G/Δ is finite.*

Proof. $\mathbf{R}(U^I)$ is a finite von Neumann algebra with a trace vector which is cyclic and separating. Hence, $\mathbf{R}(U^I)$ and $\mathbf{R}(U^I)'$ are antiisomorphic. Hence, if Q is a central projection such that $\mathbf{R}(U^I)Q$ is a Type I_m von Neumann algebra, then $\mathbf{R}(U^I)'Q$ is also a Type I_m von Neumann algebra. We show that $\mathbf{R}(U^I)$ is a II_1 von Neumann algebra if G/Δ is infinite. If G/Δ is infinite and $\mathbf{R}(U^I)$ has a Type I summand, then there is a central projection Q as above. Q will be zero, however, for there exists an infinite collection of nonzero, mutually orthogonal, mutually equivalent projections with sum the identity in the commutant of the center of $\mathbf{R}(U^I)$. For let Q_j be multiplication by the characteristic functions of the G/Δ cosets. Using the fact that central elements of $\mathbf{R}(U^I)$ are supported by Δ, a simple computation then shows that the Q_j commute with the center of $\mathbf{R}(U^I)$. Furthermore, if Q_j is multiplication by the characteristic function of $b^{-1}\Delta$, then $U^I(b)Q_jU^I(b)^{-1}$ is multiplication by the characteristic function of Δ. Since each unitary operator $U^I(b)$ certainly commutes with the center of $\mathbf{R}(U^I)$, the lemma is proved, for the sum of the Q_j is the identity.

Q.E.D.

LEMMA 5.4. *It suffices to prove Proposition* 5.1 *in case G is an FC group.*

Proof. By the proof (but not the statement) of Proposition 2.1, Kallman [6], the left regular ω-representation of Δ is Type I. Δ is of finite index in G by Lemma 5.3. Hence, if Δ has a normal Abelian subgroup of finite index, G also has a normal Abelian subgroup of finite index by Poincaré's Lemma. Hence, it suffices to prove Proposition 5.1 in case G is an *FC* group. Q.E.D.

Thus, we can (and do) assume that G is an *FC* group.

LEMMA 5.5. *Suppose that G has an n-dimensional ω-representation. Then if a and b commute, we have that $\omega(a, b)^n = \omega(b, a)^n$.*

Proof. Let π be an n-dimensional ω-representation. Then $\omega(a, b)\pi(a)\pi(b) = \pi(ab) = \pi(ba) = \omega(b, a)\pi(b)\pi(a)$. Now take the determinant of both sides of this equation and make elementary cancellations. Q.E.D.

Let S be the subgroup of the torus generated by all numbers of the form $\omega(a, b)$, where a and b are in G. Since G is countable, S is also countable. We form the group $(G, \omega)_0$ as follows. $(G, \omega)_0$ as a set is $G \times S$, with the multiplication

$$(g, t) \cdot (h, s) = \left(gh, \frac{ts}{\omega(g, h)}\right).$$

$(G, \omega)_0$ is countable and S is central. Note that there is a one-to-one correspondence between ω-representations of G and ordinary representations of $(G, \omega)_0$ whose restriction to (e, s) is s. For if π is an ω-representation of G, then $\pi^0(g, t) = t\pi(g)$ is an ordinary representation of $(G, \omega)_0$.

LEMMA 5.6. *$(G, \omega)_0$ is an FC group.*

Proof. One computes that $(a, t)^{-1} = (a^{-1}, t^{-1}\omega(a, a^{-1}))$ and that

$$(a, t) \cdot (b, s) \cdot (a, t)^{-1} = (aba^{-1}, s\omega(a, a^{-1})\omega(ab, a^{-1})^{-1}\omega(a, b)^{-1})$$

Since G is an FC group, the centralizer of any element b of G is of finite index in G. Hence, it suffices to show that the set of distinct elements of the form $(a, 1) \cdot (b, 1) \cdot (a, 1)^{-1}$, a centralizing b, is finite. But if a centralizes b,

$$(a, 1) \cdot (b, 1) \cdot (a, 1)^{-1} = (b, \omega(b, a)/\omega(a, b))$$

Now G certainly has a finite-dimensional ω-representation, for the left regular ω-representation of G is Type I and generates a finite von Neumann algebra. Hence, for some fixed positive integer n, $\omega(b, a)/\omega(a, b)$ is an nth root of unity, by Lemma 5.5, whenever a and b commute. Hence, $(G, \omega)_0$ is an FC group. Q.E.D.

If A and B are subgroups of a group C, let (A, B) be the subgroup of C generated by $[a^{-1}b^{-1}ab \mid a \text{ in } A, b \text{ in } B]$. Note that in our construction of $(G, \omega)_0$, we have an exact sequence

$$1 \to S \to (G, \omega)_0 \to G \to e.$$

Let $\pi\colon (G, \omega)_0 \to (G, \omega)_0/S = G$ be the natural quotient mapping. If H is a subgroup of G, we let $(H, \omega)_0 = \pi^{-1}(H)$. Note that in general $(H, \omega)_0 \neq (H, \omega \mid H)_0$. We denote by U_0^I the representation of $(G, \omega)_0$ corresponding to the ω-representation U^I of G.

LEMMA 5.7. *Suppose that H is a subgroup of G, and that Q is a central projection in $\mathbf{R}(U^I)$ such that $Q\mathbf{R}(U^I)$ and $Q\mathbf{R}(U^I \mid H)$ are Type I_m von Neumann* *algebras. If C is the centralizer of* $(H, \omega)_0$ *in* $(G, \omega)_0$, *then* $QU_0^I(pqp^{-1}q^{-1}) = Q$ *for every p in C and q in* $(G, \omega)_0$.

Proof. This follows from Lemma 5.6 and a simple modification of Lemma 2, Smith [11]. Q.E.D.

LEMMA 5.8. *Let H, C, and Q be as in Lemma* 5.7. *Then* $(\pi(C), G)$ *is finite.*

Proof. Let $Q = \sum_{a \in G} \lambda_a U^I(a)$. $\lambda_e \neq 0$. $QU_0^I(pqp^{-1}q^{-1}) = Q$ for all p in C and q in G. This implies that

$$|\lambda_e| = |\lambda_{\pi(pqp^{-1}q^{-1})}|.$$

Since $\sum_{a \in G} |\lambda_a|^2 < \infty$, $(\pi(C), G)$ is finite. Q.E.D.

LEMMA 5.9. (G, G) *is finite.*

Proof. Use Lemma 5.6, Lemma 5.8, and simple modifications in the third paragraph of the proof of Theorem 1, Smith [11], to obtain this result. Q.E.D.

In what follows, let Q_n be the central projection in the center of $\mathbf{R}(U^I)$ such that $Q_n\mathbf{R}(U^I)$ is a Type I_n von Neumann algebra and $\sum_{n \geq 1} Q_n = I$.

LEMMA 5.10. *The* Q_n *have support in* (G, G).

To prove this one only need copy the proof of Lemma 3, Smith [11].

Proof of Proposition 5.1. Use Lemma 5.7, Lemma 5.8, and simple modifications in the third paragraph of the proof of Theorem 2, Smith [11], to conclude that there is a subgroup C of finite index in $(G, \omega)_0$ such that

$$U_0^I(aba^{-1}b^{-1}) = I \quad \text{for all } a \text{ and } b \text{ in } C.$$

But one easily checks that this implies $\pi(aba^{-1}b^{-1})$ is the identity. Hence, $\pi(C)$ is Abelian and of finite index in G. $\pi(C)$ may be assumed to be Abelian by Poincaré's Lemma. Q.E.D.

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