AN ESTIMATE FOR LINE INTEGRALS AND AN APPLICATION TO DISTINGUISHED HOMOMORPHISMS

BY

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We prove an inequality similar to Melnikov's estimate for Cauchy integrals over closed curves [4]. As an application we prove a result relating asymptotic values to the distinguished homomorphisms defined in [3].

1. Let Γ be a smooth Jordan arc in the plane. Parametrize Γ by arc length: $\Gamma = \{\zeta(s): 0 \leq s \leq l(\Gamma)\}$ where $l(\Gamma)$ is the length of Γ. We assume that $\zeta'(s)$ is Dini continuous: $\zeta'(s)$ has modules of continuity $\omega(\delta)$ where

$$\chi_0(\Gamma) = \int_0^{l(\Gamma)} \frac{\omega(\delta)}{\delta}\, d\delta < \infty. \tag{1.1}$$

Also let K be a compact set and denote by $A(K, M)$ the set of functions analytic on $S^2 \backslash K$ such that $|f(z)| \leq M$, $z \in S^2\backslash K$; and $f(\infty) = 0$. The analytic capacity of K is

$$\gamma(K) = \sup \{\lim_{z \to \infty} |zf(z)| : f \in A(K, 1)\}.$$

THEOREM 1. *Let Γ be a Jordan arc satisfying* (1.1), *let K be a compact set, $K \cap \Gamma = \emptyset$, and let $f \in A(K, 1)$. Then*

$$\left| \int_\Gamma f(\zeta)\, d\zeta \right| \leq C(\Gamma)\gamma(K) \log\left(2 + \frac{l(\Gamma)}{\gamma(K)}\right) \tag{1.2}$$

where $C(\Gamma)$ depends only on $\chi_0(\Gamma)$.

Inequality (1.2) is sharp; this can be seen by taking K to be a disc

$$\{|z - z_0| \leq \delta\},$$

where dist $(z_0, \Gamma) = |z_0 - \zeta(0)| = 2\delta$ and letting $f = \delta(z - z_0)^{-1}$.

Theorem 1 is a fairly routine consequence of Davie's extension [1] of Melnikov's Theorem:

$$\left| \int_\Gamma f(\zeta)\, d\zeta \right| \leq C'(\chi_0(\Gamma))\gamma(K) \tag{1.3}$$

where Γ is assumed to satisfy (1.1) and Γ is *closed.* The proof follows the reasoning on pp. 163–166 of [4]. Throughout the proof $C_1, C_2, \ldots$ are universal constants and $C_j(\Gamma)$ denote constants depending only on $\chi_0(\Gamma)$.

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Proof. Since $\chi_0(\Gamma)$ and both sides of (1.2) are dilation invariant, we may assume $l(\Gamma) = 1$. We may also assume $\gamma = \gamma(K)$ is small.

Partition the plane into squares S_j of side γ and let V_j be an open square concentric with S_j but having side $5\gamma/4$. Write $K_j = K \cap \overline{V}_j$, $\gamma_j = \gamma(K_j)$. On p. 157 of [4] it is shown that

$$\sum \gamma_j \leq C_1 \gamma, \tag{1.4}$$

and as on pp. 148–151 of [4] we can write $f = \sum f_j$, $f_j \in A(K_j, C_2)$. The inequality

$$|f_j(z)| \leq \frac{C_2 \gamma_j}{\operatorname{dist}(z, K_j)}, \tag{1.5}$$

a consequence of Schwarz's Lemma, is on p. 145 of [4].

We estimate $\sum_j |\int_\Gamma f_j(\zeta)\, d\zeta|$ by taking three cases.

Case (i). If $\operatorname{dist}(K_j, \Gamma) \geq \gamma$, then by (1.5) we have

$$\left|\int_\Gamma f_j(\zeta)\, d\zeta\right| \leq C_2 \gamma_j \int_\Gamma \frac{ds}{\operatorname{dist}(\zeta(s), K_j)} \leq C_1(\Gamma)\gamma_j \log \frac{1}{\gamma}. \tag{1.6}$$

Case (ii). Let E be the endpoints of Γ. Assume $\operatorname{dist}(K_j, \Gamma) \leq \gamma$ but $\operatorname{dist}(K_j, E) > 2\gamma$. We can continue Γ to a closed curve $\tilde{\Gamma}$ such that $\tilde{\Gamma}$ satisfies (1.1) with $\chi_0(\tilde{\Gamma}) \leq C_3 \chi_0(\Gamma) + C_4$ and such that $\operatorname{dist}(K_j, \tilde{\Gamma}\backslash\Gamma) \geq \gamma$. Then $\int_{\tilde{\Gamma}\backslash\Gamma} f_j(\zeta)\, d\zeta$ can be estimated as in case (i), and using (1.3) we get (1.6) in this case also.

Case (iii). The remaining K_j satisfy $\operatorname{dist}(K_j, E) \leq 2\gamma$. Because of the smoothness of Γ there is a subset F_j of Γ such that

$$\operatorname{dist}(K_j, \Gamma\backslash F_j) > 2\gamma \quad \text{and} \quad l(F_j) \leq C_2(\Gamma)\gamma.$$

We get

$$\left|\int_\Gamma f_j(\zeta)\, d\zeta\right| \leq \left|\int_{F_j}\right| + \left|\int_{\Gamma\backslash F_j}\right| \leq C_2 C_2(\Gamma)\gamma + C_1(\Gamma)\gamma \log \frac{1}{\gamma_j}$$

just as with case (i).

Now there are at most 50 K_j for which case (iii) applies, and when we sum (1.6) over the other indices and use (1.4), we obtain (1.2).

2. $H^\infty(D)$ denotes the bounded analytic functions on a plane domain D. Let $z_0 \in \partial D$ and let A_n be the annulus $\{2^{-n-1} \leq |z - z_0| \leq 2^{-n}\}$. In [3] the Melnikov condition

$$\sum_{n=0}^{\infty} 2^n \gamma(A_n \backslash D) < \infty \tag{2.1}$$

was seen to be equivalent to the existence of a unique homomorphism ϕ_0 of $H^\infty(D)$ satisfying:

(i) $\phi_0(f) = f(z_0)$ if f extends continuously to z_0;
(ii) there is a positive measure μ on D such that

$$\phi_0(f) = \int f\, d\mu, \quad f \in H^\infty(D). \tag{2.2}$$

Some time ago M. Behrens and T. W. Gamelin asked me the following question:

Problem 1. *If* (2.1) *holds for* $z_0 \in \partial D$ *and if* $f \in H^\infty(D)$ *has limit* L *along some curve in* D *terminating at* z_0, *then is* $L = \phi_0(f)$?

Although the problem in general is unsolved we can give an affirmative answer when the curve is sufficiently smooth.

THEOREM 2. *Assume* (2.1) *holds at* $z_0 \in \partial D$. *Let* Γ *be a Jordan arc in* $D \cup \{z_0\}$ *with endpoint* z_0. *Assume* Γ *satisfies* (1.1). *If* $f \in H^\infty(D)$, *and if* $\lim_{\Gamma \ni \zeta \to z_0} f(\zeta) = L$, *then* $L = \phi_0(f)$.

Proof. We assume $|f(z)| \le 1$. We begin with a well known localization procedure [2, II 1.7]. Let $\tilde{A}_n = A_{n-1} \cup A_n \cup A_{n+1}$. Choose $\psi_n \in C_0^\infty(\tilde{A}_n)$ such that $0 \le \psi_n \le 1$, $|\text{grad}\, \psi_n| \le C_1 2^n$, $\sum \psi_n = 1$ on $\bigcup A_n$. Define $f = 0$ on $C \backslash D$ and write

$$F_n(z) = \frac{1}{\pi} \iint \frac{f(w) - f(z)}{w - z} \frac{\partial \psi_n}{\partial \bar{w}}\, du\, dv.$$

Then $F_n \in A(E_n, C)$ where $E_n = \tilde{A}_n \backslash D$, and we have

$$|F_n(z)| \le \frac{C_3 \gamma(E_n)}{\text{dist}\,(z, E_n)}, \tag{2.3}$$

which is really the same inequality as (1.5). By Theorem 1, p. 166 of [4], for example, (2.1) yields

$$\sum 2^n \gamma(E_n) < \infty. \tag{2.4}$$

From (2.3) and (2.4) follow

$$\sum_{|n-k| \ge 3} |F_n(z)| \le C_4 \sum_{|n-k| \ge 3} 2^n \gamma(E_n), \quad z \in A_k, \tag{2.5}$$

$$\sum_{n \ge n_0} |F_n(z_0)| \le C_4 \sum_{n \ge n_0} 2^n \gamma(E_n). \tag{2.6}$$

Let $\varepsilon > 0$, take n_0 so that

$$\sum_{n > n_0} 2^n \gamma(E_n) < \varepsilon, \tag{2.7}$$

and set $g = \sum_{n \geq n_0} F_n$. Then by (2.5), $g \in H^\infty(D)$ and by (2.2) and (2.5),

$$\phi_0(g) = \sum_{n \geq n_0} \phi_0(F_n) = \sum_{n \geq n_0} F_n(z_0).$$

Hence $|\phi_0(g)| \leq C\varepsilon$ by (2.6). Moreover, $f - g$ is analytic on each A_k, $k > n_0$ by II 1.7 of [2]. The singularity at z_0 is then removable and we have

$$\lim_{\Gamma \ni \zeta \to z_0} (f(\zeta) - g(\zeta)) = \phi_0(f) - \phi_0(g).$$

To complete the proof it is therefore enough to show

$$\lim_{\Gamma \ni \zeta \to z} |g(\zeta)| \leq C\varepsilon \log \frac{1}{\varepsilon}. \tag{2.8}$$

Let Γ_k be the last arc of Γ joining the two boundary curves of A_k. By (2.5) and (2.7)

$$\sum_{n \geq n_0,\, |n-k| \geq 3} |F_n(z)| < C\varepsilon \quad \text{on } \Gamma_k.$$

For $|n - k| < 3$ we apply Theorem 1 to obtain

$$\left| 2^k \int_{\Gamma_k} F_n(\zeta)\, d\zeta \right| \leq C(\Gamma) 2^k \gamma(E_n) \log \frac{1}{2^k \gamma(E_n)}.$$

Hence

$$\left| 2^k \int_{\Gamma_k} g(\zeta)\, d\zeta \right| \leq C\varepsilon \log \frac{1}{\varepsilon},$$

and because Γ is smooth this is the same as (2.8).

3. The proof of Theorem 2 shows that the hypothesis (1.1) can be weakened to $\lim \inf_{k \to \infty} \chi_0(\Gamma_k) < \infty$. More seriously, the proof shows that Problem 1 is almost equivalent to the following question.

Problem 2. *Let Γ be a continuous curve joining $|z| = \frac{1}{2}$ to $|z| = 1$. Let E be a compact subset of $\{|z| \leq 4\}$, $E \cap \Gamma = \emptyset$, and let $f \in A(E, 1)$. Assume $\gamma(E) < \delta$, $\sup_\Gamma |f(\zeta) - L| < \varepsilon$. If δ and ε are small, must L be small, uniformly in Γ?*

Clearly a yes answer to Problem 2 would solve Problem 1. If Problem 2 has a counterexample in which Γ joins $\frac{1}{2}$ to 1 and in which $E \subset \{\frac{1}{2} < |z| < 1\}$ we would have a negative answer to Problem 1.

References

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