ISOMORPHISMS OF SPACES OF CONTINUOUS VECTOR-VALUED FUNCTIONS

BY

MICHAEL CAMBERN

If X is a locally compact Hausdorff space and E is a Banach space, we denote by $C_0(X, E)$ the Banach space of continuous functions vanishing at infinity on X, taking values in E, and provided with the usual supremum norm. If X is actually compact, so that $C_0(X, E)$ consists of all continuous functions on X to E, we use the notation $C(X, E)$ to represent this function space. And if K is the scalar field associated with E, we will denote $C_0(X, K)$ by $C_0(X)$, (or by $C(X)$ if X is compact).

The well-known Banach-Stone theorem states that if X and Y are locally compact Hausdorff spaces, then the existence of an isometry T of $C_0(X)$ onto $C_0(Y)$ implies that X and Y are homeomorphic. In [2] and [3] this theorem was strengthened by showing that the conclusion holds if the requirement that T be an isometry is replaced by the requirement that T be an isomorphism with $\|T\| \, \|T^{-1}\| < 2$. Essentially the same result was obtained quite independently in [1] by D. Amir, who assumed that the spaces X and Y were compact, and that the functions were real-valued. In [4] it was shown that 2 is indeed the greatest number for which the formulation of the Banach-Stone theorem given in [3] is valid, by exhibiting a pair of locally compact Hausdorff spaces X and Y, with X compact, Y noncompact, and an isomorphism T of $C(X)$ onto $C_0(Y)$ with $\|T\| \, \|T^{-1}\| = 2$. However, it seems to be still unknown what is the best number for such a generalization in the case in which X and Y are both required to be compact. Y. Gordon has shown that if X and Y are countable compact metric spaces, then the existence of an isomorphism T of $C(X)$ onto $C(Y)$ satisfying $\|T\| \, \|T^{-1}\| < 3$ implies that X and Y are homeomorphic [6].

Here we investigate the problem of whether a generalization of this type, involving isomorphisms rather than isometries, is possible when we consider spaces of vector-valued, rather than scalar-valued functions. We establish the following:

THEOREM. *Let X and Y be locally compact Hausdorff spaces, and E a finite-dimensional Hilbert space. If there exists an isomorphism T of $C_0(X, E)$ onto $C_0(Y, E)$ satisfying $\|T\| \, \|T^{-1}\| < \sqrt{2}$, then X and Y are homeomorphic.*

We do not know if $\sqrt{2}$ is the best number for the formulation of such a theorem. The example of [4] shows that $\sqrt{2}$ cannot be replaced by any number greater than 2. We note that if T is required to be an isometry instead of merely

Received June 19, 1974.

an isomorphism with small bound, then M. Jerison has shown that, if X and Y are compact, the conclusion of the theorem is valid for a much larger class of Banach spaces E [7].

The proof of the theorem is established by two propositions and a sequence of lemmas. Lemmas 1 through 6′ do not depend upon the fact that E is a Hilbert space, nor upon the fact that we are using $\sqrt{2}$ as a bound. They require only that E be a finite-dimensional Banach space and that $\|T\|\,\|T^{-1}\| < 2$. We therefore state and prove Lemmas 1 through 6′ under these more liberal assumptions, since they in no way complicate the proofs, and since it is quite possible that a stronger theorem may eventually be established. Only following Lemma 6′ do we use the fact that E is a Hilbert space and that $\|T\|\,\|T^{-1}\| < \sqrt{2}$.

Throughout we will use the fact that the dual space $C_0(X, E)^*$ of $C_0(X, E)$ is (isometrically isomorphic to) the Banach space of all regular Borel vector measures $\mathbf{m}$ on X to E^*, with finite variation $|\mathbf{m}|$, and norm given by $\|\mathbf{m}\| = |\mathbf{m}|(X)$. This characterization of $C_0(X, E)^*$ was first proved by I. Singer [8] for the case in which X is compact. The proof for compact X also follows from Corollary 2 of [5, p. 387]. The result for locally compact X then follows readily by considering $C_0(X, E)$ as a subspace of $C(\hat{X}, E)$, where $\hat{X}$ denotes the one-point compactification of X, and using a standard theorem relating the dual space $C_0(X, E)^*$ to a quotient space of $C(\hat{X}, E)^*$ [9, p. 188]. All properties of vector measures which are used in this article may be found in [5].

Elements of E will be denoted by b, c, e, u, v, and those of E^*, for the most part, by ϕ and ψ. The value of ϕ at b is denoted by $\langle b, \phi\rangle$. We denote elements of $C_0(X, E)$ and those of $C_0(Y, E)$, respectively, by the letters F and G, often accompanied by subscripts. Elements of $C_0(X)$ and of $C_0(Y)$ will be denoted, respectively, by f and g. The norms in E and E^* will be denoted by $\|\cdot\|$, while norms in $C_0(X, E)$, $C_0(Y, E)$, $C_0(X)$ and $C_0(Y)$ are denoted by $\|\cdot\|_\infty$. The letter S will always represent the surface of the unit sphere in E,

$$S = \{e \in E\colon \|e\| = 1\}.$$

The following notational convention will be used throughout the article. We will say that a net $\{F_{x,e,i}\colon i \in I\} \subseteq C_0(X, E)$ is *regularly associated* with a pair $(x, e) \in X \times E$ if $F_{x,e,i} = f_{x,i}\cdot e$, where $\{f_{x,i}\colon i \in I\}$ is a net contained in $C_0(X)$ with $\|f_{x,i}\|_\infty = f_{x,i}(x) = 1$ for all i, and the support of $f_{x,i}$ is contained in N_i, where $\{N_i\colon i \in I\}$ is the family of neighborhoods of x and the set of indices I is directed in the usual manner by set inclusion, ($i_1 \le i_2$ if $N_{i_2} \subseteq N_{i_1}$). We write $\{F_{x,e,i}\} \leftrightarrow (x, e)$ to denote that $\{F_{x,e,i}\}$ is a net in $C_0(X, E)$ which is regularly associated with (x, e). The definition of nets $\{G_{y,e,i}\} \subseteq C_0(Y, E)$ regularly associated with pairs $(y, e) \in Y \times E$ is analogous, and we use the corresponding notation, $\{G_{y,e,i}\} \leftrightarrow (y, e)$.

PROPOSITION 1. *If E is a Hilbert space and if $e_1, e_2, \ldots, e_n$ are vectors in E with $\|e_j\| \ge \delta > 0$ for $1 \le j \le n$, then there exist scalars λ_j, $1 \le j \le n$, with $|\lambda_j| = 1$ for all j, such that $\|\sum_{j=1}^n \lambda_j e_j\| \ge \sqrt{n}\cdot\delta$.*

Proof. The proof is made by induction on the number of elements n. It is clearly true for $n = 1$. Thus assume the result holds for some $k \geq 1$ and that we are given elements $e_j \in E$, $1 \leq j \leq k + 1$, with $\|e_j\| \geq \delta > 0$ for all j. By the inductive hypothesis there exist scalars λ_j, $1 \leq j \leq k$, with $|\lambda_j| = 1$ for all j such that $\|\sum_{j=1}^{k} \lambda_j e_j\|^2 \geq k \cdot \delta^2$. Let $u = \sum_{j=1}^{k} \lambda_j e_j$. Then

$$\begin{aligned}\left\|\sum_{j=1}^{k+1} \lambda_j e_j\right\|^2 &= \|u + \lambda_{k+1} e_{k+1}\|^2 \\ &= \|u\|^2 + 2 \operatorname{Re} \lambda_{k+1}\langle e_{k+1}, u\rangle + |\lambda_{k+1}|^2 \|e_{k+1}\|^2 \\ &\geq (k + 1)\cdot \delta^2\end{aligned}$$

if λ_{k+1} is chosen so that $|\lambda_{k+1}| = 1$ and $\operatorname{Re} \lambda_{k+1}\langle e_{k+1}, u\rangle \geq 0$.

PROPOSITION 2. *If E is a finite-dimensional Banach space there exists a positive constant K_E such that if $e_1, e_2, \ldots, e_n$ are elements of E with $\|e_j\| \geq \delta > 0$ for $1 \leq j \leq n$, then there exist scalars λ_j, $1 \leq j \leq n$ with $|\lambda_j| = 1$ for all j, such that $\|\sum_{j=1}^{n} \lambda_j e_j\| \geq K_E \cdot \sqrt{n} \cdot \delta$.*

Proof. Suppose that the dimension of E is m, and let l_2^m denote m-dimensional Hilbert space over the scalar field associated with E. Let A be a linear operator taking E onto l_2^m. Now for each j, $A(e_j) \in l_m^2$ and

$$\|A(e_j)\| \geq \|e_j\|/\|A^{-1}\| \geq \delta/\|A^{-1}\|.$$

By Proposition 1, there exist scalars λ_j, $1 \leq j \leq n$ with $|\lambda_j| = 1$ for all j, such that $\|\sum_{j=1}^{n} \lambda_j A(e_j)\| \geq \sqrt{n} \cdot \delta/\|A^{-1}\|$. Thus

$$\left\|\sum_{j=1}^{n} \lambda_j e_j\right\| = \left\|A^{-1}\left(\sum_{j=1}^{n} \lambda_j A(e_j)\right)\right\| \geq \sqrt{n} \cdot \delta/\|A\| \cdot \|A^{-1}\|,$$

and we may take $K_E = 1/\|A\| \cdot \|A^{-1}\|$.

Throughout Lemmas 1 to 6′, we shall assume that E is a finite-dimensional Banach space and that T is a fixed isomorphism of $C_0(X, E)$ onto $C_0(Y, E)$ satisfying $\|T\|\;\|T^{-1}\| < 2$. There is no loss of generality in assuming that T is norm-increasing—i.e., $\|F\|_\infty \leq \|T(F)\|_\infty$ for $F \in C_0(X, E)$—and that $\|T^{-1}\| = 1$, for otherwise we may simply replace T by the isomorphism $T' = \|T^{-1}\| T$ which has these properties. Thus these assumptions concerning T will be made throughout the remainder of this article. Then throughout Lemmas 1 to 6′ M will denote a fixed real number satisfying $\|T\|/2 < M^2 < M < 1$.

For any point $x \in X$, we will denote by μ_x the scalar-valued measure which is the positive unit mass concentrated at x. Then any element $\mathbf{m} \in C_0(X, E)^*$ can be written uniquely as $\mathbf{m} = \phi \cdot \mu_x + \mathbf{n}$, where $\phi \in E^*$ and $\mathbf{n} \in C_0(X, E)^*$ with $\mathbf{n}(\{x\}) = 0$. (Let $\phi = \mathbf{m}(\{x\})$ and $\mathbf{n} = \mathbf{m} - \phi \cdot \mu_x$.) From this remark and the regularity of the measures involved, it follows that if $\{F_{x,e,i}\} \leftrightarrow (x, e) \in X \times E$, then for all $\mathbf{m} \in C_0(Y, E)^*$,

$$\lim_i \int T(F_{x,e,i})\, d\mathbf{m} = \lim_i \int F_{x,e,i}\, d(T^*\mathbf{m})$$

exists, and is equal to $\langle e, (T^*\mathbf{m})(\{x\})\rangle$. We thus obtain the following:

LEMMA 1. *If* $\{F_{x,e,i}\} \leftrightarrow (x, e) \in X \times E$ *then for each* $y \in Y$, $\lim_i (T(F_{x,e,i}))(y)$ *exists as an element of* E *(in the norm topology).*

Proof. For fixed $y \in Y$ and $\phi \in E^*$, we know that $\lim_i \int T(F_{x,e,i})\, d(\phi \cdot \mu_y)$ exists. Moreover, it is clear that this limit is equal to $\lim_i \langle (T(F_{x,e,i}))(y), \phi\rangle$. Now the map from E^* to the scalars given by $\phi \to \lim_i \langle (T(F_{x,e,i}))(y), \phi\rangle$ is clearly linear, and is bounded by $2\|e\|$. Thus there exists an element $v \in E^{**} = E$ such that for $\phi \in E^*$,

$$\lim_i \langle (T(F_{x,e,i}))(y), \phi\rangle = \langle v, \phi\rangle.$$

But this simply says that the net $\{(T(F_{x,e,i}))(y)\}$ converges to v in the weak topology on E, which, since E is finite dimensional, coincides with the norm topology.

LEMMA 2. *Let* $\{F_{x,e,i}\} \leftrightarrow (x, e) \in X \times S$. *(Note that* $\|e\| = 1$.*) For each* $i \in I$, *denote by* R_i *the subset of* Y *defined by*

$$R_i = \{y \in Y: \|(T(F_{x,e,i}))(y)\| > M\}.$$

If Y_x *denotes the subset of all* $y \in Y$ *such that there exists a net* $\{y_i\}$ *in* Y, *with* $y_i \in R_i$ *for each* i, *which has* y *as a cluster point, then* Y_x *is a finite subset of* Y.

Proof. Let $y \in Y_x$ and let $\{y_i\}$ be such a net in Y having y as a cluster point. Since for each i $(T(F_{x,e,i}))(y_i)$ lies in the compact subset of E defined by

$$\{u \in E: M \leq \|u\| \leq \|T\|\},$$

it follows readily that there exists an element $u \in E$, with $\|u\| \geq M$, and a subnet $\{y_{i(\alpha)}\} \subseteq \{y_i\}$ such that $y_{i(\alpha)} \to y$ and $(T(F_{x,e,i(\alpha)}))(y_{i(\alpha)}) \to u$.

Choose some $\phi \in E^*$ with $\|\phi\| = 1$ such that $\langle u, \phi\rangle = \|u\|$, and consider the neighborhood N_u of u defined by

$$N_u = \{v \in E: |\langle u, \phi\rangle - \langle v, \phi\rangle| < M - M^2\}.$$

Choose a real-valued $g_y \in C_0(Y)$ with $g_y(y) = \|g_y\|_\infty = 1/\|u\|$, and define $G_y \in C_0(Y, E)$ by $G_y = g_y \cdot u$. Let N_y be the neighborhood of y in Y given by

$$N_y = \{y' \in Y: \langle G_y(y'), \phi\rangle > \|T\|/2\}.$$

Then for all i such that $y_i \in N_y$ and $(T(F_{x,e,i}))(y_i) \in N_u$, we have

$$\begin{aligned}\|T(F_{x,e,i}) + G_y\|_\infty &\geq \|(T(F_{x,e,i}))(y_i) + G_y(y_i)\| \\ &\geq |\langle (T(F_{x,e,i}))(y_i) + G_y(y_i), \phi\rangle| \\ &\geq |\langle G_y(y_i), \phi\rangle + \langle u, \phi\rangle| - |\langle u, \phi\rangle - \langle (T(F_{x,e,i}))(y_i), \phi\rangle| \\ &> \|T\|/2 + \|u\| - (M - M^2) \\ &\geq \|T\|/2 + M^2.\end{aligned}$$

Thus $\|F_{x,e,i} + T^{-1}(G_y)\|_\infty > \frac{1}{2} + M^2/\|T\| > 1$.

Now $\|T^{-1}(G_y)\|_\infty \leq 1$, so that the maximum set of the function

$$\|F_{x,e,i} + T^{-1}(G_y)\|$$

is contained in the neighborhood W_i of x defined by

$$W_i = \{x' \in X: F_{x,e,i}(x') \neq 0\}.$$

Moreover, at any point x' of this maximum set, $\|(T^{-1}(G_y))(x')\|$ is bounded away from zero by the positive number $\delta = M^2/\|T\| - \frac{1}{2}$. Thus for each i such that $y_i \in N_y$ and $(T(F_{x,e,i}))(y_i) \in N_u$, there exists a point x_i in the corresponding set W_i in X with $\|(T^{-1}(G_y))(x_i)\| \geq \delta$. Since the W_i thus obtained constitute a neighborhood basis at x, we conclude that $\|(T^{-1}(G_y))(x)\| \geq \delta$.

But this clearly implies that Y_x is finite. For given any n points $y_1, \ldots, y_n$ of Y_x, we can choose the corresponding functions G_{y_j}, $1 \leq j \leq n$, with disjoint supports, so that for any choice of scalars λ_j, $1 \leq j \leq n$, with $|\lambda_j| = 1$ for each j, we have $\|\sum_{j=1}^n \lambda_j G_{y_j}\|_\infty = 1$. But for each j, $\|(T^{-1}(G_{y_j}))(x)\| \geq \delta$, so that by Proposition 2, we can choose the λ_j such that

$$\left\| T^{-1}\left(\sum_{j=1}^n \lambda_j G_{y_j}\right)\right\|_\infty \geq \left\| \sum_{j=1}^n \lambda_j (T^{-1}(G_{y_j}))(x)\right\| \geq K_E \cdot \sqrt{n} \cdot \delta.$$

LEMMA 3. *If $\{F_{x,e,i}\} \leftrightarrow (x, e) \in X \times S$, then there exists at least one point $y \in Y$ such that $\|\lim_i (T(F_{x,e,i}))(y)\| > M$.*

Proof. By Lemma 2, Y_x is finite, say $Y_x = \{y_1, \ldots, y_n\}$, and we write

$$T^{*-1}(e \cdot \mu_x) = \sum_{j=1}^n \phi_j \cdot \mu_{y_j} + \mathbf{m},$$

where the $\phi_j \in E^*$, and $\mathbf{m} \in C_0(Y, E)^*$ with $\mathbf{m}(\{y_j\}) = 0$ for $1 \leq j \leq n$. (It will follow, from the proof of the lemma, that Y_x is nonvoid, since any y satisfying the condition of the lemma must necessarily belong to Y_x. However, for the moment, we simply set $T^{*-1}(e \cdot \mu_x) = \mathbf{m}$, if Y_x is void.)

Now suppose that for all $y_j \in Y_x$, we had $\|\lim_i (T(F_{x,e,i}))(y_j)\| \leq M$. Then we could find an i_1 such that for all $i \geq i_1$ and all $y_j \in Y_x$, we would have

$$\|(T(F_{x,e,i}))(y_j)\| < M + (1 - M)/2.$$

Next, by the regularity of $|\mathbf{m}|$, we could find a compact set $K \subseteq Y - Y_x$ such that $|\mathbf{m}|(Y - K) < (1 - M)/4$. Since K is compact and disjoint from Y_x, there is an i_2 such that $i \geq i_2$ implies $\|(T(F_{x,e,i}))(y)\| \leq M$ for all $y \in K$. Hence, if i_0 is such that $i_0 \geq i_1$ and $i_0 \geq i_2$, then for all $i \geq i_0$, noting that $\sum \|\phi_j\| \leq 1$ and $\|\mathbf{m}\| \leq 1 - \sum \|\phi_j\|$, we would obtain

$$\begin{aligned} 1 &= \int F_{x,e,i}\, d(e \cdot \mu_x) \\ &= \int T(F_{x,e,i})\, d(T^{*-1}(e \cdot \mu_x)) \\ &= \sum_{j=1}^n T(F_{x,e,i})\, d(\phi_j \cdot \mu_{y_j}) + \int_K T(F_{x,e,i})\, d\mathbf{m} + \int_{Y-K} T(F_{x,e,i})\, d\mathbf{m} \\ &< (\textstyle\sum \|\phi_j\|)[M + (1 - M)/2] + M(1 - \sum \|\phi_j\|) + 2(1 - M)/4 \\ &= M + (1 + \textstyle\sum \|\phi_j\|)(1 - M)/2 \\ &\leq 1. \end{aligned}$$

This contradiction thus completes the proof of the lemma.

LEMMA 3′. *If* $(y, e) \in Y \times S$ *and* $\{G_{y,e,i}\} \leftrightarrow (y, e)$, *then there exists at least one point* $x \in X$ *such that* $\|\lim_i (T^{-1}(G_{y,e,i}))(x)\| > M/\|T\|$.

Proof. Consider the isomorphism $\hat{T}$ of $C_0(Y, E)$ onto $C_0(X, E)$ defined by $\hat{T} = \|T\|T^{-1}$. We have $\|\hat{T}\| = \|T\|$, and $\|\hat{T}^{-1}\| = 1$. Thus we may apply Lemma 3 to the mapping $\hat{T}$, providing the desired conclusion.

Before stating Lemma 4, we make the following observations. As we have previously noted, if $\{F_{x,e,i}\} \leftrightarrow (x, e) \in X \times S$, then any point y such that $\|\lim_i (T(F_{x,e,i}))(y)\| > M$ necessarily belongs to the finite set Y_x. It thus follows that

$$\sup_{y' \in Y} \left\| \lim_i (T(F_{x,e,i}))(y') \right\|$$

is attained at some point $y \in Y$. Similarly, consideration of the isomorphism $\hat{T} = \|T\|T^{-1}$ of $C_0(Y, E)$ onto $C_0(X, E)$ and Lemma 2 imply that if

$$\{G_{y,e,i}\} \leftrightarrow (y, e) \in Y \times S,$$

then $\sup_{x' \in X} \|\lim_i (T^{-1}(G_{y,e,i}))(x')\|$ is attained at some point $x \in X$.

LEMMA 4. *If* $\{F_{x,e,i}\} \leftrightarrow (x, e) \in X \times S$, *let y be a point of Y at which*

$$\left\| \lim_i (T(F_{x,e,i}))(y') \right\|$$

attains its maximum. Let

$$u = \lim_i (T(F_{x,e,i}))(y) \Big/ \left\| \lim_i (T(F_{x,e,i}))(y) \right\|.$$

Then if $\{G_{y,u,j}\} \leftrightarrow (y, u) \in Y \times S$, *it follows that for* $x' \in X$, $x' \neq x$, *we have*

$$\left\| \lim_j (T^{-1}(G_{y,u,j}))(x') \right\| \leq \frac{1}{2}.$$

Proof. Suppose, to the contrary, that there exists some $x' \in X$, $x' \neq x$, such that

$$\left\| \lim_j (T^{-1}(G_{y,u,j}))(x') \right\| > \frac{1}{2}.$$

Let $c = \lim_j (T^{-1}(G_{y,u,j}))(x')$ and choose $\psi \in E^*$ with $\|\psi\| = 1$ such that $\langle c, \psi \rangle = \|c\|$. Then write $T^{*-1}(\psi \cdot \mu_{x'}) = \phi \cdot \mu_y + \mathbf{m}$, where $\phi \in E^*$ and $\mathbf{m} \in C_0(Y, E)^*$ is such that $\mathbf{m}(\{y\}) = 0$. Then

$$\begin{aligned} \|c\| = \langle c, \psi \rangle &= \lim_j \int T^{-1}(G_{y,u,j})\, d(\psi \cdot \mu_{x'}) \\ &= \lim_j \int G_{y,u,j}\, d(T^{*-1}(\psi \cdot \mu_{x'})) \\ &= \langle u, \phi \rangle. \end{aligned}$$

Since $\|u\| = 1$, we have $\|\phi\| \geq \|c\| > \frac{1}{2}$, and hence, since $\|T^{*-1}(\psi \cdot \mu_{x'})\| \leq 1$,

$$\|\mathbf{m}\| = \|T^{*-1}(\psi \cdot \mu_{x'})\| - \|\phi\| \leq \|T^{*-1}(\psi \cdot \mu_{x'})\| - \|c\| \leq 1 - \|c\| < \tfrac{1}{2}.$$

Now let $v = \lim_i (T(F_{x,e,i}))(y)$ (so that $u = v/\|v\|$). We have

$$\lim_i \langle (T(F_{x,e,i}))(y), \phi\rangle = \langle v, \phi\rangle = \langle \|v\|u, \phi\rangle = \|v\| \, \|c\|.$$

Since $\|c\| > 1 - \|c\|$, we can choose a positive number ε such that

$$(\|v\| - \varepsilon)\|c\| > (\|v\| + \varepsilon)(1 - \|c\|).$$

Next, write $\mathbf{m} = \sum_{k=1}^{r} \phi_k \mu_{y_k} + \mathbf{n}$, where $\{y, y_1, \ldots, y_r\}$ is the set Y_x, the $\phi_k \in E^*$, $1 \leq k \leq r$ and $\mathbf{n}$ is an element of $C_0(Y, E)^*$ with $\mathbf{n}(\{y\}) = \mathbf{n}(\{y_k\}) = 0$, $1 \leq k \leq r$. By our choice of y, we can find an i_1 such that for all $i \geq i_1$, we have

$$|\langle (T(F_{x,e,i}))(y), \phi\rangle| > (\|v\| - \varepsilon)\|c\|$$

and

$$|\langle (T(F_{x,e,i}))(y_k), \phi_k\rangle| < (\|v\| + \varepsilon)\|\phi_k\| \quad \text{for } 1 \leq k \leq r.$$

Now since $|\mathbf{n}|(Y_x) = 0$, we can find a compact set $K \subseteq Y - Y_x$ such that

$$|\mathbf{n}|(Y - K) \leq [\|v\| + \varepsilon - M]\|\mathbf{n}\|/2.$$

Because K is compact and disjoint from Y_x, there exists an i_2 such that if $i \geq i_2$, $\|(T(F_{x,e,i}))(y')\| \leq M$ for all $y' \in K$. We choose an i_0 such that $i_0 \geq i_1$ and $i_0 \geq i_2$, and such that for $i \geq i_0$ the support of $F_{x,e,i}$ does not contain the point x'. Then for $i \geq i_0$ we have

$$\begin{aligned}
0 = \int F_{x,e,i}\, d(\psi \cdot \mu_{x'}) &= \int T(F_{x,e,i})\, d(T^{*-1}(\psi \cdot \mu_{x'})) \\
&= \int T(F_{x,e,i})\, d(\phi \cdot \mu_y) + \sum_{k=1}^{r} T(F_{x,e,i})\, d(\phi_k \cdot \mu_{y_k}) \\
&\quad + \int_{Y-K} T(F_{x,e,i})\, d\mathbf{n} + \int_K T(F_{x,e,i})\, d\mathbf{n} \\
&= \langle (T(F_{x,e,i}))(y), \phi\rangle + \sum_{k=1}^{r} \langle (T(F_{x,e,i}))(y_k), \phi_k\rangle \\
&\quad + \int_{Y-K} T(F_{x,e,i})\, d\mathbf{n} + \int_K T(F_{x,e,i})\, d\mathbf{n}.
\end{aligned}$$

But for all $i \geq i_0$, the modulus of the first term on the right is greater than $(\|v\| - \varepsilon)\|c\|$, while the modulus of the sum of the remaining terms is less than

$$\begin{aligned}
(\|v\| + \varepsilon)\left(\sum_{k=1}^{r} \|\phi_k\|\right) &+ [\|v\| + \varepsilon - M]\|\mathbf{n}\| + M\|\mathbf{n}\| \\
&= (\|v\| + \varepsilon)\|\mathbf{m}\| \leq (\|v\| + \varepsilon)(1 - \|c\|).
\end{aligned}$$

Since this contradicts our choice of ε, the proof of the lemma is complete.

If we again consider the isomorphism $\hat{T} = \|T\|T^{-1}$, and note that $\hat{T}^{-1} = T/\|T\|$ we obtain the companion result:

LEMMA 4′. *If $\{G_{y,e,i}\} \leftrightarrow (y, e) \in Y \times S$ let x be a point of X at which*

$$\left\| \lim_i (T^{-1}(G_{y,e,i}))(x') \right\|$$

attains its maximum. Let

$$b = \lim_i (T^{-1}(G_{y,e,i}))(x) \Big/ \left\| \lim_i (T^{-1}(G_{y,e,i}))(x) \right\| .$$

Then if $\{F_{x,b,j}\} \leftrightarrow (x, b) \in X \times S$, it follows that for all $y' \in Y$, $y' \neq y$, we have $\|\lim_j (T(F_{x,b,j}))(y')\| \leq \|T\|/2$.

LEMMA 5. *Let x, y, u, and $\{G_{y,u,j}\}$ be as in the statement of Lemma 4. Then*

$$\left\| \lim_j (T^{-1}(G_{y,u,j}))(x) \right\| > M/\|T\|.$$

Proof. By Lemma 3′, there is some point $x_0 \in X$ such that

$$\left\| \lim_j (T^{-1}(G_{y,u,j}))(x_0) \right\| > M/\|T\|,$$

and, by Lemma 4, the only candidate for x_0 is x.

Similarly, by using Lemmas 3 and 4′ we obtain:

LEMMA 5′. *Let y, x, b, and $\{F_{x,b,j}\}$ be as in the statement of Lemma 4′. Then*

$$\left\| \lim_j (T(F_{x,b,j}))(y) \right\| > M.$$

LEMMA 6. *Let x, y, u, and $\{G_{y,u,j}\}$ be as in the statement of Lemma 4. If*

$$b = \lim_j (T^{-1}(G_{y,u,j}))(x) \Big/ \left\| \lim_j (T^{-1}(G_{y,u,j}))(x) \right\|$$

and if $\{F_{x,b,i}\} \leftrightarrow (x, b)$, then we have $\|\lim_i (T(F_{x,b,i}))(y)\| > M$, and for all $y' \in Y$, $y' \neq y$,

$$\left\| \lim_i (T(F_{x,b,i}))(y') \right\| \leq \|T\|/2.$$

Proof. By Lemmas 4 and 5, we know that x is the unique point of X at which

$$\left\| \lim_j (T^{-1}(G_{y,u,j}))(x') \right\|$$

attains its maximum. Thus by Lemma 4′ (with u replacing e) and Lemma 5′, the desired conclusion follows.

Similarly by using Lemmas 4′ and 5′, followed by Lemma 4 (with b replacing e) and Lemma 5, we obtain:

LEMMA 6′. *Let y, x, b, and $\{F_{x,b,j}\}$ be as in the statement of Lemma 4′. If*

$$u = \lim_j (T(F_{x,b,j}))(y) \Big/ \left\| \lim_j (T(F_{x,b,j}))(y) \right\|$$

and if $\{G_{y,u,i}\} \leftrightarrow (y, u)$, then we have $\|\lim_i (T^{-1}(G_{y,u,i}))(x)\| > M/\|T\|$, and for all $x' \in X$, $x' \neq x$,

$$\left\| \lim_i (T^{-1}(G_{y,u,i}))(x') \right\| \leq \frac{1}{2}.$$

Lemmas 4, 5, and 6 show that starting with any point $x \in X$, there is a point $y \in Y$ and elements b, $u \in S$, such that if $\{F_{x,b,i}\} \leftrightarrow (x, b)$ then

$$(1) \qquad \left\| \lim_i (T(F_{x,b,i}))(y) \right\| > M,$$

while

$$(2) \qquad \left\| \lim_i (T(F_{x,b,i}))(y') \right\| \leq \|T\|/2, \quad y' \in Y - \{y\},$$

and if $\{G_{y,u,j}\} \leftrightarrow (y, u)$ then

$$(3) \qquad \left\| \lim_j (T^{-1}(G_{y,u,j}))(x) \right\| > M/\|T\|$$

while

$$(4) \qquad \left\| \lim_j (T^{-1}(G_{y,u,j}))(x') \right\| \leq \frac{1}{2}, \quad x' \in X - \{x\}.$$

Lemmas 4′, 5′, and 6′ show conversely that starting with any point $y \in Y$, there is an $x \in X$ and elements b, $u \in S$ such that (1), (2), (3), and (4) are satisfied.

We now place further restrictions on the space E and on the bound of T which will insure that the relations (1) through (4) define a correspondence between points of X and Y which is, in fact, a homeomorphism. From now on, we shall assume that E is a finite-dimensional Hilbert space. Recall that the conclusions of Lemmas 1 through 6′ hold under the assumptions that T is any isomorphism of $C_0(X, E)$ onto $C_0(Y, E)$ with $\|T\| < 2$ and $\|T^{-1}\| = 1$, and that M is any real number with $\|T\|/2 < M^2 < M < 1$. We shall henceforth assume, in addition, that $\|T\| < \sqrt{2}$ and that $\|T\|/\sqrt{2} < M$.

For $y \in Y$, define $x = \rho(y)$ if there exists a $b \in S$ such that x is related to y by (1) and (2). Then ρ is a well-defined function from Y to X. For if not, for some $y \in Y$ there would exist points $x_1, x_2 \in X$, $x_1 \neq x_2$ and elements $b_1, b_2 \in S$, such that if $\{F_{x_1,b_1,i}\} \leftrightarrow (x_1, b_1)$ and $\{F_{x_2,b_2,j}\} \leftrightarrow (x_2, b_2)$ then

$$\|(T(F_{x_1,b_1,i}))(y)\| > M \quad \text{for all } i \geq \text{some } i_0,$$

and

$$\|(T(F_{x_2,b_2,j}))(y)\| > M \quad \text{for all } j \geq \text{some } j_0.$$

If we choose $i \geq i_0$ and $j \geq j_0$ such that the supports of $F_{x_1, b_1, i}$ and $F_{x_2, b_2, j}$ are disjoint, then for all choices of scalars λ_1, λ_2 with $|\lambda_1| = |\lambda_2| = 1$, we have

$$\|\lambda_1 F_{x_1, b_1, i} + \lambda_2 F_{x_2, b_2, j}\|_\infty = 1.$$

But by Proposition 1, we could choose such scalars λ_1, λ_2 so that

$$\begin{aligned}\|T(\lambda_1 F_{x_1, b_1, i} + \lambda_2 F_{x_2, b_2, j})\|_\infty &\geq \|\lambda_1(T(F_{x_1, b_1, i}))(y) + \lambda_2(T(F_{x_2, b_2, j}))(y)\| \\ &> \sqrt{2}\, M \\ &> \|T\|,\end{aligned}$$

and this contradiction shows that ρ is indeed a well defined function.

Similarly, if for $x \in X$, we define $y = \tau(x)$ if there exists a $u \in S$ such that y is related to x by (3) and (4), then τ is a well-defined function from X to Y. The remarks of the paragraph following Lemma 6′ show that $y = \tau(x)$ if, and only if, $x = \rho(y)$, so that τ is a one-one function mapping X onto all of Y and $\rho = \tau^{-1}$.

LEMMA 7. *τ is a homeomorphism of X onto Y.*

Proof. We show that τ is continuous. The proof that $\rho = \tau^{-1}$ is continuous is analogous.

Suppose, to the contrary, that there exists a net $\{x_\alpha : \alpha \in A\}$ in X such that $x_\alpha \to x_0$, but that $y_\alpha = \tau(x_\alpha) \nrightarrow \tau(x_0) = y_0$. Then there exists some compact neighborhood N of y_0 such that for every $\alpha_0 \in A$, there is an $\alpha \geq \alpha_0$ such that y_α lies outside N. By the definition of τ, there exists a $u \in S$ such that if $\{G_{y_0, u, i}\} \leftrightarrow (y_0, u)$, then for some i_0, $\|(T^{-1}(G_{y_0, u, i_0}))(x_0)\| > M/\|T\|$ and the support of G_{y_0, u, i_0} is contained in N.

Since $x_\alpha \to x_0$ and $T^{-1}(G_{y_0, u, i_0})$ is continuous, there exists an $\alpha_0 \in A$ such that if $\alpha \geq \alpha_0$ then $\|(T^{-1}(G_{y_0, u, i_0}))(x_\alpha)\| > M/\|T\|$. Thus fix an $\alpha \geq \alpha_0$ such that $y_\alpha = \tau(x_\alpha)$ lies outside N. Again by the definition of τ, there exists a $v \in S$ such that if $\{G_{y_\alpha, v, j}\} \leftrightarrow (y_\alpha, v)$, then for some j_0,

$$\|(T^{-1}(G_{y_\alpha, v, j_0}))(x_\alpha)\| > M/\|T\|$$

and the supports of G_{y_0, u, i_0} and G_{y_α, v, j_0} are disjoint. Thus for all scalars λ_k, $k = 1, 2$, with $|\lambda_k| = 1$, we have $\|\lambda_1 G_{y_0, u, i_0} + \lambda_2 G_{y_\alpha, v, j_0}\|_\infty = 1$. But again using Proposition 1, for a proper choice of such scalars λ_k, we have

$$\begin{aligned}\|T^{-1}(\lambda_1 G_{y_0, u, i_0} + \lambda_2 G_{y_\alpha, v, j_0})\|_\infty &\geq \|\lambda_1(T^{-1}(G_{y_0, u, i_0}))(x_\alpha) \\ &\quad + \lambda_2(T^{-1}(G_{y_\alpha, v, j_0}))(x_\alpha)\| \\ &> \sqrt{2}\, M/\|T\| \\ &> 1,\end{aligned}$$

which contradicts the fact that $\|T^{-1}\| = 1$.

Remark. If, for any fixed finite-dimensional Banach space E, one could show that (1) and (2) hold for all $b \in S$, instead of simply for some $b \in S$, one could then establish that the conclusion of the theorem remains valid for all isomorphisms T satisfying $\|T\| \, \|T^{-1}\| < 2$.

References

1. D. Amir, *On isomorphisms of continuous function spaces*, Israel J. Math., vol. 3 (1965), pp. 205–210.
2. M. Cambern, *A generalized Banach-Stone theorem*, Proc. Amer. Math. Soc., vol. 17 (1966), pp. 396–400.
3. ———, *On isomorphisms with small bound*, Proc. Amer. Math. Soc., vol. 18 (1967), pp. 1062–1066.
4. ———, *Isomorphisms of $C_0(Y)$ onto $C(X)$*, Pacific J. Math., vol. 35 (1970), pp. 307–312.
5. N. Dinculeanu, *Vector measures*, Pergamon Press, New York, 1967.
6. Y. Gordon, *On the distance coefficient between isomorphic function spaces*, Israel J. Math., vol. 8 (1970), pp. 391–397.
7. M. Jerison, *The space of bounded maps into a Banach space*, Ann. of Math. (2), vol. 52 (1950), pp. 309–327.
8. I. Singer, *Linear functionals on the space of continuous mappings of a compact space into a Banach space* (Russian), Revue Roumaine Math. Pures Appl., vol. 2 (1957), pp. 301–315.
9. A. E. Taylor, *Introduction to functional analysis*, Wiley, New York, 1958.

University of California
Santa Barbara, California