SURGERY IN A FIBER BUNDLE

BY

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Introduction

For an oriented smooth compact manifold M of dimension m let $\mathscr{L}_m(M)$ be the surgery space which is defined by Quinn in [9], [14, p. 240]. This space has the property that its homotopy groups are the Wall groups of $\pi_1(M)$. Let $\xi = (E, B, p, F)$ be an oriented smooth fiber bundle, i.e., E, B, F are oriented smooth compact manifolds, $p\colon E \to B$ is smooth and the orientation of E is induced by the orientations of B and F. Let $b = \dim B$ and $f = \dim F$. There is a pull back map $p^{\#}: \mathscr{L}_b(B) \to \mathscr{L}_{b+f}(E)$ [14, p. 240]. In this paper, we will study this map in the case that $\pi_1(B) = \pi_1(E) = Z_p$, a cyclic group of odd prime order p. At first, recall the following:

THEOREM 0.1 [14, p. 240]. *For a finitely generated group π, let $L_k(\pi)$ be the Wall group of π with trivial homomorphism $1\colon \pi \to Z_2$.*

(i) $\pi_i(\mathscr{L}_m(M)) = L_{m+i}(\pi_1(M))$.

(ii) $L_{4k}(1) = Z$ *and* $L_{4k+2}(1) = Z_2$.

(iii) *Let p be an odd prime, then* $L_{2k+1}(Z_p) = 0$, $L_{2k}(Z_p) = L_{2k}(1) \oplus \tilde{L}_{2k}(Z_p)$ *and* $\tilde{L}_{2k}(Z_p)$ *is a free abelian group of rank* $\frac{1}{2}(p-1)$.

Our main result is the following:

THEOREM 0.2. *Let ξ be a smooth fiber bundle as above with structure group H. If the identity component of H has a finite index, then for $i > 0$, $\pi_i(p^{\#})$ is given by*

$$\pi_i(p^{\#})x = \begin{cases} (I(F)x_1, I(F)x_2) & \text{if } f \equiv 0 \pmod 4 \text{ and } b + i \equiv 0 \pmod 4, \\ (\chi(F)x_1, I(F)x_2) & \text{if } f \equiv 0 \pmod 4 \text{ and } b + i \equiv 2 \pmod 4, \\ 0 & \text{otherwise,} \end{cases}$$

where $(x_1, x_2) = x \in L_{b+i}(1) \oplus \tilde{L}_{b+i}(Z_p)$, and $I(F)(\chi(F)$, respectively) is the index (Euler characteristic, respectively) of F.

This paper is organized as follows: In Section 1, we study the G-signatures of G-fibered manifolds. In Section 2, we apply the result of Section 1 and the theory of free G-bordism of Conner-Floyd to study the Atiyah-Singer invariants of free G-fibered manifolds. In Section 3, we will apply the result of Section 2 and the results of Sullivan to prove our main result.

1. The G-signatures of G-fibered manifolds

Let $p\colon E^{2n} \to B^{2m}$ be an oriented smooth fiber bundle with fiber F^{2k} where B, E may have nonempty boundaries. Then for each integer t, there is a bundle

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$\mathscr{H}^t(F; R)$ of coefficients over X with fiber $H^t(F; R)$. Let $H^s(B, \partial B; \mathscr{H}^t(F; R))$ be the s-dimensional cohomology module of $(B, \partial B)$ with coefficients in $\mathscr{H}^t(F; R)$ [13]. Let $\hat{H}^s(B; \mathscr{H}^t(F; R))$ be the image of the natural homomorphism

$$j^*: H^s(B, \partial B; \mathscr{H}^t(F; R)) \to H^s(B; \mathscr{H}^t(F; R)).$$

It is easy to see that the bilinear form ϕ on $\hat{H}^m(B; \mathscr{H}^k(F; R))$ defined by

$$\phi(j^*(a), j^*(b)) = ((a \cup b) \cap [B]) \cap [F]$$

is nondegenerate and $\phi(j^*(a), j^*(b)) = (-1)^{m+k}\phi(j^*(a), j^*(b))$.

Let G be a compact Lie group which acts smoothly on E and B such that p is equivariant. It is clear that there is an induced G action on $\mathscr{H}^k(F; R)$ and consequently $\hat{H}^m(B; \mathscr{H}^k(F; R))$ is a G module. It is also clear that the bilinear form ϕ is G-invariant. Then we can define the G-signature of ϕ as in [1] which will be denoted by Sign $(G; \xi)$ where ξ stands for the bundle (E, B, p, F).

THEOREM 1.1. *Let G be a compact Lie group and let $\xi = (E, B, p, F)$ be an oriented smooth fiber bundle as above. Suppose G acts smoothly on E and B such that p is equivariant. Then*

$$\text{Sign } (G, \xi) = \text{Sign } (G, E).$$

where Sign (G, E) *is the G-signature of the G action on E* [1].

Proof. By [12], there is a covergent E_2 cohomology spectral sequence of bigraded algebra with $E_2^{s,t} \approx H^s(B, \partial B; \mathscr{H}^t(F; R))$. Let $\hat{E}_2^{s,t} \approx \hat{H}^s(B; \mathscr{H}^t(F; R))$. This spectral sequence is functorial on the category of fiber bundles and fiber preserving maps. It is clear that for each $g \in G$, g acts on ξ as a fiber preserving map. Thus g induces a map $g^*: \hat{E}_*^{s,t} \to \hat{E}_*^{s,t}$ which is a spectral sequence isomorphism and which induces an isomorphism of some filtration of $\hat{H}^*(E; R)$ and which in turn induces an isomorphism of $\hat{H}^*(E; R)$ which coincides with the induced action of g on $\hat{H}^*(E; R)$. Hence $\hat{E}_*^{s,t}$ are G-modules and differentials of the spectral sequence are G-homomorphisms, etc. Now it is easy to follow formally the arguments of [3] to prove the theorem. ∎

COROLLARY 1.2. *Let $p: E^{2n} \to B^{2m}$ be a smooth fiber bundle with fiber F^{2k} as above. Let H be the structure group. Let G be a compact Lie group which acts on E^{2n} as bundle isomorphisms* [5]. *If the identity component of H has finite index, then*

$$\text{Sign } (G, E) = I(F) \text{ Sign } (G, B).$$

Proof. Let $\pi: P \to B$ be the associated principal H bundle. Note that $E \cong P \times_H F$. There is a G action on P which commutes with the principal H action such that $g(p, f) = (gp, f)$ where $g \in G$, $p \in P$, and $f \in F$ [5]. If H is connected, then it is easy to see that

$$\hat{H}^s(B; \mathscr{H}^t(F; R)) \cong \hat{H}^s(B; R) \otimes H^t(F; R)$$

and the G action on $\hat{H}^s(B; R) \otimes H^t(F; R)$ is given by $g(u \otimes v) = gu \otimes v$ where $g \in G$, $u \in \hat{H}^s(B; R)$ and $v \in H^t(F; R)$. It is clear that

$$\operatorname{Sign}(G, \xi) = \operatorname{Sign}(G, \hat{H}^m(B; R) \otimes H^k(F; R)) = I(F) \operatorname{Sign}(G, B).$$

Thus $\operatorname{Sign}(G, E) = I(F) \operatorname{Sign}(G, B)$ by Theorem 1.1.

If H is not connected, but the identity component of H has a finite index, then we can use an equivariant version of [11] to prove the Corollary 1.2 without difficulty. ■

Remark 1.3. The assumption in the Corollary 1.2 that G acts as bundle isomorphisms is necessary. For example, let B and F be nontrivial G manifolds and let G acts on $B \times F$ by $g(b, f) = (gb, gf)$, then, by [1],

$$\operatorname{Sign}(G, B \times F) = \operatorname{Sign}(G, B) \otimes \operatorname{Sign}(G, F).$$

2. The Atiyah-Singer invariants of free Z_p-fibered manifolds

Let $\xi = (E^{2n-1}, B^{2m-1}, p, F^{2k})$ be an oriented smooth fiber bundle where E, B, F are closed. Let Z_p, p an odd prime, act smoothly and freely on E as bundle isomorphisms such that the induced action on B is also smooth and free. Then for each $g \in Z_p$, $g \neq 1$, the Atiyah-Singer invariants $\sigma(g, E)$ and $\sigma(g, B)$ are defined as in [1]. In this section, we will prove the following.

THEOREM 2.1. *Let $\xi = (E, B, p, F)$ be an oriented smooth fiber bundle with free Z_p action as above. Let H be the structure group. If the identity component of H has a finite index, then for each $g \in Z_p$, $g \neq 1$,*

$$\sigma(g, E) = I(F)\sigma(g, B).$$

Proof. Let $\pi: P \to B$ be the associated principal H-bundle. Then there is an induced Z_p action on P which commutes with the principal H action on P and E is equivariant diffeomorphic to $P \times_H F$ where the Z_p action on $P \times_H F$ is given by $g(p, f) = (gp, f)$ for $g \in Z_p$, $p \in P$ and $f \in F$. Note that the induced Z_p action on P is also free and thus P is a principal oriented $Z_p \times H$-manifold [4]. For a compact Lie group K, let $\Omega_*(K)$ be the principal K-bordism group of Conner-Floyd [4]. By [6], we have an exact sequence.

$$0 \to \Omega_*(Z_p) \otimes_\Omega \Omega_*(H) \to \Omega_*(Z_p \times H) \to \Omega_*(Z_p) *_\Omega \Omega_*(H) \to 0.$$

Note that $\Omega_*(Z_p) = \Omega_* \oplus \tilde{\Omega}_*(Z_p)$ and $\tilde{\Omega}_*(Z_p)$ is a torsion group [4]. Since

$$\Omega_* *_\Omega \Omega_*(H) = 0,$$

$\Omega_*(Z_p) *_\Omega \Omega_*(H)$ is a torsion group. Thus there exists a principal oriented $Z_p \times H$-manifold W such that $\partial W = kP \cup (\bigcup_{i=1}^s X_i \times V_i)$ as principal $Z_p \times H$-manifolds for some k where X_i (respectively V_i) are principal H (respectively Z_p) manifolds. Let $M = W \times_H F$, $N_i = X_i \times_H F$, $Q = W/H$, $Z_i = X_i/H$. Let $\tilde{\pi}: M \to Q$ be the induced projection. Then $(M, Q, \tilde{\pi}, F)$ is an oriented smooth fiber bundle with Z_p acting as bundle isomorphisms. By

Corollary 1.2, Sign $(g, M) = I(F)$ Sign (g, Q). Note that $\tilde{\pi}|_{N_i}: N_i \to Z_i$ are oriented smooth fiber bundles with fiber F and structure group H. By [11], $I(N_i) = I(F)I(Z_i)$. It is easy to see that

$$\partial M = kE \cup \left(\bigcup_{i=1}^{s} N_i \times V_i\right), \quad \partial Q = kB \cup \left(\bigcup_{i=1}^{s} Z_i \times V_i\right).$$

By definition of Atiyah-Singer invariants, we have

$$\text{Sign}\,(g, M) = k\sigma(g, E) + \sum_{i=1}^{s} \sigma(g, N_i \times V_i),$$

$$\text{Sign}\,(g, Q) = k\sigma(g, B) + \sum_{i=1}^{s} \sigma(g, Z_i \times V_i)$$

$$\sigma(g, N_i \times V_i) = I(N_i)\sigma(g, V_i), \quad \sigma(g, Z_i \times V_i) = I(Z_i)\sigma(g, V_i).$$

Thus $\sigma(g, E) = I(F)\sigma(g, B)$. ∎

Remark 2.2. If dim $F \equiv 2 \pmod 4$, $I(F)$ is defined to be 0 as usual.

3. Surgery in a fiber bundle

For a closed oriented smooth manifold M of dimension m let $\mathscr{S}_{G/O}(M)$, $(G/O)^M$, $\mathscr{L}_m(M)$ be the spaces defined as in [14, p. 240]. By definition,

$$\pi_i(\mathscr{S}_{G/O}(M)) = \mathscr{S}^s(M \times D^i, M \times S^{i-1}), \quad \pi_i((G/O)^M) = [\Sigma^i M_+, G/O]$$

and

$$\pi_i(\mathscr{L}_m(M)) = L_{m+i}(\pi_1(M))$$

where $\mathscr{S}^s(M \times D^i, M \times S^{i-1})$ is the set of simple homotopy smoothings of $M \times D^i$, $[\Sigma^i M_+, G/O]$ is the group of normal invariants, and $L_{m+i}(\pi_1(M))$ is the Wall group. Then there is a homotopy fibration [14, p. 34]

$$\mathscr{S}_{G/O}(m) \xrightarrow{\eta} (G/O)^M \xrightarrow{\theta} \mathscr{L}_m(M).$$

Let $\xi = (E, B, p, F)$ be the oriented smooth fiber bundle as in the Introduction where E, B, F are closed. There are maps $p^b: \mathscr{S}_{G/O}(B) \to \mathscr{S}_{G/O}(E)$, $p^*: (G/O)^B \to (G/O)^E$ and $p^\#: \mathscr{L}_{b+f}(E)$ such that the diagram

$$\begin{array}{ccccc}
\mathscr{S}_{G/O}(B) & \longrightarrow & (G/O)^B & \longrightarrow & \mathscr{L}_b(B) \\
\downarrow p^b & & \downarrow p^* & & \downarrow p^\# \\
\mathscr{S}_{G/O}(E) & \longrightarrow & (G/O)^E & \longrightarrow & \mathscr{L}_{b+f}(E)
\end{array}$$

commutes [14, p. 242] where $b = \dim B$ and $f = \dim F$. Then we have a commutative diagram of long exact sequences of homotopy groups

$$\begin{array}{ccccccccc}
\cdots \to & \pi_i(\mathscr{L}_b(B)) & \to & \pi_{i-1}(\mathscr{S}_{G/O}(B)) & \to & \pi_{i-1}((G/O)^B) & \to & \pi_{i-1}(\mathscr{L}_b(B)) & \to \cdots \\
& \downarrow \pi_i(p^\#) & & \downarrow \pi_{i-1}(p^b) & & \downarrow \pi_{i-1}(p^*) & & \downarrow \pi_{i-1}(p^\#) & \\
\cdots \to & \pi_i(\mathscr{L}_{b+f}(E)) & \to & \pi_{i-1}(\mathscr{S}_{G/O}(E)) & \to & \pi_{i-1}((G/O)^E) & \to & \pi_{i-1}(\mathscr{L}_{b+f}(E)) & \to \cdots
\end{array}$$

or, equivalently,

$$\begin{array}{ccccccccc}
\cdots \to & L_{b+i}(Z_p) & \to & \mathscr{S}^s(B \times D^{i-1}, B \times S^{i-2}) & \to & [\Sigma^{i-1}B_+, G/O] & \to & L_{b+i-1}(Z_p) & \to \cdots \\
 & \downarrow{\scriptstyle \pi_i(p^{\#})} & & \downarrow{\scriptstyle \pi_{i-1}(p^{b})} & & \downarrow{\scriptstyle \pi_{i-1}(p^{*})} & & \downarrow{\scriptstyle \pi_{i-1}(p^{\#})} & \\
\cdots \to & L_{b+f+i}(Z_p) & \to & \mathscr{S}^s(E \times D^{i-1}, E \times S^{i-2}) & \to & [\Sigma^{i-1}E_+, G/O] & \to & L_{b+f+i-1}(Z_p) & \to \cdots
\end{array}$$

It is obvious that $\pi_i(p^{\#}) = 0$ if $b + i \equiv 1 \pmod 2$ or $b + f + i \equiv 1 \pmod 2$. So we only consider the case that $b + i \equiv 0 \pmod 2$ and $b + f + i \equiv 0 \pmod 2$. Write

$$L_{b+i}(Z_p) = L_{b+i}(1) \oplus \tilde{L}_{b+i}(Z_p) \quad \text{and} \quad L_{b+f+i}(Z_p) = L_{b+f+i}(1) \oplus \tilde{L}_{b+f+i}(Z_p).$$

By an observation of Wall [14, p. 198] that if $i > 0$, $L_{b+i}(1)(L_{b+f+i}(1)$, respectively) acts trivially on $\mathscr{S}^s(B \times D^{i-1}, B \times S^{i-2})(\mathscr{S}^s(E \times D^{i-1}, E \times S^{i-2})$, respectively), we can write $\pi_i(p^{\#}) = p_1 \oplus p_2$ and the above commutative diagram breaks into two commutative diagrams of short exact sequences:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathscr{S}^s(B \times D^i, B \times S^{i-1}) & \xrightarrow{\eta_1} & [\Sigma^i B_+, G/O] & \xrightarrow{\theta_1} & L_{b+i}(1) & \longrightarrow 0 \\
 & \downarrow{\scriptstyle \pi_i(p^b)} & & \downarrow{\scriptstyle \pi_i(p^*)} & & \downarrow{\scriptstyle p_1} & \\
0 \longrightarrow & \mathscr{S}^s(E \times D^i, E \times S^{i-1}) & \xrightarrow{\eta_2} & [\Sigma^i E_+, G/O] & \xrightarrow{\theta_2} & L_{b+f+i}(1) & \longrightarrow 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 \longrightarrow & \tilde{L}_{b+i}(Z_p) & \xrightarrow{\omega_1} & \mathscr{S}^s(B \times D^{i-1}, B \times S^{i-2}) & \xrightarrow{\eta_1} & [\Sigma^{i-1} B_+, G/O] & \longrightarrow 0 \\
 & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \pi_{i-1}(p^b)} & & \downarrow{\scriptstyle \pi_{i-1}(p^*)} & \\
0 \longrightarrow & \tilde{L}_{b+f+i}(Z_p) & \xrightarrow{\omega_2} & \mathscr{S}^s(E \times D^{i-1}, E \times S^{i-2}) & \xrightarrow{\eta_2} & [\Sigma^{i-1} E_+, G/O] & \longrightarrow 0
\end{array}$$

The following result is due to Sullivan [14, p. 177].

THEOREM 3.1. *Let* $[g] \in [M, G/O]$. *Then*

$$\theta([g]) = \tfrac{1}{8}\sigma(M, g) = \tfrac{1}{8}l(M)g^*l(G/O)[M]$$

if $\dim M \equiv 0 \pmod 4$ *where* $l(M)$ *is the total Hirzebruch class of* M *and* $l(G/O) \in H^{4*}(G/O; R)$ *is defined as in* [14, p. 177] *or* $\theta[g]) = c(M, g) = W(M)g^*\kappa[M]$ *if* $\dim M \equiv 2 \pmod 4$ *where* $W(M)$ *is the total Stiefel-Whitney class of* M *and* $\kappa \in H^{2*}(G/O, Z_2)$ *is defined as in* [14, p. 178].

PROPOSITION 3.2. *The map* $p_1 \colon L_{b+i}(1) \to L_{b+f+i}(1)$ *is given by*

$$p_1(x) = \begin{cases} 0 & \text{if } f \not\equiv 0 \pmod 4, \\ I(F)x & \text{if } f \equiv 0 \pmod 4 \text{ and } b + i \equiv 0 \pmod 4, \\ \chi(F)x & \text{if } f \equiv 0 \pmod 4 \text{ and } b + i \equiv 2 \pmod 4. \end{cases}$$

Proof. It is obvious that ith reduced suspension of the fiber bundle ξ, denoted by $\Sigma^i\xi_+$, is $(\Sigma^i E_+, \Sigma^i B_+, \Sigma^i p_+, F)$ and if $[g] \in [\Sigma^i B_+, G/O]$,

$$\pi_i(p^*)([g]) = [g\Sigma^i p_+].$$

Let τ be the tangent bundle along the fiber of $\Sigma^i\xi_+$. Then

$$l(\Sigma^i E_+) = l(\tau)(\Sigma^i p_+)^* l(\Sigma^i B_+) \quad \text{and} \quad W(\Sigma^i E_+) = W(\tau)(\Sigma^i p_+)^* W(\Sigma^i B_+).$$

Let

$$\mu\colon H^k(\Sigma^i E_+;\, G) \to H^{k-q}(\Sigma^i B_+;\, H^q(F;\, G))$$

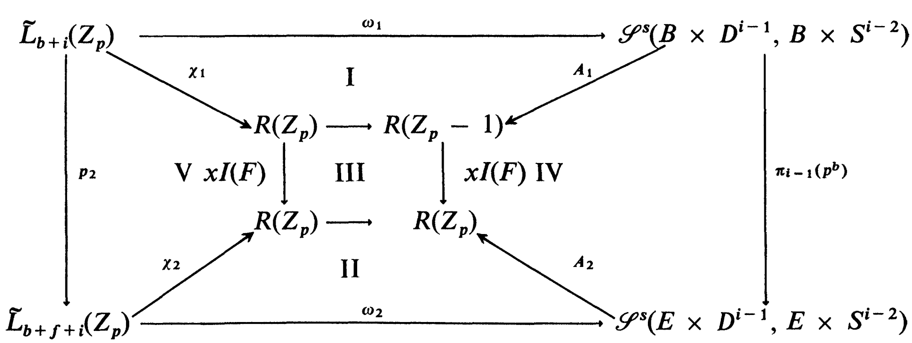
be the integration over the fiber [2] where $G = R$ or Z_2. We denote $\mu(x)$ by $(x)^\mu$. It is easy to see that $(W(\tau))^\mu = \chi(F)\cdot 1 \in H^*(\Sigma^i B_+;\, Z_2)$. Note that H is also the structure group of $\Sigma^i\xi_+$ whose identity component has a finite index by assumption. By a result of Schafer [11], $I(\Sigma^i E_+) = I(F)I(\Sigma^i B_+)$. It follows from a result of Borel and Hirzebruch [2] that the total Hirzebruch class is strictly multiplicative in $\Sigma^i\xi_+$. Thus $(l(\tau))^\mu = I(F)\cdot 1 \in H^*(\Sigma^i B_+;\, R)$. It is obvious that $p_1(x) = 0$ if $b + i \equiv f \equiv 2 \pmod 4$. Next we consider the case $b + i + f \equiv 2 \pmod 4$. There is $[g] \in [\Sigma^i B_+, G/O]$ such that $x = \theta_1([g])$.

$$\begin{aligned} p_1(x) &= p_1\theta_1([g]) \\ &= \theta_2\pi_i(p^*)([g]) \\ &= \theta_2([g\Sigma^i p_+]) \\ &= c(\Sigma^i E_+, g\Sigma^i p_+) \\ &= W(\Sigma^i E_+)(g\Sigma^i p_+)^*\kappa[\Sigma^i E_+] \\ &= (W(\tau)(\Sigma^i p_+)^* W(\Sigma^i B_+)(\Sigma^i p_+)^* g^*\kappa)^\mu[\Sigma^i B_+] \\ &= (W(\tau))^\mu W(\Sigma^i B_+) g^*\kappa[\Sigma^i B_+] \\ &= \chi(F)c(\Sigma^i B_+, g). \end{aligned}$$

If $b + i \equiv 0 \pmod 4$, then $f \equiv 2 \pmod 4$. Since we assume F is oriented, F is a Z_2-manifold in the sense of Sullivan [10]. By an observation in [10], $\chi(F) = 0$. Thus $p_1(x) = 0$. On the other hand, if $b + i \equiv 2 \pmod 2$, then $c(\Sigma^i B_+, g) = \theta_1([g]) = x$. Hence $p_1(x) = \chi(F)x$. Finally, if $b + i \equiv f \equiv 0 \pmod 4$, we can show $p_1(x) = I(F)x$ in a similar way. ∎

PROPOSITION 3.3. *The map* $p_2\colon \tilde{L}_{b+2}(Z_p) \to \tilde{L}_{b+f+i}(Z_p)$ *is given by* $p_2(x) = I(F)x$.

Proof. The diagram



(where the maps χ_i, A_i, γ_i are defined as in [7], [8]) should facilitate the proof. It has been shown in [7], [8] that the subdiagrams I, II, III are commutative. In order to show the subdiagram IV commutes, recall the definition of A_i. Let $[N, h]$ represent an element

$$x \in \mathscr{S}^s(B \times D^{i-1}, B \times S^{i-2}).$$

Let $\tilde{B}$ be the universal covering of B and let $h^*(\tilde{B} \times D^{i-1})$ be the induced covering over N induced by the map h. It is obvious that Z_p acts freely on $\tilde{B} \times D^{i-1}$ and $h^*(\tilde{B} \times D^{i-1})$. Since $h|_{\partial M}\colon \partial M \to B \times S^{i-2}$ is a diffeomorphism, $\partial(h^*\tilde{B} \times D^{i-1})$ is equivariantly diffeomorphic to $\tilde{B} \times S^{i-2}$. Then $A_1(x)$ is defined to be

$$\sum_{g \in Z_p - 1} (\sigma(g, h^*(\tilde{B} \times D^{i-1}) \cup \tilde{B} \times D^{i-1}) - \sigma(g, \tilde{B} \times S^{i-1})).$$

Let $\tilde{E}$ be the universal covering of E and let $\bar{h}\colon h^*(E \times D^{i-1}) \to E \times D^{i-1}$ be the induced map which covers h. It is clear that

$$[h^*(E \times D^{i-1}), \bar{h}] = \pi_{i-1}(p^b)([N, h])$$

and

$$A_2([h^*(E \times D^{i-1}), \bar{h}]) = \sum_{g \in Z_p - 1} (\sigma(g, (\bar{h}^*(\tilde{E} \times D^{i-1}) \cup \tilde{E} \times D^{i-1})) - \sigma(g, \tilde{E} \times S^{i-1})).$$

It is also clear that

$$\bar{h}^*(\tilde{E} \times D^{i-1}) \cup \tilde{E} \times D^{i-1} \to h^*(\tilde{B} \times D^{i-1}) \cup \tilde{B} \times D^{i-1}$$

and

$$\tilde{E} \times S^{i-1} \to \tilde{B} \times S^{i-1}$$

are fiber bundles with fiber F, structure group H, and with free Z_p actions which act as fiber bundle isomorphisms. Hence by Theorem 2.1,

$$\sigma(g, \bar{h}^*(\tilde{E} \times D^{i-1}) \cup \tilde{E} \times D^{i-1}) = I(F)\sigma(g, h^*(\tilde{B} \times D^{i-1}) \cup \tilde{B} \times D^{i-1})$$

and

$$\sigma(g, \tilde{E} \times S^{i-1}) = I(F)\sigma(g, \tilde{B} \times S^{i-1}).$$

Then we have $A_2\pi_{i-1}(p^b)(x) = I(F)A_1(x)$, i.e., subdiagram IV commutes. Since the big diagram commutes, so does subdiagram V. Now our claim follows easily from the fact that $\ker \chi_i = 0$ [14, p. 168]. ∎

Now the following theorem which is Theorem 0.2 in the Introduction is clear.

THEOREM 3.4. *Let $\xi = (E, B, p, F)$ be an oriented smooth fiber bundle where E, B, F are closed. If $\pi_1(B) = \pi_1(E) = Z_p$ and the identity component of the structure group H has a finite index, then, for $i > 0$, $\pi_i(p^{\#})$ is given by*

$$\pi_1(p^*)x = \begin{cases} (I(F)x_1, I(F)x_2) & \text{if } f \equiv 0 \pmod 4 \text{ and } b + i \equiv 0 \pmod 4, \\ (\chi(F)x_1, I(F)x_2) & \text{if } f \equiv 0 \pmod 4 \text{ and } b + i \equiv 2 \pmod 4, \\ 0 & \text{otherwise,} \end{cases}$$

where $x = (x_1, x_2) \in L_{b+i}(1) \oplus \tilde{L}_{b+i}(Z_p) = L_{b+i}(Z_p)$.

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