ON THE DIFFERENTIABILITY OF FUNCTIONS OF SEVERAL REAL VARIABLES

BY

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Introduction

This paper is concerned with two problems. In the first place, we show that if a function f belongs to $L_n^1(R^n)$ (see definition below), then $f(x)$ possesses total differential of order n at almost all the points of R^n (see definition below).

In the second place, if we restrict our attention to functions arising from Bessel Potentials of order n, that is

$$f(x) = \int_{R^n} G_n(x - y)g(y)\, dy$$

where $\hat{G}_n = (1 + |x|^2)^{-n/2}$, $\hat{G}_n$ is the Fourier Transform of G_n and $g \in L^1(R^n)$, our result in this case is that if $g \in L^1(R^n) \cap L^1 \log^+ L^1$ then, f possesses total differential of order n at almost all the points of R^n. This result is the best possible in the sense that given an Orlicz Class $L_\phi(R^n)$ that contains a function g for which

$$\int_{R^n} |g| \log^+ |g|\, dx = \infty,$$

then there exists a function $g_0 \in L^1(R^n) \cap L_\phi(R^n)$ such that $G_n * g_0$ fails to have total differential of any order at almost all the points of R^n.

These two results complete the ones in [5], [6], [7]. We are indebted to the referee for a simplification of the proof of Theorem A.

1. Notation and definitions

As usual α will denote the n-tuple of integers $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and

$$D^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} f, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

$$\alpha! = \alpha_1!\, \alpha_2! \ldots \alpha_n!, \quad D^0 f = f$$

The Taylor's expansion of order m will be written as

$$f(x) = \sum_{|\alpha| \le m} \frac{D^\alpha f(z)}{\alpha!} (x - z)^\alpha + \sum_{|\alpha| = m+1} \frac{(m + 1)}{\alpha!} (x - z)^\alpha \int_0^1 (1 - t)^m D^\alpha f(z + t(x - z))\, dt$$

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(1.1) $L_k^p(R^n)$, k integer, $k \geq 0$, $1 \leq p \leq \infty$, denotes the Sobolev space of functions such that

$$D^\alpha f \in L^p(R^n), \quad 0 \leq |\alpha| \leq k.$$

Here, the derivatives are taken in the distribution sense and

$$\|f\|_{L_k p} = \sum_{0 \leq |\alpha| \leq k} \|D^\alpha f\|_p.$$

(1.2) $\mathscr{L}_k^p(R^n)$ denotes the Bessel Potential Class of functions f represented by the convolution

$$f = G_k * g, \quad g \in L^p(R^n),\ 1 \leq p \leq \infty,$$

$$\hat{G}_k = (1 + |x|^2)^{-k/2}, \quad |x| = \left(\sum_1^n x_i^2\right)^{1/2}, \ k \text{ integer},\ k \geq 0.$$

G_k is called the Bessel kernel of order k. See [9 p. 130]. We define $\|f\|_{\mathscr{L}_k p} = \|g\|_p$. Notice that if $1 < p < \infty$, then $\mathscr{L}_k^p \equiv L_k^p$. (See [9, Chapter V].)

(1.3) Given a vector $h \in R^n$, we define $\Delta_h f(x) = f(x + h) - f(x)$ and

$$\Delta_h^{(k)} f(x) = \Delta_h(\Delta_h^{(k-1)} f)(x) \quad (\Delta_h^{(1)} \equiv \Delta_h)$$

(1.4) Following [5] we shall consider the maximal operator

$$M_k^* f(x) = \operatorname*{Sup}_{h \in R^n} \left| \frac{\Delta_h^{(k)} f(x)}{|h|^k} \right|$$

where f is any real valued function defined on R^n. Likewise, we are going to consider also the Hardy-Littlewood maximal operator

$$\operatorname*{Sup}_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |f|\, dy = M(f)(x).$$

Here, f is measurable and locally integrable.

(1.5) Let $f(x)$ be a real valued measurable function defined on R^n. We say that f has total differential or order k at x_0, if there exists a homogeneous polynomial $P(x)$ of degree k, $P(x)\colon R^n \to R$, such that

$$\lim_{|h| \to 0} \frac{1}{|h|^k} |\Delta_h^{(k)} f(x_0) - P(h)| = 0.$$

2. Statement of results

(2.1) THEOREM A. *Let f belong to $L_n^p(R^n)$, $1 \leq p \leq \infty$. Then*

(i) $\|M_n^* f\|_p \leq C_p \sum_{|\alpha|=n} \|D^\alpha f\|_p$, $1 < p \leq \infty$.

Here C_p depends on n and on p only.

(ii) *If $p = 1$ we have*

$$|E(M_n^* f > \lambda)| < \frac{C_0}{\lambda} \sum_{|\alpha|=n} \|D^\alpha f\|_1$$

(iii) *If* $1 \le p \le \infty$, *f possesses total differential of order n at almost all the points of* R^n.

(2.2) THEOREM B. (i) *Let f be given by*

$$f = G_n * g.$$

Then, if $g \in L^1(R^n) \cap L^1(R^n) \log^+ L^1(R^n)$, *$f$ possesses total differential of order n at almost all the points of* R^n.

(ii) *Let* $L_\phi(R^n)$ *be an Orlicz class that contains a function g for which* $\int_{R^n} |g| \log^+ |g| \, dy = \infty$. *Then there exists a function* $g_0 \in L^1(R^n) \cap L_\phi(R^n)$ *such that* $G_n * g_0$ *fails to have total differential of any order at almost all the points of* R^n.

3. Proof of Theorem A

We are going to show (i) and (ii) for functions in $C_0^\infty(R^n)$ since the general case follows from a standard density argument. C will denote a constant depending on n, not necessarily the same at each occurrence.

Given x and h consider the points $x_0 = x, x_1 = x_0 + h, \ldots, x_n = x_0 + nh$. We also consider a variable point z in the ball $B = \{z \mid |z - x_0| \le 3n|h|\}$. Let $f \in C_0^\infty(R^n)$ and consider the Taylor expansion of f about the point z, namely

$$(3.4) \quad f(s) = \sum_{0 \le |\alpha| \le n-1} \frac{D^\alpha f(z)}{\alpha!} (s - z)^\alpha + n \sum_{|\alpha| = n} (s - z)^\alpha \int_0^1 t^{n-1} D^\alpha f(s + t(z - s)) \, dt.$$

Observe that

$$\Delta_h^{(n)} \left\{ \sum_{0 \le |\alpha| \le n-1} \frac{D^\alpha f(z)}{\alpha!} (x - z)^\alpha \right\} (x) = 0.$$

Consequently

$$|\Delta_h^{(n)} f(x)| \le C \sum_{j=0}^{n} \sum_{|\alpha| = n} \int_0^{|z - x_j|} \rho^{n-1} \left| D^\alpha f\left(x_j + \rho \frac{z - x_j}{|z - x_j|} \right) \right| d\rho.$$

Integrating over $|z - x| \le 3n|h|$, we have

$$|\Delta_h^{(n)} f(x)| \le \frac{C}{|h|^n} \sum_{j=0} \sum_{|\alpha| = n} \int_0^{6n|h|} \rho^{n-1} \int_{|z - x| \le 3n|h|} \left| D^\alpha f\left(x_j + \rho \frac{z - x_j}{|z - x_j|} \right) \right| dz \, d\rho.$$

Observe that

$$\int_{|z - x| \le 3n|h|} \left| D^\alpha f\left(x_j + \rho \frac{z - x_j}{|z - x_j|} \right) \right| dz \le \int_{|z| \le 6n|h|} \left| D^\alpha f\left(x_j + \rho \frac{z}{|z|} \right) \right| dz \le C|h|^n \int_\Sigma |D^\alpha f(x_j + \rho\sigma)| \, d\sigma.$$

Hence

$$|\Delta_h^{(n)}f(x)| \leq C \sum_{j=0}^{n} \sum_{|\alpha|=n} \int_{|z|\leq 6n|h|} |D^\alpha f(x_j + z)|\, dz$$
$$\leq C \sum_{j=0} \sum_{|\alpha|=n} \int_{|z-x|\leq 12n|h|} |D^\alpha f(x + z)|\, dz.$$

We conclude that

$$|M_n^* f(x)| \leq CM\left(\sum_{|\alpha|=n} |D^\alpha f|\right)(x). \tag{3.5}$$

Parts (i) and (ii) of the thesis follow from (3.5) and the Hardy-Littlewood maximal theorem.

The differentiability a.e. follows from the maximal inequalities and from the fact that the property holds for a dense subset $C_0^\infty(R^n)$; see argument in [5, p. 892, Corollary I].

4. Proof of Theorem B

In the first place, we are going to show that if f is given by the Bessel potential

$$f(x) = G_n * g \tag{4.1.1}$$

where $g \in L^1(R^n) \cap L^1(R^n) \log^+ L^1(R^n)$ then, $f(x)$ possesses total differential of order n at almost all the points of R^n.

We shall prove first an auxiliary lemma.

(4.2) LEMMA. *Let $K(f)$ be a singular integral operator, that is*

$$K(f) = af + \text{p.v.} \int_{R^n} K(x - y)f(y)\, dy, \quad \textit{where} \quad K(x)$$

is homogeneous of degree $-n$, has mean value 0 *on the unit sphere and is C^∞ in $R^n - \{0\}$. Here, a stands for a fixed constant. Suppose that*

$$f \in L^1(R^n) \cap L^1(R^n) \log^+ L^1(R^n).$$

Then f admits the decomposition

$$f = f_1 + f_2 \tag{4.2.1}$$

where $f_1 \in L^2(R^n)$ and f_2 satisfies

$$\int_{R^n} |f_2|\, dx < \infty, \quad \int_{R^n} |K(f_2)|\, dx < \infty. \tag{4.2.2}$$

Proof. There is no loss of generality if we assume that $f \geq 0$. Fix a real number $\lambda_0 > 0$ and consider the corresponding Calderon-Zygmund decomposition for f. (See [9, Chapter 1, p. 17].) There exists a family of non overlapping cubes $\{Q_k\}$, $k = 1, 2, \ldots$ with edges parallel to coordinate axes, and such that

$$\lambda_0 < \frac{1}{|Q_k|} \int_{Q_k} f\,dx \leq 2^n \lambda_0. \tag{4.2.3}$$

Let $G_0 = \bigcup_1^\infty Q_k$; then $f(x) \leq \lambda_0$ a.e. in $R^n - G_0$. Call m_k the mean value $(1/|Q_k|) \int_{Q_k} f\,dx$, Ψ_k the characteristic function of the cube Q_k. Ψ_0 will denote the characteristic function of $R^n - G_0$.

Now define

$$f_1 = \Psi_0(x) f(x) + \sum_1^\infty m_k \Psi_k(x). \tag{4.2.4}$$

Consequently

$$f_2 = \sum_1^\infty (f(x) - m_k)\Psi_k(x). \tag{4.2.5}$$

Clearly, $f_1 \in L^2(R^n) \cap L^1(R^n)$. Let now $5G_0$ be $\bigcup_1^\infty 5Q_k$, where $5Q_k$ is the dilation of Q_k 5 times about its center. By using the smoothness of $K(x)$ outside the origin, its homogeneity and the fact that $f_2(x)$ has mean value 0 over Q_k, we have

$$\int_{R^n - 5Q_k} |K[(f - m_k)\Psi_k]|\,dx \leq C_0 \int_{Q_k} |f|\,dx. \tag{4.2.6}$$

Consequently

$$\int_{R^n - 5G_0} |K(f_2)|\,dx \leq C_0 \int_{R^n} |f|\,dx. \tag{4.2.7}$$

Now, using the fact that $K(f)$ is of weak type 1-1 we have

$$\int_{5G_0} |K(f_2)|\,dx \leq C_1 + C_2 \int_{R^n} (|f_2| + 1) \log^+ |f_2|\,dx \tag{4.2.8}$$

where, the constant C_1 depends on the measure of $5G_0$ which does not exceed $(5^n/\lambda_0) \int_{R^n} f\,dx$ and C_2 depends on the operator K only. This finishes the proof of lemma (4.2). Let us now return to the proof of Theorem B and consider $f = G_n * g$. Consider now the decomposition of g as $g_1 + g_2$ as introduced in Lemma (4.2). From Theorem 3, p. 185 in [9], it follows that $G_n * g_1 \in L_n^2(R^n)$ and consequently, it possesses total differential of order n at almost all the points of R^n.

Our next step will be to show that $G_n * g_2$ belongs to $L_n^1(R^n)$. Observe that $G_n * g_2$ is an L^1-function, therefore $D^\beta G_n g_2$ is a tempered distribution. Denoting by $\hat{T}$ the Fourier transform of T we have, for $0 \leq |\beta| \leq n$,

$$(D^\beta G_n^* g_2)^\wedge = \frac{x^\beta}{|x|^\beta} \cdot |x|^{|\beta|} \cdot (1 + |x|^2)^{-|\beta|/2} \cdot (1 + |x|^2)^{-(n-|\beta|)/2} \cdot \hat{g}_2. \tag{4.3.1}$$

Here $x^\beta/|x|^{|\beta|}$ is the symbol of a singular integral operation K_β satisfying the conditions of Lemma (4.2) (see [2 Chapter 5], [3]), $|x|^{|\beta|}(1 + |x|^2)^{-|\beta|/2}$ is the Fourier transform of a finite measure μ_β (see [9 pp. 133–134]) and finally, $(1 + |x|^2)^{-(n-|\beta|)/2}$ is the Fourier transform of $G_{n-|\beta|}$ if $|\beta| < n$ and that of δ if $|\beta| = n$. Consequently, we have for all $\phi \in \mathcal{S}$ (L. Schwartz space of rapidly decreasing C^∞ functions),

$$\langle D^\beta G_n * g_2, \phi\rangle = \langle \hat{K}_\beta \cdot \hat{\mu}_\beta \cdot \hat{G}_{n-|\beta|} \cdot \hat{g}_2, \phi\rangle \tag{4.3.2}$$

$$= \langle \hat{\mu}_\beta \cdot \hat{G}_{n-|\beta|} \cdot \widehat{K_\beta(g_2)}, \hat{\phi}\rangle$$

Taking anti-Fourier transforms we have

$$D^\beta G_n * g_2 = \mu_\beta * G_{n-|\beta|} * K_\beta(g_2) \quad \text{if } |\beta| < n, \tag{4.3.3}$$

$$D^\beta G_n * g_2 = \mu_\beta * K_\beta(g_2) \quad \text{if } |\beta| = n. \tag{4.3.4}$$

The identities (4.3.3) and (4.3.4) should be interpreted in the distributions sense.

Finally, from lemma (2.4) we have

$$\|K_\beta(g_2)\|_1 < \infty \quad \text{for} \quad |\beta| \le n. \tag{4.3.5}$$

This concludes the proof of the first part of Theorem B.

(4.4) *Proof of the second part of Theorem B.* Let g be a function belonging to $L^1(R^n) \cap L_\phi(R^n)$ such that

$$\int_{R^n} |g|\, dx < \infty \quad \text{and} \quad \int_{R^n} \phi(|g|)\, dx < \infty. \tag{4.4.1}$$

Here ϕ is a non-negative convex function, such that $\phi(0) = 0$ and $\phi(2t) < C\phi(t)$ $t > 0$. Assume also that

$$\int_{R^n} |g| \log^+ |g|\, dx = \infty. \tag{4.4.2}$$

Without loss of generality, we may assume

$$|E(|g| > \lambda_1)| \ne |E(|g| > \lambda_2)| \quad \text{if} \quad \lambda_1 \ne \lambda_2. \tag{4.4.3}$$

Our next step will be to construct the function g_0 of the thesis. Call g_1 the radial non-increasing rearrangement of $|g|$ and S_k the set $\{g_1 > 2^k\}$. Calling $\Psi_k(x)$ the characteristic function of the sphere S_k we have

$$\sum_{-\infty}^{\infty} 2^{k-1}\Psi_k(x) \le g_1(x) \le \sum_{-\infty}^{\infty} 2^k\Psi_k(x). \tag{4.4.4}$$

That shows that $\bar{g} = \sum_{-\infty}^{\infty} 2^k\Psi_k(x) \in L^1(R^n) \cap L_\phi(R^n)$.
From (4.42) a summation by parts yields

$$\sum_{1}^{\infty} k2^k|S_k| = \infty. \tag{4.4.5}$$

Consider now $(G_n * \bar{g})(0)$. Notice that $G_n(x) \geq -C(n) \log |x|$ if $|x| \leq \frac{1}{2}$, $C(n) > 0$, and in general $G_n(x) \geq 0$.
Consequently,

$$(G_n * \bar{g})(0) \geq C(n) \left|\sum\right| \sum_{k_0}^{\infty} 2^i \int_0^{|S_k|^{1/n}} (-\log r) r^{n-1}\, dr. \tag{4.4.6}$$

Here $|\sum|$ is the "area" of the unit sphere in R^n; k_0 has been chosen so that $|S_{k_0}|^{1/n} < \frac{1}{2}$ and $|S_k|2^k < 1$ for all $k \geq k_0$. The right hand member of (4.4.6) equals

$$C_0 C(n) \sum_{k_0}^{\infty} 2^k |S_k| \{-\log |S_k|\}. \tag{4.4.7}$$

Notice that

$$-\log |S_k| = -\log (|S_k| 2^k 2^{-k}) = -\log |S_k| 2^k + k \log 2 \geq k \log 2. \tag{4.4.8}$$

By using this last remark and (4.4.5) we see that the series (4.4.7) is divergent and moreover

$$\lim_{h \to 0} (G_n * \bar{g})(h) = \infty. \tag{4.4.9}$$

Let C_j be a sequence of positive numbers such that

$$\sum_1^{\infty} C_j = 1. \tag{4.4.10}$$

Select a denumerable family of points $\{x_j\}$ in R^n, such that the set $\{x_j\}$ is dense in R^n; let

$$g_0(x) = \sum_{j=1} C_j \bar{g}(x - x_j). \tag{4.4.11}$$

Clearly, $g_0 \in L^1(R^n) \cap L_\phi(R^n)$ and on account of (4.49) $g_n * g_0$ is essentially unbounded on each neighborhood of R^n. Thus, $G_n * g_0$ fails to possess total differential of any order at almost all the points of R^n. This concludes the proof of Theorem B.

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