TORSION IN HOMOTOPY ASSOCIATIVE H-SPACES

BY

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1. Introduction

For the purposes of this paper an *H-space* (X, μ) is a pointed topological space X which has the homotopy type of a connected CW complex of finite type together with a basepoint preserving map $\mu: X \times X \to X$ with two sided homotopy unit. An H-space is *homotopy associative* if the maps $\mu(\mu \times 1)$ and $\mu(1 \times \mu)$ are homotopic. An H-space is *mod p finite* if $H^* = H^*(X; Z/p)$ is a finite dimensional Z/p module (Z/p are the integers reduced mod p).

Let p be odd and let (X, μ) be a homotopy associative H-space which is mod p finite. Then H^* and $H_* = H_*(X; Z/p)$ are dual Hopf algebras. By Theorem D of [13], $H^*(X; Z)$ has no p torsion if, and only if, H^* is primitively generated. But, by Theorem 1.1 of [5], H^* is primitively generated if, and only if, H_* is commutative. Thus:

THEOREM 1.1. *Let p be odd. Let (X, μ) be a homotopy associative H-space which is* mod *p finite. Then $H^*(X; Z)$ has no p torsion if, and only if, H_* is commutative.*

This theorem suggests that one might study homotopy associative H-spaces which are mod p finite and have integral p torsion by determining exactly how the resulting lack of commutativity in H_* occurs. The purpose of this paper is to begin such a study.

Now commutativity or the lack of it in H_* is measured by the Lie bracket product $[\ ,\ ]: H_* \otimes H_* \to H_*$ which is defined by the rule

$$[\alpha, \beta][\alpha, \beta] = \alpha\beta - (-1)^{|\alpha|\,|\beta|}\beta\alpha \quad \text{for } \alpha, \beta \in H_*$$

($|\ \ |$ denotes the degree of an element). Let $\{Q_k\}_{k \geq 0}$ be the Milnor elements in the Steenrod algebra $A^*(p)$. (See [8].) In particular Q_0 is the usual Bockstein operation β_p. The action of $A^*(p)$ on H_* is a right action obtained by duality from its left action on H^*.

THEOREM 1.2. *Let p be odd. Let (X, μ) be a homotopy associative H-space which is* mod *p finite. Given integers $k \geq 0$ and $m \geq 1$ suppose $x \in (\ker Q_0\mathscr{P}^m)^{2m+1}$ is primitive and $y = Q_k(x)$ is non decomposable. Then we can find nonzero primitive elements $\{\alpha_s\}_{1 \leq s \leq p-1}$ and β in H_* where $\langle \beta, y\rangle \neq 0$ and α_s is defined recursively by the rule $\alpha_1 = \beta Q_{k+1}$, $\alpha_s = [\alpha_{s-1}, \beta]$ for $s \geq 2$.*

Received May 19, 1975.

If (X, μ) is a mod p finite H-space then $H^*(X; Q)$ is an exterior algebra on r odd dimensional generators for some $r \geq 0$ and X is said to be a *rank r H-space*. Using Theorem 1.2 we show the following.

THEOREM 1.3. *Let p be odd. Let (X, μ) be a 1-connected homotopy associative H-space which is* mod *p finite. Then:*

(a) *if* rank $X < 2(p - 1)$ *then $H^*(X; Z)$ has no p torsion.*

(b) *if* rank $X = 2(p - 1)$ *and $H^*(X; Z)$ has p torsion then H_* is primitively generated and $P(H_*)$ has a Z/p basis*

$$\{\alpha_s\}_{1 \leq s \leq p-1} \cup \{\bar{\alpha}_s\}_{1 \leq s \leq p-1} \cup \{\beta\}$$

where $|\beta| = 2p + 2$ while α_s and $\bar{\alpha}_s$ are defined recursively by the rule $\alpha_1 = \beta Q_1$, $\bar{\alpha}_1 = \beta Q_0$, $\alpha_s = [\alpha_{s-1}, \beta]$ for $s \geq 2$, $\bar{\alpha}_s = [\bar{\alpha}_{s-1}, \beta]$ for $s \geq 2$.

Under the restrictions on (X, μ) stated, Theorem 1.3 completely determines the structure of H_* and H^* as Hopf algebras over $A^*(p)$ for the only possible case of rank $2p - 2$ or less in which integral p torsion occurs. The mod 3 cohomology of F_4 and the mod 5 cohomology of E_8 show that this possibility does occur. Furthermore, in all of 1.1, 1.2, and 1.3 the necessity of some type of homotopy associativity holding for X is essential. This is demonstrated by the H-spaces constructed in [3] or [10].

We prove 1.2 and 1.3 by a combination of arguments involving secondary operations and homological algebra. In Section 2 we discuss finite H-spaces and Hopf algebras. In Section 3 we discuss secondary operations in the cohomology of homotopy associative H-spaces. In Section 4 we prove Theorem 1.2. In Section 5 we prove Theorem 1.3.

Finally I would like to add that both the theorems and the proofs in this paper originated from a consideration of the work of Zabrodsky in [13]. This paper should be viewed as a sequel to that one.

2. Hopf algebras and finite H-spaces

The basic reference for Hopf algebras is [9]. We are only concerned with graded, connected Hopf algebras of finite type over Z/p. Let (X, μ) be an H-space. Then $H^* = H^*(X; Z/p)$ and $H_* = H_*(X; Z/p)$ have natural structures as Hopf algebras over $A^*(p)$ induced by μ and the diagonal map $\Delta: X \to X \times X$. The action of $A^*(p)$ on H_* is a right one and is obtained by duality from the usual left action of $A^*(p)$ on H^*.

Given a Hopf algebra Γ we use $P(\Gamma)$ and $Q(\Gamma)$ to indicate primitives and indecomposables respectively. If Γ^* is the dual of Γ then $P(\Gamma^*)$ and $Q(\Gamma)$ are dual in the sense of a quotient module of Γ being dual to a submodule of Γ^*. In the case $\Gamma = H^*$, $Q(H^*)$ and $P(H_*)$ possess dual Steenrod module structures as well.

Given a Hopf algebra Γ we define the Lie algebra product

$$[\ ,\ ] : \Gamma \otimes \Gamma \to \Gamma$$

as in Section 1. We define the Frobenius map $\xi_p: \Gamma \to \Gamma$ by the rule $\xi_p(x) = x^p$ for any $x \in \Gamma$. These two maps provide Γ with a *restricted Lie algebra* structure. Further $P(\Gamma)$ is a sub restricted Lie algebra of Γ. Given a restricted Lie algebra L we can construct a primitively generated Hopf algebra $V(L)$ such that $P[V(L)]$ and L are isomorphic as restricted Lie algebras. We call $V(L)$ *the universal enveloping Hopf algebra* of L. See [9] for details on $V(L)$.

Given a Hopf algebra Γ we can define its cohomology $H^{**}(\Gamma)$. An element of $H^{s,t}(\Gamma)$ will be said to have external degree s and internal degree t. We will consider it only as a bigraded Z/p module and ignore its other algebraic structure. Let $\psi: \Gamma \to \Gamma \otimes \Gamma$ be the comultiplication of Γ and let $\bar{\psi}: \Gamma \to \Gamma \otimes \Gamma$ be the reduced comultiplication defined by the rule

$$\bar{\psi}(x) = \psi(x) - x \otimes 1 - 1 \otimes x \quad \text{for any } x \in \Gamma.$$

Define a filtration $\{F_q\}_{q \geq 0}$ by the rule $F_0 = Z/p$, $F_1 = P(\Gamma)$ and $F_{q+1} = \{x \in \Gamma \mid \bar{\psi}(x) \in F_q \otimes F_q\}$. Let $E^0(\Gamma)$ be the associated bigraded Hopf algebra. As usual we can consider $E^0(\Gamma)$ to be a graded Hopf algebra by assuming elements of $E^0(\Gamma)^{s,t}$ have degree $s + t$. The filtration $\{F_q\}_{q \geq 0}$ induces a spectral sequence $\{E_r^{**}\}_{r \geq 1}$ of bigraded Z/p modules with $E_2^{**} = H^{**}(E^0(\Gamma))$ and $E_\infty^{**} = H^{**}(\Gamma)$. See [7] for the properties of $H^{**}(\Gamma)$ and the above spectral sequence. The above, of course, is only a simplified version of the spectral sequence constructed in [7].

For the rest of this section we will assume that p is an odd prime and that (X, μ) is an H-space which is mod p finite. It can be shown (see Corollary 3.12 of [2]) that:

LEMMA 2.1. *The dimension of $Q(H^{\text{odd}})$ (and hence $P(H_{\text{odd}})$) as a Z/p module equals the rank of X as an H-space.*

We will also have need of a number of structure theorems regarding $Q(H^*)$ and $P(H_*)$.

LEMMA 2.2. $Q(H^{\text{even}}) = \sum_{m \geq 1} Q_0 \mathscr{P}^m Q(H^{2m+1})$

LEMMA 2.3. *Given $\alpha, \beta \in P(H_{\text{odd}})$ then $\alpha^2 = \beta^2 = 0$, $\alpha\beta = -\beta\alpha$ and $\alpha Q_0 = \beta Q_0 = 0$.*

LEMMA 2.4. *Given $\alpha, \beta \in P(H_{\text{even}})$ then $\alpha^p = \beta^p = 0$, $\alpha\beta = \beta\alpha$ and $\alpha\mathscr{P}^1 = \beta\mathscr{P}^1 = 0$.*

Lemma 2.2 is a result of Lin (see [6]). Lemma 2.3 is a result of Browder (see [5]). Lemma 2.4 can be deduced from 2.2, specifically the fact that $Q(H^{2m}) = 0$ unless $m \equiv 1 \bmod p$. For, by duality, $P(H_{2m}) = 0$ unless $m \equiv 1 \bmod p$. Thus, for dimension reasons, $\mathscr{P}^1$ acts trivially on $P(H_{\text{even}})$ and the restricted Lie algebra structure on $P(H_{\text{even}})$ is trivial.

The Milnor elements $\{Q_k\}_{k \geq 0}$ satisfy the relations

$$Q_k Q_l = \begin{cases} 0, & k = l \\ -Q_l Q_k, & k \neq l \end{cases} \tag{2.5}$$

In particular, for any $k \geq 0$, H_* is a Q_k differential Hopf algebra. For any $0 \neq \beta \in P(H_{\text{even}})$ we now define $\Gamma_k(\beta)$, a sub Q_k-differential Hopf algebra of H_*. Let $S = \{\beta\} \cup \{\alpha_s\}_{s \geq 1}$ where α_s is recursively defined by the rule $\alpha_1 = \beta Q_k$ and $\alpha_s = [\alpha_{s-1}, \beta]$ for $s \geq 2$. Let $L(S)$ be the Z/p module generated by S. By 2.3 and 2.4, $L(S)$ has a well defined restricted Lie algebra structure. Then $\Gamma_k(\beta)$ is the universal enveloping Hopf algebra of $L(S)$. To completely determine $\Gamma_k(\beta)$ it remains to determine which of the elements α_s are nonzero. Obviously $\alpha_s = 0$ implies $\alpha_t = 0$ for $t > s$.

LEMMA 2.6. *For any $k \geq 0$ and any $\beta \in P(H_{\text{even}})$, $\alpha_p = 0$ in $\Gamma_k(\beta)$.*

Proof. By 2.4, $\beta^p = 0$. It follows that $\sum_{i=0}^{p-1} \beta^i \alpha \beta^{p-i-1} = (\beta^p)Q_k = 0$. Thus it suffices to show $\alpha_p = \sum_{i=0}^{p-1} \beta^i \alpha \beta^{p-i}$. It follows by induction on s that

$$\alpha_{s+1} = \sum_{i=0}^{s} (-1)^i \binom{s}{i} \beta^i \alpha \beta^{s-i}.$$

One uses the definition $\alpha_{s+1} = \alpha_s \beta - \beta \alpha_s$ and the binomial identities

$$\binom{s}{i} = \binom{s-1}{i} + \binom{s-1}{i-1} \quad \text{for } i < s.$$

And, since

$$(-1)^i \binom{p-1}{i} \equiv 1 \mod p,$$

it then follows that α_p is of the required form. Q.E.D.

Of course 2.6 is not necessarily the strongest bound on the size of $\Gamma_k(\beta)$ for a given k and β. It is entirely possible that $\alpha_i = 0$ for $i < p$ as well. However Theorem 1.2 assures us that under suitable hypotheses this will not happen. For, the hypotheses enable one to construct secondary cohomology operations by which it can be proved, using homological algebra, that $\alpha_s \neq 0$ for $1 \leq s \leq p - 1$. These secondary operations are constructed in the next section.

3. Secondary operations

Besides 2.5 the Milnor elements satisfy the additional relations

(3.1) $$Q_k \mathscr{P}^{m+p^k} = \mathscr{P}^{m+p^k} Q_k - Q_{k+1} \mathscr{P}^m \quad \text{for any } m \geq 0,\ k \geq 0.$$

It follows that for $k, m \geq 0$ we have the relation

$$Q_0 \mathscr{P}^{m+p^k} Q_k = Q_0 Q_k \mathscr{P}^{m+p^k} + Q_0 Q_{k+1} \mathscr{P}^m.$$

For elements of dimension $2m + 2$ this reduces to the relation

(3.2) $$Q_0 \mathscr{P}^{m+p^k} Q_k = -Q_{k+1} Q_0 \mathscr{P}^m.$$

Now (unstable) relations in the Steenrod algebra give rise to (unstable) secondary operations. We will construct an unstable secondary operation ϕ in dimension $2m + 1$ associated with 3.2 such that:

THEOREM 3.3. *Let (X, μ) be a homotopy associative H-space. Let $k \geq 0$ and $m \geq 1$ be integers. Let $x \in P(H^{2m+1}) \cap \ker(Q_0\mathscr{P}^m)$. Let $y = Q_k(x)$. Then, in $H^* \otimes H^*$,*

$$\bar{\mu}^*\phi(x) = \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} y^{p-1} \otimes y^{i} + Q_{k+1}(z)$$

where $z \in \ker(\bar{\mu}^* \otimes 1 - 1 \otimes \bar{\mu}^*)$.

This theorem is analogous to Proposition 2.1 of [13]. Section 2 of [13], in particular, is a useful preliminary to our proof. The rest of this section will be devoted to proving Theorem 3.3.

All coefficients will be understood to be Z/p. We use K_s to denote the Eilenberg MacLane space $K(Z/p, s)$ and ι_s to denote the fundamental class in $H^s(K_s)$.

Consider the following homotopy commutative diagram of infinite loop spaces and maps.

$$\begin{array}{ccc}
K_{2mp+1} & \xrightarrow{\Omega h} & K_{2mp+2p^{k+1}} \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i_0} \\
E & \xrightarrow{h_2} & E_0 \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j_0} \\
K_{2m+1} & \xrightarrow{h_1} & K_{2m+2p^k} \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g_0} \\
K_{2mp+2} & \xrightarrow{h} & K_{2mp+2p^{k+1}+1}
\end{array}$$

Here

$$h^*(\iota_{2mp+2p^k+1}) = Q_{k+1}(\iota_{2mp+2}), \quad h_1^*(\iota_{2m+2p^k}) = Q_k(\iota_{2m+1}),$$
$$g^*(\iota_{2mp+2}) = Q_0\mathscr{P}^m(\iota_{2m+1}) \quad \text{and} \quad g_0^*(\iota_{2mp+2p^{k+1}+1}) = Q_0\mathscr{P}^{m+p^k}(\iota_{2m+2p^k}).$$

The remaining vertical maps arise as the fibre sequence of g and g_0 respectively while h_2 and Ωh are the induced maps between the two fibre sequences. In $H^*(E)$ let $u = j^*(\iota_{2m+1})$ and $w = Q_k(u)$.

LEMMA 3.5. *There exists $v \in H^{2mp+2p^{k+1}}(E)$ such that*

(a) $$i^*(v) = Q_{k+1}(\iota_{2mp+1})$$

(b) $$\bar{\mu}_E^*(v) = \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} w^{p-i} \otimes w^i$$

Proof. By Proposition 3.1 of [11], $E_0 \simeq K_{2m+2p^k} \times K_{2mp+2p^{k+1}}$ and we can choose the equivalence such that

$$\bar{\mu}_{E_0}^*(1 \otimes \iota_{2mp+2p^{k+1}}) = \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} (\iota_{2m+2p^k} \otimes 1)^{p-i} \otimes (\iota_{2m+2p^k} \otimes 1)^i.$$

Since h_1 is an H-map we let $v = h_1^*(1 \otimes \iota_{2mp+2p^{k+1}})$. Q.E.D.

LEMMA 3.6. $w \neq 0$.

Proof. Since $\Omega g \sim *$ it follows that $\Omega E \sim K_{2m} \times K_{2mp}$. Since $Q_k(\iota_{2m}) \neq 0$ it follows by the naturality of the loop map $\Omega\colon H^*(E) \to H^*(\Omega E)$ that $w \neq 0$.
Q.E.D.

Secondary operations are defined by means of universal example (see Section 1 of [11]). We define a secondary operation ϕ by the universal example (E, μ, v).

To see that ϕ satisfies 3.3 pick $f\colon X \to K_{2m+1}$ such that $f^*(\iota_{2m+1}) = x$ and let $\bar{f}\colon X \to E$ be a lifting of f.

(3.7)
$$\begin{array}{ccccc} & & K_{2mp+1} & & \\ & & \downarrow{\scriptstyle i} & & \\ & \overset{\bar{f}}{\nearrow} & E & & \\ & & \downarrow{\scriptstyle j} & & \\ X & \xrightarrow{f} & K_{2m+1} & \xrightarrow{g} & K_{2mp+2} \end{array}$$

Since x is primitive f is an H-map. Now $\bar{f}$ is not an H-map but its H-map deviation can be calculated. That is, let η be the composition

$$E \times K_{2mp+1} \xrightarrow{1 \times i} E \times E \xrightarrow{\mu_E} E.$$

Then there exists $\omega\colon X \times X \to K_{2mp+1}$ such that the following diagram is commutative

(3.8)
$$\begin{array}{ccccc} X \times X & \xrightarrow{\Delta_{X \times X}} & (X \times X) \times (X \times X) & \xrightarrow{\mu_E(\bar{f} \times \bar{f}) \times \omega} & E \times K_{2mp+1} \\ \downarrow{\scriptstyle \mu} & & & & \downarrow{\scriptstyle \eta} \\ X & & \xrightarrow{\quad\bar{f}\quad} & & E \end{array}$$

(see Section 2 of [13]). It follows that

$$\bar{\mu}^*\bar{f}^*(v) = \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} u^{p-i} \otimes y^i + Q_{k+1}\omega^*(\iota_{2mp+1}). \tag{3.9}$$

Letting $z = \omega^*(\iota_{2mp+1})$ we need only show $z \in \ker(\bar{\mu}^* \otimes 1 - 1 \otimes \bar{\mu}^*)$. To do this we give a different interpretation of ω. Now choosing $\bar{f}$ amounts to choosing a null homotopy $l\colon gf \sim *$. And ω is then defined in terms of l. But l also induces a homotopy equivalence between the fibre F of gf and the fibre $X \times K_{2mp+1}$ of the trivial map $*$. However the H-space structure on $X \times K_{2mp+1}$ induced from F by this equivalence is not simply the product H-space structure. There is the twisting factor $\omega\colon X \times X \to K_{2mp+1}$ as well. That is, the H-space structure on $X \times K_{2mp+1}$ is given by the composition

$$\begin{aligned} X \times K_{2mp+1} \times X \times K_{2mp+1} & \\ \xrightarrow{1 \times T \times 1}\ & X \times X \times K_{2mp+1} \times K_{2mp+1} \\ \xrightarrow{\Delta_{X \times X} \times 1 \times 1}\ & (X \times X) \times (X \times X) \times K_{2mp+1} \times K_{2mp+1} \\ \xrightarrow{\mu \times \omega \times \mu_0}\ & X \times K_{2mp+1} \times K_{2mp+1} \\ \xrightarrow{1 \times \mu_0}\ & X \times K_{2mp+1} \end{aligned}$$

(Again, see Section 2 of [13] for this second interpretation of ω.) (Here T is the twist map and μ_0 is the multiplication on K_{2mp+1}.) Thus to show $z \in \ker(\bar{\mu}^* \otimes 1 - 1 \otimes \bar{\mu}^*)$ it suffices to show:

LEMMA 3.10. *F is a homotopy associative H-space.*

Proof. If A and B are homotopy associative H-spaces and $h: A \to B$ is an H-map then there is an invariant $\alpha(h)$ $[A \times A \times A, \Omega B]$ which is trivial if and only if the induced H-space structure on the fibre of h is homotopy associative (see Section 2 of [13]). Also

$$\alpha(gf) = (\Omega g)_{\#}\alpha(f) + (f \times f \times f)^{\#}\alpha(g)$$

where $(\ )^{\#}$ and $(\ )_{\#}$ denote the mappings induced on homotopy classes. But $\Omega g \sim *$ while $\alpha(g) = 0$ since g is an infinite loop map. Q.E.D.

Remark. John Harper has considerably generalized Theorem 3.3 by using a different argument. His results apply to any decomposition $\sum a_i b_i = Q_0\mathscr{P}^m$ of $Q_0\mathscr{P}^m$.

4. Proof of Theorem 1.2

In this section we will prove Theorem 1.2. Theorem 3.3 is actually the first part of this proof—the part involving secondary operations. In this section we will be concerned with the part of the proof involving homological algebra.

Let x, y, k, and m be as in 1.2. Pick $\beta \in P(H_{2m+2p^k})$ such that $\langle \beta, y \rangle \neq 0$. Let $\Gamma = \Gamma_{k+1}(\beta)$ be the Q_{k+1} differential Hopf algebra defined as in Section 2. To prove 1.2 we need only show $\alpha_{p-1} \neq 0$ in Γ. The inclusion map $\rho: \Gamma \subset H_*$ dualizes to a map of Q_{k+1} differential Hopf algebra $\rho^*: H^* \to \Gamma^*$. Pick a Z_p basis $B = \{b_i\}$ of H^* such that

(i) a subset $B' \subset B$ is a basis of $\ker \rho^*$,
(ii) $A = \{y^i\}_{1 \le i \le p-1} \subset B$,
(iii) for $1 \le s \le p-1$ $\langle \beta^s, b_i \rangle = 0$ unless $b_i = y^s$.

By identifying elements with their images we will speak of elements of H^* as being in Γ^*. Thus $C = B - B'$ is a basis of Γ^* and

$$C \otimes C = \{c' \otimes c'' \mid c', c'' \in C\}$$

is a basis of $\Gamma^* \otimes \Gamma^*$. (Throughout this section tensor product of sets will be defined in a similar manner.)

LEMMA 4.1 *For any $w \in (\Gamma^*)^{2mp+2p^{k+1}}$, $\bar{\mu}^*(w)$ can be expanded in $\Gamma^* \otimes \Gamma^*$ in terms of the elements $C \otimes C - A \otimes A$.*

Proof. The only elements of $A \otimes A$ in dimension $2mp + 2p^{k+1}$ are

$$\{y^{p-s} \otimes y^s\}_{1 \le s \le p-1}.$$

But, for any $1 \leq s \leq p - 1$,

$$\langle \beta^{p-s} \otimes \beta^s, \bar{\mu}^*(w) \rangle = \langle \bar{\mu}_*(\beta^{p-s} \otimes \beta^s), w \rangle = \langle \beta^p, w \rangle = 0$$

since $\beta^p = 0$ by 2.4. Q.E.D.

Let

$$t(y) = \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} y^{p-i} \otimes y^i.$$

Theorem 3.3 implies that in $H^* \otimes H^*$, and hence in $\Gamma^* \otimes \Gamma^*$, $\bar{\mu}^*\phi(x) = t(y) + Q_{k+1}(z)$. Further, $z \in \ker(\bar{\mu}^* \otimes 1 - 1 \otimes \bar{\mu}^*)$. Thus z represents an element $\{z\}$ in $H^{2,2mp+1}(\Gamma)$ when one considers $H^{**}(\Gamma)$ as being defined via the cobar construction.

LEMMA 4.2. $\{z\} \neq 0$.

Proof. We consider $H^{**}(\Gamma)$ as being defined via the cobar construction. Then Q_{k+1} induces a map on $H^{**}(\Gamma)$ and it suffices to show $\{Q_{k+1}(z)\} \neq 0$ in $H^{2,2mp+2p^{k+1}}(\Gamma)$. But this follows from the relation

$$Q_{k+1}(z) = -t(y) + \bar{\mu}^*\phi(z)$$

since, by 4.1, $-t(y) + \mu^*\phi(z)$ has a non zero image in $\ker(\bar{\mu}^* \otimes 1 - 1 \otimes \bar{\mu}^*)/$ image $\bar{\mu}^*$. Q.E.D.

Filter Γ as in Section 2 and let $E^0(\Gamma)$ be the associated graded Hopf algebra. By 4.2 and the spectral sequence of Section 2 it follows that:

LEMMA 4.3. $H^{2,2mp+1}[E^0(\Gamma)] \neq 0$.

But $H^{**}[E^0(\Gamma)]$ is easy to compute. First, $E^0(\Gamma)$ is isomorphic, as a Hopf algebra, to the Hopf algebra $\otimes_{\alpha_s \neq 0} E(\alpha_s) \otimes P(\beta)/\langle \beta^p \rangle$ where E indicates exterior algebra, P indicates polynomial algebra, and all generators are assumed to be primitive. Secondly cohomology respects tensor products; that is, given a tensor product $\otimes_{i=1}^n \Gamma_i$ of Hopf algebras, then $H^{**}[\otimes_{i=1}^n \Gamma_i]$ and $\otimes_{i=1}^n H^{**}(\Gamma_i)$ are isomorphic as bigraded Z/p modules. Finally, we can calculate the cohomology of the factors $E(\alpha_s)$ and $P(\beta)/\langle \beta^p \rangle$.

(a) $H^{**}[E(\alpha_s)]$ and $P(\bar{\alpha}_s)$ are isomorphic as bigraded Z/p modules where $\bar{\alpha}_s$ has bidegree $(1, |\alpha_s|)$.
(b) $H^{**}[P(\beta)/\langle \beta^p \rangle]$ and $E(\bar{\beta}) \otimes P(\bar{\bar{\beta}})$ are isomorphic as bigraded Z/p modules where $\bar{\beta}$ and $\bar{\bar{\beta}}$ have bidegree $(1, 2m + 2p^k)$ and $(2, 2mp + 2p^{k+1})$ respectively.

It now follows from 4.3 that $\alpha_{p-1} \neq 0$. For the only possible non zero element in $H^{2,2mp+1}[E^0(\Gamma)]$ is $\bar{\alpha}_{p-1}\bar{\beta}$.

5. Proof of Theorem 1.3

In this section we prove Theorem 1.3. Let (X, μ) be a 1-connected H-space which is homotopy associative and mod p finite. To prove 1.3 we use 1.2. Hence we first establish:

LEMMA 5.1. *There exists $m \geq 1$ and $x \in P(H^{2m+1}) \cap \ker Q_0\mathscr{P}^m$ such that $y = Q_0(x)$ is non decomposable.*

Proof. By 4.9 of [1], $Q(H^{\text{even}}) \neq 0$. Pick the minimal $n \geq 0$ such that $Q(H^{2n+2}) \neq 0$. By Theorem 6.11 of [1] and the 1-connectedness of X it follows that $n \geq 1$. Using 4.9 of [1] and the fact that H_* is associative it follows that in dimensions $2n + 1$ or less, H^* is isomorphic, as a Hopf algebra, to a primitively generated exterior Hopf algebra. By 2.2, $Q(H^{2n+2}) = Q_0Q(H^{2n+1})$. Hence there exists $\bar{x} \in P(H^{2n+1})$ such that $Q_0(x)$ is non decomposable. Pick the minimal s such that $\bar{x} \in \ker Q_0\mathscr{P}^{p^sn} \cdots \mathscr{P}^{p^n}\mathscr{P}^n$. Let $x = \bar{x}$ if $s = 0$ and $x = \mathscr{P}^{p^{s-1}n} \cdots \mathscr{P}^n(\bar{x})$ if $s > 0$. Let $m = p^sn$. Then $x \in P(H^{2m+1}) \cap \ker Q_0\mathscr{P}^m$. Also $Q_0(x)$ is non decomposable. For $s > 0$ this follows from 4.21 of [9] since $Q_0(x)$ is primitive and lies in dimension $2m + 2$ where $m \equiv 0 \bmod p$. Q.E.D.

Pick $\beta \in P(H_{2m+2})$ such that $\langle \beta, y \rangle \neq 0$ where y and m are as in 5.1.

LEMMA 5.2. *H_* contains a sub Hopf algebra Ω which is primitively generated and $P(\Omega)$ has a Z/p basis*

$$S = \{\alpha_s\}_{1 \leq s \leq p-1} \cup \{\bar{\alpha}_s\}_{1 \leq s \leq p-1} \cup \{\beta\}$$

where $\alpha_1 = \beta Q_1$ $\bar{\alpha}_1 = \beta Q_0$, while $\alpha_s = [\alpha_{s-1}, \beta]$ and $\bar{\alpha}_s = [\bar{\alpha}_{s-1}, \beta]$ for $s \geq 2$.

Proof. By 2.3, 2.4, and 2.6 the Z/p module L generated by S has a well defined restricted Lie algebra structure. Hence there exists a primitively generated Hopf algebra $\Omega \subset H_*$ such that $P(\Omega) = L$. It is left to show $\bar{\alpha}_{p-1} \neq 0$ and $\alpha_{p-1} \neq 0$. Since $\beta\mathscr{P}^1 = 0$ by 2.4 it follows that $\alpha_{p-1} = \bar{\alpha}_{p-1}\mathscr{P}^1$. Hence it suffices to show $\alpha_{p-1} \neq 0$. This follows from 5.1 and 1.2. Q.E.D.

Now part (a) of 1.3 follows from 2.1 and 5.2. To prove part (b) we first observe that if rank $X = 2p - 2$ then, by 2.1 and 2.3, $P(\Omega_{\text{odd}})$ and $P(H_{\text{odd}})$ are isomorphic as Steenrod modules. In particular $\mathscr{P}^q$ acts trivially on $P(H_{\text{odd}})$ if $q > 1$. Then, by the dual of 2.2, β has dimension $2p + 2$ plus $P(\Omega)$ and $P(H_*)$ are isomorphic as Steenrod modules. It remains to show $P(H_*)$ is primitively generated. By 2.1 of [2] it suffices to show $x^p = 0$ if $x \in H^{2p+2}$. But $x^p = \mathscr{P}^{p+1}(x) = P^1\mathscr{P}^p(x)$. Let $y = \mathscr{P}^p(x)$. Now x must be primitive and thus y is primitive. By 4.21 of [9] y will be non decomposable if y is non zero. But this is not possible since $Q(H^{2s}) = 0$ if $s \neq p + 1$. Thus $x^p = \mathscr{P}^1(y) = 0$.

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