A WEAK THEORY OF VECTOR VALUED KÖTHE FUNCTION SPACES

BY

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1. Introduction

Let E be a complete locally convex topological vector space. Let Z be a locally compact, σ-compact, topological space with a positive Radon measure π. Let Λ be a Köthe space of real valued measurable functions on Z. In [11] we have investigated the space $\Lambda(E)$ which is, roughly speaking, the space of measurable functions $f\colon Z \to E$ such that $p(f) \in \Lambda$ for every continuous seminorm p on E. The space $\Lambda(E)$ is topologized by the seminorms $q \circ p(f)$ as p and q run through the families of continuous seminorms on E and Λ respectively.

In this paper, we study the space $\Lambda[E]$ which is the completion of a space of measurable functions $f\colon Z \to E$ such that for every $x' \in E'$, we have

$$\langle f(\cdot), x' \rangle \in \Lambda.$$

The space $\Lambda[E]$ will be topologized by the seminorms

$$\operatorname{Sup}\, \{q(\langle f(z), x' \rangle)\colon x' \in U^\circ\}$$

as q runs through the family of continuous seminorms on Λ and U runs through the family of neighborhoods of zero in E.

The space $\Lambda[E]$ has been extensively studied by Pietsch [12] when Z is the natural numbers and π the counting measure; Cac [3] has chosen a slightly different definition for $\Lambda[E]$ and studied the spaces so obtained.

In Section 2, we review the relevant material about Köthe spaces. In Section 3, we study properties of the spaces $\Lambda[E]$. In Section 4, the topological dual of $\Lambda[E]$ is investigated. In Section 5, we see how certain spaces of linear maps can be represented by $\Lambda[E]$, thus extending or complementing the results of several authors.

2. Definitions and notation

We recall briefly the theory of Köthe spaces as presented in [4]. The space Ω is the set of locally integrable, real valued measurable functions on Z and is topologized by the seminorms $\int_K |a|\, d\pi$ as K runs through the compact sets of Z. A set $A \subseteq \Omega$ is *solid* if it contains with every $a \in A$ also ab where b is in the unit ball of L^∞. A *Köthe space* Λ will be a solid subspace of Ω containing the characteristic functions of relatively compact measurable sets. A topology on Λ is *solid* if it has a base of solid neighborhoods of zero. If Λ has a solid topology, Q will be the set of continuous seminorms which are gauges of solid

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neighborhoods. We say that (Λ, Σ) is a dual pair of Köthe spaces if $\int |ab|\, d\pi < \infty$ for all $a \in \Lambda$ and $b \in \Sigma$. The integral $\int ab\, d\pi$ will always be understood to be the bilinear form connecting the spaces. The Köthe dual Λ^* of Λ is defined by

$$\Lambda^* = \left\{ b \in \Omega: \int |ab|\, d\pi < \infty \text{ for all } a \in \Lambda \right\}.$$

The *solid hull* of $A \subseteq \Omega$ is the smallest solid set containing A.

If (Λ, Σ) is a dual pair of Köthe spaces, then the *normal topology* on Λ is the topology of uniform convergence on solid hulls of points of Σ. This is a solid topology with $\Lambda' = \Sigma$. If in addition $\Lambda = \Sigma^*$, then Λ is complete under the normal topology and thus also under the Mackey and strong topologies. These latter topologies are also solid.

The topological dual of Ω is the space Φ of all essentially bounded measurable functions of essentially compact support.

Let E be a locally convex space. Let P be the set of continuous seminorms on E. If $p \in P$, let E_p be the completion of the normed space $E/p^{-1}(0)$ and $\theta_p\colon E \to E_p$ be the canonical map. A function f from Z into a topological space is *measurable* [1, p. 169] if given a compact set $K \subseteq Z$ and $\varepsilon > 0$ there is a compact set $K' \subseteq K$ with $\pi(K - K') < \varepsilon$ and $f|_{K'}$ continuous. A function $f\colon Z \to E$ is *p-measurable* if $\theta_p \circ f$ is measurable for every $p \in P$. The function f is *weakly measurable* if it is measurable when E is given the weak topology $\sigma(E, E')$ and is *scalarly measurable* if $\langle f(\cdot), x' \rangle$ is measurable for every $x' \in E'$.

Consider the space of functions $f\colon Z \to E$ which are p-measurable and such that $\int_K p(f)\, d\pi < \infty$ for every compact K and $p \in P$. Define $\Omega_o(E)$ to be the separated space associated with this space when equipped with the seminorms $\int_K p(f)\, d\pi$ and $\Omega(E)$ to be its completion. We define $\bar{\Omega}(E')$ to be the set of $\sigma(E', E)$ scalarly measurable functions $g\colon Z \to E'$ satisfying the following condition: For every compact set K, $g|_K = bg_o$ where b is real valued and integrable and g_o is a $\sigma(E', E)$ scalarly measurable function which takes values in an equicontinuous set. We identify g_1 and g_2 if $g_1 = g_2$ scalarly a.e. (i.e., if $\langle x, g_1(\cdot) \rangle = \langle x, g_2(\cdot) \rangle$ a.e. for all $x \in E$). The spaces $\Omega(E)$ and $\bar{\Omega}(E')$ have been studied in [9], [10], and [11]. It is shown there that if $f \in \Omega_o(E)$ and $g \in \bar{\Omega}(E')$, then $\langle f(z), g(z) \rangle$ is a well defined measurable function. Furthermore, if $f \in \Omega(E)$ but $f \notin \Omega_o(E)$ then for $p \in P$ and for $g \in \bar{\Omega}(E')$, we can define $p(f)$ and $\langle f, g \rangle$ in a natural way as real valued measurable functions.

If Λ is a Köthe space with a solid topology, we set

$$\Lambda(E) = \{ f \in \Omega(E) : p(f) \in \Lambda \text{ for all } p \in P \}.$$

We topologize $\Lambda(E)$ with the seminorms $\{ q \circ p(f) : q \in Q, p \in P \}$. A class of functions in $\bar{\Omega}(E')$ is in $\Lambda^\circ(E')$ if there is a function g in the class such that $g = bg_o$ where $b \in \Lambda$ and g_o is scalarly measurable and equicontinuous valued. If (Λ, Σ) is a dual pair of Köthe spaces with $\Lambda^* = \Sigma$ and Λ is given a solid polar topology from Σ, then $\Lambda' = \Sigma$ iff $\Lambda(E') = \Sigma^\circ(E')$ [11, Theorem 2.3].

The following result, which is contained in [15, p. 85], will be needed often.

LEMMA 2.1. *Let E be a locally convex space. Let T_π be an equicontinuous net of linear operations on E into a locally convex space and suppose $T_\pi \to 0$ pointwise. Then $T_\pi \to 0$ uniformly on precompact sets.*

If E and F are locally convex spaces, $\mathscr{L}(E', F)$ will denote the space of continuous linear maps from E' into F. The topology on E' will always be specified and will often be the topology of uniform convergence on the precompact sets in E (denoted E'_π). With an abuse of notation, $\mathscr{L}(U^\circ, F)$ will denote the space of linear maps from E' into F continuous on each equicontinuous set U° (here U° is given the weak topology from E). The space $\mathscr{L}(E', F)$ will be given the topology of uniform convergence on the equicontinuous sets U° in E'. Seminorms generating the topology on $\mathscr{L}(E', F)$ are given by

$$\begin{aligned}\phi \to &\operatorname{Sup} \{|\langle \phi(x'), y'\rangle| : x' \in U^\circ, y' \in V^\circ\}\\ &= \operatorname{Sup} \{q(\phi(x')) : x' \in U^\circ\}\end{aligned}$$

where U and V are arbitrary neighborhoods of zero in E and F, respectively, and q is the gauge of V.

We shall find it convenient to have available the following lemma, much of which is implicit in [7].

LEMMA 2.2. *Let E and F be complete locally convex spaces with neighborhood bases $\{U\}$ and $\{V\}$ respectively. Let $T: E' \to F$ be a linear map. Then the following are equivalent:*

(a) $T \in \mathscr{L}(E'_\pi, F)$.
(b) $T \in \mathscr{L}(U^\circ, F)$.
(c) $T(U^\circ)$ *is compact for every* U *and* $T \in \mathscr{L}(U^\circ, F_\sigma)$.
(d) $T^* \in \mathscr{L}(F'_\pi, E)$.
(e) $T^* \in \mathscr{L}(V^\circ, E)$.
(f) $T^*(V^\circ)$ *is compact for every* V *and* $T^* \in \mathscr{L}(V^\circ, E_\sigma)$.

Proof. (a) $\Rightarrow$ (b). The topology π induces the same topology on U° as does σ [14, p. 106].

(b) $\Rightarrow$ (c). $T(U^\circ)$ is compact since U° is compact and T is continuous on U°.

(c) $\Rightarrow$ (d). For a fixed $y' \in F'$, the form $\langle Tx', y'\rangle$ is continuous on each U° and so, since E is complete, is represented by an element $T^*y' \in E$ [14, p. 107]. Since T^* is obviously the adjoint map, we have $T^{*-1}(U) = T(U^\circ)^\circ$, an F'_π neighborhood, whence $T^* \in \mathscr{L}(F'_\pi, E)$.

The implications (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (a) are similar. ∎

3. The space $\Lambda[E]$

From now on Λ will be a complete Köthe space with a solid topology and E a complete locally convex space. We define $\Lambda[E] = \Lambda \hat{\otimes}_\varepsilon E$ where $\Lambda \hat{\otimes}_\varepsilon E$ is the completion of the tensor product of Λ and E equipped with the topology of biequicontinuous convergence.

Recall that a locally convex space has the approximation property (a.p.) if the identity operator can be approximated, uniformly on precompact sets, by continuous linear operators of finite dimensional range.

PROPOSITION 3.1. (i) $\Lambda[E] \subseteq \mathscr{L}(U^\circ, \Lambda)$.
(ii) *If* Λ *or* E *has a.p., then* $\Lambda[E] = \mathscr{L}(U^\circ, \Lambda)$.

Proof. (i) By [15, IV, 9.1], $\Lambda[E] \subseteq \mathscr{L}(E'_\tau, \Lambda)$ a complete space. By Lemma 2.2, $\mathscr{L}(U^\circ, \Lambda) = \mathscr{L}(E'_\pi, \Lambda) \subseteq \mathscr{L}(E'_\tau, \Lambda)$. Also, $\mathscr{L}(U^\circ, \Lambda)$ is closed in $\mathscr{L}(E'_\tau, \Lambda)$ (uniform limits of continuous functions on U° are continuous) so $\mathscr{L}(U^\circ, \Lambda)$ is complete. An element

$$f = \sum a_i(z) \otimes x_i \in \Lambda \otimes E$$

induces the map $L_f\colon E' \to \Lambda$ given by $L_f x' = \sum a_i(z)\langle x_i, x'\rangle$ which certainly belongs to $\mathscr{L}(U^\circ, \Lambda)$. Thus $\Lambda[E] = \Lambda \hat{\otimes}_\varepsilon E \subseteq \mathscr{L}(U^\circ, \Lambda)$.

(ii) This follows from [15, III, 9.1] (in (c) of that theorem, replace E, E', and F with E'_π, E, and Λ, respectively). ■

Among the spaces which enjoy the a.p. are the L^p spaces [7, p. 185] and the nuclear spaces [15, p. 110]. By modifying the proof that the L^p spaces have the a.p. it is easy to show that any Köthe space with a normal topology has the a.p. The same method may be used to show that if the simple functions are dense in an Orlicz space or a space with the property J of [6], then the space has the a.p. (If the dual of a Köthe space Λ is a Köthe space then the simple functions are dense in Λ since they separate the dual.)

COROLLARY 3.2. *If* Λ *is given a polar topology from a Köthe space* Σ, *then* $\Lambda[E] \subseteq \Omega[E]$.

Proof. For every compact set $K \subseteq Z$, we have $\chi_K(z) \in \Sigma$. Thus since the topology on Λ is solid, the topology on Λ is stronger than the subspace topology from Ω. Thus

$$\Lambda[E] \subseteq \mathscr{L}(U^\circ, \Lambda) \subseteq \mathscr{L}(U^\circ, \Omega) = \Omega[E]. \qquad ■$$

Any element $f = \sum a_i(z) \otimes x_i \in \Lambda \otimes E$ may be considered as a function $f\colon Z \to E$ by setting $f(z) = \sum a_i(z)x_i$. The map in $\mathscr{L}(U^\circ, \Lambda)$ with which f is associated is given by $L_f(x') = \langle f(z), x'\rangle \in \Lambda$. Thus we see that $\Lambda[E]$ is indeed the completion of a space of functions as described in the introduction. There are, however, more functions "in" $\Lambda[E]$. If $f\colon Z \to E$ has the property that $\langle f(z), x'\rangle \in \Lambda$ for all $x' \in E'$ we define $L_f\colon E' \to \Lambda$ by $L_f x' = \langle f(z), x'\rangle$. By identifying f with L_f, we may ask if f is in $\Lambda[E] \subseteq \mathscr{L}(U^\circ, \Lambda)$. The question can be answered in the affirmative in a number of situations as the next two propositions show.

We write $R_j \uparrow R$ to mean that $R_1 \subseteq R_2 \subseteq \cdots$ are measurable sets and $\bigcup R_j = R$. Given Λ, the set of $a \in \Lambda$ such that $a|_{R_j} \to a|_R$ whenever $R_j \uparrow R$ will be denoted Λ_r. By [10, Proposition 3.3], if Λ' is a Köthe space then $\Lambda = \Lambda_r$.

PROPOSITION 3.3. *Suppose Λ' is a Köthe space. Suppose $f\colon Z \to E$ is p-measurable and $p(f) \in \Lambda$ for every $p \in P$. Then $f \in \mathscr{L}(U^\circ, \Lambda)$.*

Thus if Λ or E has a.p., then $f \in \Lambda[E]$ by Proposition 3.1.

Proof. Suppose a net $(x'_\alpha) \subseteq U^\circ$ satisfies $x'_\alpha \to 0$. Let p be the gauge of U and let $q \in Q$. By the comments above, $\Lambda = \Lambda_r$. Thus given an $\varepsilon > 0$ and using a sequence of compact sets $K_j \uparrow Z$ there is a compact set K such that $q(p(f)|_{Z-K}) < \varepsilon/3$ and so $q(\langle f(z), x'\rangle|_{Z-K}) < \varepsilon/3$ for $x' \in U^\circ$. By the p-measurability of f and the fact that $\Lambda = \Lambda_r$ there is a compact set $K' \subseteq K$ such that $\theta_p \circ f|_{K'}$ is continuous and $q \circ p(f|_{K-K'}) < \varepsilon/3$ and so

$$q(\langle f(z), x'\rangle|_{K-K'}) < \varepsilon/3 \quad \text{for } x' \in U^\circ.$$

Now for every z, $\langle f(z), x'_\alpha\rangle \to 0$ and since $\theta_p \circ f(K')$ is compact in E_p, we have $\langle f(z), x'_\alpha\rangle \to 0$ uniformly on $\theta_p \circ f(K')$ (Lemma 2.1). Choose α_o so that if $\alpha \geq \alpha_o$, then for $z \in K$,

$$|\langle f(z), x'_\alpha\rangle| \leq \varepsilon/3q(\chi_{K'}).$$

Then for $\alpha \geq \alpha_o$,

$$\begin{aligned} q(\langle f(z), x'_\alpha\rangle) &\leq q(\langle f(z), x'_\alpha\rangle|_{K'}) + q(\langle f(z), x'_\alpha\rangle|_{K-K'}) \\ &\quad + q(\langle f(z), x'_\alpha\rangle|_{Z-K}) \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon. \end{aligned}$$

Thus $\langle f(z), x'_\alpha\rangle \to 0$ in Λ and $L_f \in \mathscr{L}(U^\circ, \Lambda)$. ∎

Remark. If we replace the hypothesis $p(f) \in \Lambda$ with $L_f x' \in \Lambda$ for all $x' \in E'$ and $L_f(U^\circ)$ is relatively compact then the conclusion of the proposition is still true. For with the aid of Lemma 2.1 the inequalities

$$q(\langle f(z), x'\rangle|_{Z-K}) < \varepsilon/3 \quad \text{and} \quad q(\langle f(z), x'\rangle|_{K-K'}) < \varepsilon/3$$

for $x' \in U^\circ$ are still valid and the proof is as above.

PROPOSITION 3.4. *Suppose $f\colon Z \to E$ has the property that $\langle f(z), x'\rangle \in \Lambda$ for every $x' \in E'$. Suppose that Λ has a normal topology. Then $f \in \Lambda[E]$ if*

(i) *$L_f \in \mathscr{L}(U^\circ, \Lambda_\sigma)$ and f is p-measurable,*

(ii) *$L_f \in \mathscr{L}(U^\circ, \Lambda_\sigma)$ and E is separable, nuclear, or a reflexive Banach space, or*

(iii) *E is weakly sequentially complete and f is weakly measurable.*

Proof. We have already observed that Λ has a.p. so by Proposition 3.1, we need only show that $L_f \in \mathscr{L}(U^\circ, \Lambda)$.

(i) Let $b \in \Lambda'$. Then the map $\psi\colon U^\circ \to L^1$ defined by $\psi(x') = \langle f(z), x'\rangle b(z)$ is continuous into the weak topology of L^1. Thus if $R_j \uparrow R$, then by Lemma 2.1 and the fact that $\Lambda = \Lambda_r$,

$$\langle f(z), x'\rangle b(z)|_{R-R_j} \to 0 \quad \text{uniformly for } x' \in U^\circ.$$

The proof now proceeds as in Proposition 3.3.

(ii) The proof is similar to that of [5, Corollary 9.3.12] (using Lemma 2.2(c) and [15, p. 198, Ex. 31]).

(iii) By a proof similar to that of [5, Proposition 9.3.13] we have $L_f \in \mathscr{L}(U^\circ, \Lambda_\sigma)$. Thus by (i), $f \in \Lambda[E]$. (A weakly measurable function is p-measurable since a weakly measurable function into a Banach space is measurable [2, p. 96, Ex. 25].) ■

If $f \in \Lambda[E] \subseteq \mathscr{L}(U^\circ, \Lambda)$ and $a \in L^\infty$, define $aL_f \in \mathscr{L}(U^\circ, \Lambda)$ by $(aL_f)(x') = a(L_f x')$. We can then say that $A \subseteq \Lambda[E]$ is *solid* if $af \in A$ whenever $f \in A$ and $a \in L^\infty$ satisfies $\|a\|_\infty \leq 1$. If R is a measurable set, we set $f|_R = \chi_R f$. The definition of $\Lambda[E]_r$ is now analogous to that of Λ_r.

PROPOSITION 3.5. *$\Lambda[E]$ is solid.*

Proof. Since $\Lambda[E]$ is the closure of $\Lambda \otimes E$ in $\mathscr{L}(U^\circ, \Lambda)$ we may, given an $f \in \Lambda[E]$, a solid neighborhood V in Λ, and a neighborhood U in E, find a $\sum a_i(z)x_i \in \Lambda \otimes E$ such that

$$L_f x' - \sum a_i(z)\langle x_i, x'\rangle \in V$$

for all $x' \in U^\circ$. If $\|a\|_\infty \leq 1$ we then have

$$aL_f(x') - \sum a(z)a_i(z)\langle x_i, x'\rangle \in V.$$

Thus aL_f is in the closure of $\Lambda \otimes E$ and so in $\Lambda[E]$. ■

Given a solid neighborhood V in Λ and a neighborhood U in E, then $\{f\colon L_f x' \in V \text{ for all } x' \in U^\circ\}$ is a typical element of a neighborhood base in $\Lambda[E]$. Thus it is easy to see that $\Lambda[E]$ has a solid topology and from this it is easy to see that the solid hull of a bounded set in $\Lambda[E]$ is again bounded.

PROPOSITION 3.6. *If $\Lambda = \Lambda_r$ then $\Lambda[E] = \Lambda[E]_r$.*

Proof. Let $f \in \Lambda[E]$ be given and suppose $R_j \uparrow R$. Let a solid neighborhood $V \subseteq \Lambda$ and a neighborhood $U \subseteq E$ be given. Since $L_f(U^\circ)$ is compact, $L_f(x')|_{R_j} \to L_f(x')|_R$ uniformly for $x' \in U^\circ$ (Lemma 2.1). But this says exactly that $f|_{R_j} \to f|_R$ in $\Lambda[E]$. ■

4. The dual of $\Lambda[E]$

Schaefer [15, IV, 9.2], gives a representation for elements of the dual of a tensor product which is symmetric in the factors of the product. We now give an alternative representation which is not symmetric and is very suggestive in the case of $\Lambda[E]$. Note that the following discussion and theorem do not really use the fact that Λ is a Köthe space.

We may construct an element of $\Lambda[E]'$ as follows. Let U be a neighborhood in E and μ a positive Radon measure on U°. Let L^1 and L^∞ be L^1 and L^∞ for the measure μ on U° and let $B \in L^{\infty\circ}(\Lambda')$ (see Section 2; B is a class of functions from U° into Λ'). Any $f \in \Lambda[E] \subseteq \mathscr{L}(U^\circ, \Lambda)$ may be considered as an element of $L^1(\Lambda)$, since as a map from U° into Λ, L_f is continuous and so measurable and if q is a continuous seminorm on Λ, then

$$\begin{aligned}\int_{U^\circ} q(L_f x')\, d\mu(x') &\leq \int_{U^\circ} \operatorname*{Sup}_{x' \in U^\circ} q(L_f x')\, d\mu(x') \\ &\leq \mu(U^\circ) \operatorname*{Sup}_{x' \in U^\circ} q(L_f x') \\ &< \infty.\end{aligned}$$

Also, $\int (L_f x')B(x')\, d\pi$ is μ measurable and almost everywhere defined [9, Theorem 3.2]. Set

$$(*) \qquad \phi(f) = \int_{U^\circ} \int_Z L_f x' B(x')\, d\pi\, d\mu.$$

Let B take values in an equicontinuous set whose polar has gauge q. Then

$$\begin{aligned}|\phi(f)| &\leq \int_{U^\circ} \int_Z |L_f x' B(x')|\, d\pi\, d\mu \\ &\leq \mu(U^\circ) \operatorname*{Sup}_{x' \in U^\circ} q(L_f x')\end{aligned}$$

and ϕ is continuous on $\Lambda[E]$.

THEOREM 4.1. *Every $\phi \in \Lambda[E]'$ can be represented as in* (*).

Proof. Given ϕ, choose neighborhoods U and V in E and Λ such that

$$|\phi(f)| \leq \operatorname{Sup}\left\{\left|\int (L_f x')b\, d\pi\right| : x' \in U^\circ,\ b \in V^\circ\right\}.$$

For $f \in \Lambda[E]$ define a scalar valued function h_f on $U^\circ \times V^\circ$ by $h_f(x', b) = \int (L_f x')b\, d\pi$. Then $h_f \in \mathscr{C}(U^\circ \times V^\circ)$ (the space of continuous functions on $U^\circ \times V^\circ$). For let a net $((x'_\alpha, b_\alpha)) \subseteq U^\circ \times V^\circ$ satisfy $(x'_\alpha, b_\alpha) \to (x', b)$. Then, if q is the gauge of V,

$$\begin{aligned}\left|\int ((L_f x')b - (L_f x'_\alpha)b_\alpha)\, d\pi\right| &\leq \left|\int (L_f x')(b - b_\alpha)\, d\pi\right| + \left|\int L_f(x' - x'_\alpha)b_\alpha\, d\pi\right| \\ &\leq \left|\int (L_f x')(b - b_\alpha)\, d\pi\right| + q(L_f(x' - x'_\alpha)) \\ &\to 0;\end{aligned}$$

this proves the continuity of h_f. Define a continuous linear form μ_o on $h_{\Lambda[E]} \subseteq \mathscr{C}(U^\circ \times V^\circ)$ by $\mu_o(h_f) = \phi(f)$. Then

$$|\mu_o(h_f)| = |\phi(f)| \leq \operatorname{Sup}\left\{\left|\int (L_f x')b\, d\pi\right| : x' \in U^\circ,\ b \in V^\circ\right\} = \|h_f\|.$$

Thus μ_o is well defined and continuous. By the Hahn-Banach Theorem extend μ_o to a Radon measure μ_o on $\mathscr{C}(U^\circ \times V^\circ)$. Set $\mu = |\mu_o|$. Finally, any

$c \in \mathscr{C}(U^\circ)$ can be considered as a $c' \in \mathscr{C}(U^\circ \times V^\circ)$ by setting $c'(x', b) = c(x')$. Thus μ induces a Radon measure on $\mathscr{C}(U^\circ)$ which we again denote μ. Now

$$|h_f(x', b)| = \left|\int (L_f x')b \, d\pi\right| \leq q(L_f x')$$

and so

$$|\phi(f)| = |\mu_o(h_f)| \leq \mu(h_f) \leq \mu(qL_f).$$

Thus ϕ is continuous on $\Lambda[E]$ when given the subspace topology induced from $L^1(\Lambda)$. Thus, by the Hahn-Banach Theorem and [11, Theorem 2.3], there is a $B \in L^{\infty\circ}(\Lambda')$ such that

$$\phi(f) = \int_{U^\circ} \int_Z (L_f x')B(x') \, d\pi \, d\mu. \qquad \blacksquare$$

The great temptation is to try to define a function $g: Z \to E'$ by

$$\langle x, g(z)\rangle = \int_{U^\circ} \langle x, x'\rangle B(x')(z) \, d\mu;$$

for then we have formally, for functions $f \in \Lambda[E]$,

$$\begin{aligned} \int_Z \langle f, g\rangle \, d\pi &= \int_Z \int_{U^\circ} \langle f(z), x'\rangle B(x') \, d\mu \, d\pi \\ &= \int_{U^\circ} \int_Z L_f(x')B(x') \, d\pi \, d\mu \\ &= \phi(f) \end{aligned}$$

and g gives a representation of the functional. In case that $Z = N$, the natural numbers, this can easily be done and we have a new proof of [12, Satz 4.13]. We have also been able to do this in several special cases, all of which however are contained in Theorem 4.9 which is obtained by a slightly different method. We now build the necessary machinery to obtain the result.

Definition. A p-measurable function $f: Z \to E$ will be in $\Lambda_o[E]$ if $L_f \in \Lambda[E]$. We identify f_1 and f_2 if

$$\langle f_1(z), x'\rangle = \langle f_2(z), x'\rangle \text{ a.e.} \quad \text{for all } x' \in E',$$

i.e., if f_1 and f_2 are scalarly a.e. equal. (See Proposition 3.3.)

Definition. A function $g: Z \to E'$ will be in $\Lambda^\circ[E']$ if there is a neighborhood U in E, a positive Radon measure μ on U°, a $b \in \Lambda$, and a scalarly measurable function $g_o: Z \to U^\circ$ with $g = bg_o$ and

$$(*) \qquad |\langle x, g_o(z)\rangle| \leq \int_{U^\circ} |\langle x, x'\rangle| \, d\mu \quad \text{a.e.}$$

Scalarly a.e. equal functions are identified.

PROPOSITION 4.2. *If E is separable, then the function g_o in the definition of $\Lambda^\circ[E']$ may be chosen so that (*) holds everywhere.*

Proof. Let $\{x_n\} \subseteq E$ be dense. By altering g_o on a set of measure zero we may assume that

$$|\langle x_n, g_o(z)\rangle| \leq \int_{U^\circ} |\langle x_n, x'\rangle| \, d\mu$$

everywhere for all n. If $x_\alpha \to x \in E$, then for a fixed z,

$$|\langle x_\alpha, g_o(z)\rangle| \to |\langle x, g_o(z)\rangle|$$

and

$$\int_{U^\circ} |\langle x_\alpha, x'\rangle| \, d\mu \to \int_{U^\circ} |\langle x, x'\rangle| \, d\mu$$

since $\| \langle x_\alpha, x'\rangle| - |\langle x, x'\rangle\| \leq |\langle x - x_\alpha, x'\rangle| \leq p(x - x_\alpha) \to 0.$ ■

PROPOSITION 4.3. *If $f \equiv 0$ in $\Lambda_o[E]$, then $p(f) = 0$ a.e. for every $p \in P$.*

Proof. The function $\theta_p \circ f$ is measurable and scalarly a.e. equal to zero in E_p. By [7, p. 21], $\theta_p \circ f = 0$ a.e. which gives the result. ■

We now compare the spaces introduced in [9] and [11] with those in this paper.

PROPOSITION 4.4. (i) *If $\Lambda' = \Lambda^*$ then $\Lambda(E) \subseteq \Lambda[E]$,*
(ii) *If $\Lambda' = \Lambda^*$, $\Lambda^{**} = \Lambda$, and E is nuclear, $\Lambda(E) = \Lambda[E]$.*
(iii) $\Lambda^\circ[E'] \subseteq \Lambda^\circ(E')$.
(iv) *If E is nuclear, $\Lambda^\circ[E'] = \Lambda^\circ(E')$.*

Proof. Let $\Lambda \otimes_\alpha E$ and $\Lambda \otimes_\pi E$ be $\Lambda \otimes E$ equipped with the subspace topology from $\Lambda(E)$ and with the projective topology respectively. Then the identity maps

$$i\colon \Lambda \otimes_\pi E \to \Lambda \otimes_\alpha E \quad \text{and} \quad i'\colon \Lambda \otimes_\alpha E \to \Lambda \otimes_\varepsilon E$$

are continuous. For let $p \in P$ and $q \in Q$ and let $f = \sum a_i(z)x_i \in \Lambda \otimes E$. Then

$$\begin{aligned} q(p(f)) &= qp(\textstyle\sum a_i(z)x_i) \\ &\leq \textstyle\sum q(a_i(z)p(x_i)) \\ &= \textstyle\sum p(x_i)q(a_i(z)). \end{aligned}$$

Thus $q(p(f)) \leq \text{Inf}\,\{\sum p(x_i)q(a_i(z)) \colon f = \sum a_i(z)x_i\}$ which is a typical semi-norm for the topology $\Lambda \otimes_\pi E$ [15, III, 6.3]. Thus i is continuous. Also,

$$\sup_{x' \in U^\circ} q(\langle f(z), x'\rangle) \leq qp(f)$$

showing that i' is continuous.

(i) $\Lambda \otimes_\alpha E$ is dense in $\Lambda(E)$ since it is easy to show that it separates points of $\Lambda(E)'$ ($= \Lambda'^{\circ}(E')$ by [11, Theorem 2.3]). Thus i' has a continuous extension from $\Lambda(E)$ into $\Lambda[E]$. By [9, Proposition 6.2], the extension is one-to-one.

(ii) By [15, IV, 9.4, Cor. 2], $\Lambda \otimes_\pi E = \Lambda \otimes_e E$ so $\Lambda \otimes_\alpha E = \Lambda \otimes_e E$. By [11, Theorem 2.4], $\Lambda[E]$ is complete. Then $\Lambda(E) = \Lambda \hat{\otimes}_\alpha E = \Lambda \hat{\otimes}_e E = \Lambda[E]$.

(iii) This follows immediately from the definitions.

(iv) This follows immediately from the definitions and the fact that for any equicontinuous set in E' there is a radon measure μ satisfying (*) [15, IV, 10.2]. ∎

PROPOSITION 4.5. *If $f \in \Lambda_o[E]$ and μ is a Radon measure on U°, $\langle f(z), x'\rangle$ is $\pi \times \mu$ measurable and $\langle f(z), x'\rangle b(z)$ is $\pi \times \mu$ integrable for any $b \in \Lambda^*$.*

Proof. Let $\varepsilon > 0$ and $p \in P$ be given. Since $\theta_p \circ f$ is measurable we may find a compact $K' \subseteq K$ such that $\pi(K - K') < \varepsilon(\mu(U^\circ))^{-1}$ and $\theta_p \circ f|_{K'}$ is continuous. Then

$$(\pi \times \mu)(K \times U^\circ - K' \times U^\circ) < \varepsilon$$

and we claim that $\langle f(z), x'\rangle$ is continuous on $K' \times U^\circ$. Let a net $((z_\alpha, x'_\alpha)) \subseteq K' \times U^\circ$ satisfy $(z_\alpha, x'_\alpha) \to (z, x')$. Set $x_\alpha = f(z_\alpha)$, $x = f(z)$. Then

$$\begin{aligned} |\langle x, x'\rangle - \langle x_\alpha, x'_\alpha\rangle| &\leq |\langle x, x' - x'_\alpha\rangle| + |\langle x - x_\alpha, x'_\alpha\rangle| \\ &\leq |\langle x, x' - x'_\alpha\rangle| + p(x - x_\alpha) \\ &\to 0. \end{aligned}$$

This establishes the desired measurability. By the Tonelli Theorem,

$$\begin{aligned} \int_{Z \times U^\circ} |\langle f(z), x'\rangle b(z)|\, d(\pi \times \mu) &= \int_{U^\circ} \int_Z |\langle f(z), x'\rangle b(z)|\, d\pi\, d\mu \\ &\leq \mu(U^\circ) \operatorname*{Sup}_{x' \in U^\circ} \int_Z |\langle f(z), x'\rangle b(z)|\, d\pi \\ &< \infty. \end{aligned}$$ ∎

Functions in $\Lambda_o[E]$ and $\Lambda^\circ[E']$ are not a.e. defined. However, we have the following result.

PROPOSITION 4.6. *If (Λ, Σ) is a dual pair of Köthe spaces and $f \in \Lambda_o(E)$ and $g \in \Sigma^\circ(E')$, then $\langle f(z), g(z)\rangle$ is a well defined measurable function.*

Proof. By Proposition 4.3, if $f \equiv 0$ in $\Lambda_o[E]$, then $p(f) = 0$ a.e. for all $p \in P$. The proof is now exactly as in [9, Theorem 3.2]. ∎

PROPOSITION 4.7. *If (Λ, Σ) is a dual pair of Köthe spaces, and $g \in \Sigma^\circ[E']$ then the map $f \to \langle f, g\rangle$ is continuous from $\Lambda_o[E]$ into L^1.*

Proof. Let $g = bg_o$ as in the definition of $\Sigma^\circ[E']$. Then

$$\int |\langle f, g\rangle| \, d\pi = \int |\langle f, g_o\rangle b| \, d\pi$$
$$\leq \int_Z \int_{U^\circ} |\langle f(z), x'\rangle b(z)| \, d\mu \, d\pi$$
$$= \int_{U^\circ} \int_Z |\langle f(z), x'\rangle b(z)| \, d\pi \, d\mu$$
$$\leq \mu(U^\circ) \operatorname*{Sup}_{x' \in U^\circ} \int_Z |\langle f(z), x \rangle b(z)| \, d\pi$$
$$< \infty.$$

This shows that $\langle f, g\rangle \in L^1$ and that $\int |\langle f, g\rangle| \, d\pi$ is dominated by a continuous seminorm on $\Lambda_o[E]$.

For a fixed $g \in \Sigma^\circ[E']$ we now define $\langle f, g\rangle \in L^1$ for every $f \in \Lambda[E]$ by extending the continuous map of the above proposition. One can then easily prove the following result.

PROPOSITION 4.8. (i) *The form* $\langle f, g\rangle$ *is bilinear.*

(ii) *For* $a \in L^\infty$, $a\langle f, g\rangle = \langle af, g\rangle = \langle f, ag\rangle$ (*see definition preceding Proposition* 3.5).

(iii) *If* $f \in \Lambda[E]$ *and* $g = bg_o \in \Sigma^\circ[E]$, *then*

$$\int |\langle f, g\rangle| \, d\pi \leq \mu(U^\circ) \operatorname*{Sup}_{x' \in U^\circ} \int |L_f x' b(z)| \, d\pi.$$

THEOREM 4.9. *If* Λ *is given the normal topology from* Σ, *then* $\Lambda[E]' = \Sigma^\circ[E']$.

Proof. Proposition 4.8 (iii) shows immediately that $f \to \int \langle f, g\rangle \, d\pi$ is a continuous linear functional on $\Lambda[E]$.

Now let $\phi \in \Lambda[E]'$ be given. Then there is a $b_o \in \Sigma$ with $b_o \geq 0$ and a neighborhood U in E such that

$$(*) \qquad |\phi(f)| \leq \operatorname{Sup} \left\{ \int |L_f(x')| b_o(z) \, d\pi \colon x' \in U^\circ \right\}.$$

For $f \in \Lambda[E]$ and $b \in B^\infty$, the unit ball of L^∞, define

$$h_f(b, x') = \int b(z) L_f x' b_o(z) \, d\pi.$$

Then as in the proof of Theorem 4.1, $h \in \mathscr{C}(B^\infty \times U^\circ)$, the space of continuous functions on $B^\infty \times U^\circ$. Define a continuous linear form μ_o on $h_{\Lambda[E]} \subseteq \mathscr{C}(B^\infty \times U^\circ)$ by $\mu_o(h_f) = \phi(f)$. As in the proof of Theorem 4.1, μ_o is well defined and continuous. Set $\mu = |\mu_o|$ and also denote by μ the restriction of μ to $\mathscr{C}(U^\circ)$ as in the proof of Theorem 4.1. Then for $f \in \Lambda[E]$,

$$(**)\qquad \begin{aligned} |\phi(f)| &= |\mu_o(h_f)| \\ &\leq \int_{B^\infty \times U^\circ} |h_f(b, x')|\, d\mu \\ &= \int_{B^\infty \times U^\circ} \left| \int_Z bL_f x' b_o\, d\pi \right| d\mu \\ &\leq \int_{U^\circ} \int_Z |L_f x'| b_o(z)\, d\pi\, d\mu. \end{aligned}$$

For any element of $L^1(E)$ (not $L^1[E]$) of the form $b_o \sum a_i(z)x_i$ where $a_i \in \Lambda$ and $x_i \in E$ define

$$\psi(b_o \textstyle\sum a_i(z)x_i) = \phi(\sum a_i(z)x_i).$$

Then (*) shows that ψ is well defined. By (**),

$$\begin{aligned} |\psi(b_o \textstyle\sum a_i(z)x_i)| &= |\phi(\sum a_i(z)x_i)| \\ &\leq \int_{U^\circ} \int_Z |\langle b_o \textstyle\sum a_i(z)x_i, x'\rangle|\, d\pi\, d\mu \\ &\leq \mu(U^\circ) \int_Z p(b_o \textstyle\sum a_i(z)x_i)\, d\pi. \end{aligned}$$

Thus ψ is continuous on a subspace of $L^1(E)$. Extend ψ to all of $L^1(E)$ by the Hahn-Banach Theorem. Then by [11, Theorem 2.3] there is a scalarly measurable $g_o\colon Z \to U^\circ$ such that

$$(***)\qquad \phi(a(z)x) = \psi(b_o a(z)x) = \int \langle a(z)x, b_o(z)g_o(z)\rangle\, d\pi$$

Comparing this with (**) we have

$$\begin{aligned} \left| \int_Z \langle a(z)x, b_o(z)g_o(z)\rangle\, d\pi \right| &\leq \int_{U^\circ} \int_Z |\langle a(z)x, x'\rangle| b_o(z)\, d\pi\, d\mu \\ &= \int_Z |a(z)| b_o(z) \int_{U^\circ} |\langle x, x'\rangle|\, d\mu\, d\pi. \end{aligned}$$

Since Λ is solid, we have

$$\int_Z |ab_o\langle x, g_o(z)\rangle|\, d\pi \leq \int_Z |ab_o| \int_{U^\circ} |\langle x, x'\rangle|\, d\mu\, d\pi.$$

If R is any measurable set then setting $a = \chi_R$ in the above we have,

$$\int_R |b_o\langle x, g_o(z)\rangle|\, d\pi \leq \int_R b_o \int_{U^\circ} |\langle x, x'\rangle|\, d\mu\, d\pi.$$

Thus

$$b_o(z)|\langle x, g_o(z)\rangle| \leq b_o(z) \int_{U^\circ} |\langle x, x'\rangle|\, d\mu \quad \text{a.e.}$$

Since we may assume $g_o(z) = 0$ whenever $b_o(z) = 0$ without changing (***) we have

$$|\langle x, g_o(z)\rangle| \leq \int_{U^\circ} |\langle x, x'\rangle| \, d\mu \quad \text{a.e.}$$

Thus $g = b_o g_o \in \Sigma^\circ[E']$. Using (***), and the linearity of ϕ we see that g represents ϕ on $\Lambda \otimes E$. By the continuity of ϕ the representation extends to all of $\Lambda[E]$.

It is not hard to show that $\Lambda[E]$ separates $\Sigma^\circ[E']$ and so different elements of $\Sigma^\circ[E']$ induce different elements of $\Lambda[E]'$. ∎

5. $\Lambda[E]$ as a space of linear maps

Recall [5, p. 591] that a Radon measure on Z into E is a continuous linear map of $\mathscr{K}(Z)$, the space of continuous functions on Z with compact support equipped with the usual inductive limit topology, into E. We shall say that a Radon measure ϕ is *absolutely continuous* with respect to π if for each $x' \in E'$ the (scalar valued) Radon measure $\langle \phi(\cdot), x'\rangle$ is absolutely continuous with respect to π.

Let $f \in \Omega[E]$. By Lemma 2.2, $L_f^* \in \mathscr{L}(\Phi, E)$. For $b \in \Phi$ we set $L_f^*(b) = \int bf \, d\pi$. Thus

$$\left\langle \int bf \, d\pi, x' \right\rangle = \int L_f x' b \, d\pi$$

and if f is a function,

$$\left\langle \int bf \, d\pi, x' \right\rangle = \int \langle f(z), x'\rangle b \, d\pi.$$

Thus every $f \in \Omega[E]$ induces a map of $\mathscr{K}(Z)$ ($\subseteq \Phi$) into E.

THEOREM 5.1. *$\Omega[E]$ can be identified with the space of Radon measures into E whose restrictions to compact sets in Z are compact linear maps and which are absolutely continuous with respect to π.*

Proof. Let $f \in \Omega[E]$. By Lemma 2.2 the map $b \to \int bf \, d\pi$ maps the equicontinuous sets in Φ into compact sets in E (the equicontinuous sets in Φ are those whose supports are contained in a fixed compact set and are uniformly bounded). By restricting this map to $\mathscr{K}(Z)$, we see that f induces a compact Radon measure on compact sets in Z.

If $\pi(R) = 0$, then $\chi_R \equiv 0$ in Φ and so the measure is absolutely continuous with respect to π.

Conversely, let $m: \mathscr{K}(Z) \to E$ be a Radon measure of the supposed type. By [5, p. 592], m can be extended to Φ whence by Lemma 2.2 and Proposition 3.1, $m^* \in \Omega[E]$. ∎

COROLLARY 5.2. *If E is a nuclear Fréchet space, the space of Radon measures into E absolutely continuous with respect to π may be identified with the set of measurable functions f: $Z \to E$ such that $\int_K p(f) \, d\pi < \infty$ for every continuous seminorm p and compact set K.*

Proof. Since E is nuclear, the relatively compact sets in E are precisely the bounded sets [15, p. 101].

Now $\mathscr{K}(Z)$ is a strict inductive limit of Banach spaces so the continuous maps from $\mathscr{K}(Z)$ into E are just those bounded on each $\mathscr{K}(K)$ where K is compact. Putting these facts together with the theorem, we see that $\Omega[E]$ can be identified with the space of Radon measures into E. But by Proposition 4.4, $\Omega[E] = \Omega(E)$ and by [9], $\Omega(E)$ is the space of functions described. ∎

The above result is contained in [16, p. 65] where a different proof is given. We hope to explore the relationship between other results in [16] with those given here in a later paper. Rao [13, p. 158] and Edwards [5, 8.19.4 and 8.19.5] also have similar results.

THEOREM 5.3. *Suppose* $\Lambda = \Lambda^{**}$. *Let* Λ *be given the Mackey topology from* Λ^* *and* Λ^* *be given the topology of precompact convergence from* Λ. *Suppose that* E *or* Λ^* *has a.p. Then* $\mathscr{L}(\Lambda, E) = \Lambda^*[E]$.

Proof. By [4, Théorème 6], Λ is complete and by [10, Proposition 3.7], Λ^* is quasicomplete. Thus by [8, p. 309], the topological dual of Λ^* equipped with the topology of precompact convergence is Λ. Thus by Lemma 2.2 and Proposition 3.1,

$$\Lambda^*[E] = \mathscr{L}(U^\circ, \Lambda^*) = \mathscr{L}(\Lambda_\pi^{**}, E) = \mathscr{L}(\Lambda, E). \qquad ∎$$

Any $\phi \in \mathscr{L}(\Lambda, E)$ induces an additive set function on the relatively compact measurable sets in Z defined by $R \to \phi(\chi_R)$. Rao [13, Theorem 3.2] has characterized those set functions which arise in this manner with the assumption that Λ is a Banach space but without the assumption that Λ' is a Köthe space. Thus the present result complements Rao's. Note that if Λ' is a Köthe space then $\Lambda = \Lambda_r$ and so a set function which represents an element of $\mathscr{L}(\Lambda, E)$ is countably additive.

COROLLARY 5.4. *Suppose* $\Lambda = \Lambda^{**}$. *Let* Λ *be given the Mackey topology from* Λ^* *and* Λ^* *be given the topology of precompact convergence from* Λ. *Suppose* E *is a nuclear Fréchet space. Then* $\mathscr{L}(\Lambda, E)$ *can be identified with the space of measurable functions* $f\colon Z \to E$ *such that* $p(f) \in \Lambda^*$ *for every continuous seminorm* p *on* E.

Proof. We have by the theorem and Proposition 4.4, $\mathscr{L}(\Lambda, E) = \Lambda^*[E] = \Lambda^*(E)$ and $\Lambda^*(E)$ is the space of functions described above. ∎

By a proof similar to that of Theorem 5.3, we may prove:

THEOREM 5.5. *If* E *is polar reflexive and if* Λ *or* E'_π *has a.p., then* $\mathscr{L}(E, \Lambda) = \Lambda[E'_\pi]$.

Polar reflexivity is defined in [8, p. 308], where it is shown that all Fréchet spaces and all reflexive spaces are polar reflexive.

Results similar to the theorem above are found in [6].

BIBLIOGRAPHY

1. N. BOURBAKI, *Intégration*, Éléments de Mathématique, Livre VI, Chapitres 1–4, Seconde Édition, Hermann, Paris, 1965.
2. ———, *Intégration*, Éléments de Mathématique, Livre VI, Chapitre 6, Paris, 1959.
3. N. CAC, *Generalized Köthe function spaces II*, preprint.
4. J. DIEUDONNÉ, *Sur les espaces de Köthe*, J. Analyse Math., vol. 1, (1951), pp. 81–115.
5. R. E. EDWARDS, *Functional analysis*, Holt, Rinehart and Winston, New York, 1965.
6. N. GRETSKY, *Representation theorems on Banach function spaces*, Mem. Amer. Math. Soc., no. 84, 1968.
7. A. GROTHENDIECK, *Produits tensoriels et espaces nucléares*, Mem. Amer. Math. Soc., no. 16, 1955.
8. G. KÖTHE, *Topological vector spaces I*, Springer-Verlag, Berlin, 1969.
9. A. MACDONALD, *Vector valued Köthe function spaces I*, Illinois J. Math., vol. 17 (1973), pp. 533–545.
10. ———, *Vector valued Köthe function spaces II*, Illinois J. Math., vol. 17 (1973), pp. 546–557.
11. ———, *Vector valued Köthe function spaces III*, Illinois J. Math., vol. 18 (1974), pp. 136–146.
12. A. PIETSCH, *Verallgemeinerte Vollkommene Folgenraume*, Akademie-Verlag, Berlin, 1962.
13. M. RAO, *Linear operations, tensor products, and contractive projections in function spaces*, Studia Math., vol. 38 (1970), pp. 131–186.
14. A. P. ROBERTSON AND W. J. ROBERTSON, *Topological vector spaces*, Cambridge University Press, Cambridge, 1966.
15. H. SCHAEFER, *Topological vector spaces*, Collier, MacMillan Ltd., New York, 1966.
16. E. THOMAS, *The Lebesgue-Nikodym Theorem for vector valued Radon measures*, preprint.

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